II CONFERENZA

COMPLEX QUATERNIONIC ANALYSIS, CONNECTION TO MATH. PHYSICS.

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(II.1) The connection with Atiyah-Hitchin-Singer work on instantons.

The equations, discussed in the first lecture, can be given another interesting interpretation. In the recent work by Atiyah, Hitchin and Singer ([AHS]) the instanton solutions of selfdual Yang--Mills field equation on Eucidean spacetimes are described. During the description of Penrose's twistor method, applied to the Riemannian case, two important differential operators on conformal manifolds are described - the Dirac operator and the twistor oeprator.

If we restrict ourselves to the simple flat case, i.e. if we take for the Riemannian manifold X simply R_4 and if we identify the spinor spaces $V_+ \cong V_- \cong C_2$ with H

(e.g. as $\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \in C_2 \sim \begin{bmatrix}\phi_1 & -\bar{\phi}_2\\ \phi_2 & \bar{\phi}_1\end{bmatrix} = x_0 - i\sigma_1 x_1 - i\sigma_2 x_2 - i\sigma_3 x_3 \cong x_0 + i_1 x_1 + i_2 x_2 + i_3 x_3$).

then the operators $\mathcal{D}$, $\vec{\mathcal{D}}$ can be identified with the Dirac, resp. the twistor operator from [AHS] .

To describe the identification in more details, let us first recall the definition of these operators.

Let $e_j = dx_j$, $j=0,1,2,3$ be an orthonormal basis of 1-forms on R_4 and let $\mathcal{C}$ be the corresponding Clifford algebra (the Clifford algebra is generated by e_j which are subjected to the relations $e_i . e_j + e_j . e_i = -2\delta_{ij}$). Its complexification $\mathcal{C}_c$ is isomorphic to the algebra $C(4)$ of 4×4 complex matrices. The total spin space $V \cong C_4$ (on which elements of $C(4)$ are acting) can be split onto two pieces $V = V_+ \oplus V_-$, $V_+ \cong V_- \cong C_2$ in such a way that the generators $e_0, .., e_3$ form a basis of $Hom(V_+, V_-)$ and $Hom(V_-, V_+)$, respectively.

If we consider now a spinor field ϕ on R_4 with values in V_-, two operators are defined in [AHS] :

a) Dirac operator: $D: \phi \longmapsto D\phi = \sum_0^3 e_i . \frac{\partial \phi}{\partial x_i}$

Conferenza tenuta nel mese di Maggio 1982

b) twistor operator: $\bar{D} : \phi \longmapsto \bar{D}\phi = d\phi + \frac{1}{4}\sum_{0}^{3}(e_i . D\phi)dx_i$

Identifying $C_2 = V_+ = V_- = H$ suitably, the action of e_i on will become the multiplication by $-1, -i_1, -i_2, -i_3$ respectively, which leads to the identification of D with -4∂ and $\bar{D}$ with $d - dq.\partial = \vec{\mathcal{D}}$.

It brings us to two important conclusions: first, there is an unexpected strong connection between the basic equations of quaternionic analysis and equations, which appeared in the mathematical treatment of instantons; second, that there should be a way how these basic quaternionic equations can be treated in much more general setting. There is clearly a lot of unanswered questions at the moment.

(II.2) Why complex quaternions?

Quaternionic analysis could be treated in the natural setting, described above, and developed further and further. Nevertheless, there are some reasons why the enlargement of the basic setting can be very useful.

First, all 'holomorphic' functions in quaternionic analysis are real-analytic mappings from R_4 to R_4 . Very common and very useful procedure is to study such real-analytic mappings through their complexification and to restrict the attention to the real slice after.

Second, we have seen that differentiable and regular functions of quaternionic variable were solutions of 'twistor' and 'Dirac' equations from [AHS]. The names of the operators are coming from mathematical physics even if they are not, properly speaking, the same operators as the Dirac and the twistor operators on Minkowski space. They are - as it is usually expressed by physicists - an analytic continuation of the Dirac and the twistor operators from Minkowski spacetime to Euclidean spacetime. It means simply, that there are operators acting on mappings, living on complex Minkowski space, such that the restriction of the operators to 'Minkowski' slice coincide with the physical version of the operators, while the restriction of them to 'Eucidean' slice are just the Dirac and twistor operators in the sense of [AHS] .

All this means that, after the complexification, the direct connection of the equations of quaternionic analysis to problems of mathematical physics can be gained. It can be useful in both way. The methods of quaternionic analysis can be used in mathematical physics and, on the other hand, some parts of mathematical physics can offer many ideas and models to quaternionic analysis. The connection between both is handled best via the study of complex quaternionic analysis. The use of complex Minkowski space in mathematical phyiscs is more and more common (see [Pe1], [PR], [ATT]). So the mappings, living on the algebra of complex quaternions, can be easily identified with some fields from mathematical physics.

It is probably worth to review first shortly some basic notions from mathematical physics, needed in the sequel, to be able to show the connection of complex quaternionic analysis to them later.

(II.3) Spinors.

Let us review very shortly some basic facts on spinors (for details see [Pe2], [PR], [Pi]). The spinors (in the sense used usually in mathematical physics) are representation spaces of the group $SL(2,\mathbb{C})$. The importance of this group in physics follows immediately from the fact that $SL(2,\mathbb{C})$ is the universal covering group for the (proper) Lorentz group. Thereare four different spinor spaces - S_A, $S_{A'}$, S^A, $S^{A'}$. All of them are $\mathbb{C}_2$'s and the action of matrices from $SL(2,\mathbb{C})$ on them is described by a matrix multiplication.

a) The multiplication of $\omega \in S^A$ by $L \in SL(2,\mathbb{C})$ is the multiplication of a (column) vector by a matrix. In coordinates, if $L=L^A{}_B$, $A=0,1$; $B=0,1$ and if $\omega = \omega^A$, $A=0,1$, then $(L.\omega)^A = L^A{}_B\omega^B$ (the summation convention is used).

b) The space S_A is the dual space to S^A with corresponding contragradient representation of $SL(2,\mathbb{C})$. It is convenient to consider elements of S_A as row vectors. In coordinates, the multiplication is given by the multiplication by L^{-1} from the right.

c) The space $S^{A'}$ is again $\mathbb{C}_2$ (as row vectors) with the action of $SL(2,\mathbb{C})$ given by the multiplication of $\omega \in S^{A'}$ by $L^{\dagger}$ from the right ($\dagger$ denoting the Hermitian conjugation). In the coordinate description is $\omega \in S^{A'}$ represented by $\omega^{A'}$, $A'=0',1'$.

d) The space $S_{A'}$ is again the dual of $S^{A'}$ with the corresponding action of $SL(2, C)$.

There are various useful mappings among the spinor spaces, compatible with $SL(2, C)$ action.

e) The Hermitian conjugation gives the map from S^A to $S^{A'}$ (and similarly for duals). In coordinates it is defined by

$\bar{\omega}^{0'} := \overline{\omega^0}$, $\bar{\omega}^{1'} = \overline{\omega^1}$ (bar denoting the complex conjugation),

f) A skew-symmetric form on S^A gives the identification of S^A with S_A, in suitable coordinates the identification has the standard form

$$\omega^A \longmapsto \omega_B = \omega^A \epsilon_{AB}, \text{ where } \epsilon_{00} = \epsilon_{11} = 0,\ \epsilon_{01} = -\epsilon_{10} = 1 ,$$

i.e. $\omega^0 = \omega_1$, $\omega^1 = -\omega_0$.

It is necessary to consider not only the described spinor spaces, but also tensor products of them. The most important of them is $S_A \otimes S_{A'}$. Its importance lies in the fact that it can be identified with complex Minkowski space through the map:

$$z_\mu = [z_0, z_1, z_2, z_3] \in CM \longleftrightarrow z_{AA'} = \begin{bmatrix} z_0+z_3, & z_1-iz_2 \\ z_1+iz_2, & z_0-z_3 \end{bmatrix} \in S_A \otimes S_{A'}.$$

Under this identification $z^\mu = [z_0, -z_1, -z_2, -z_3]$ correspond to $z^{AA'}$, $z_\mu z^\mu = z_0^2 - z_1^2 - z_2^2 - z_3^2$ is equal to $\det(z_{AA'}) = \frac{1}{2}(z_{AA'} z^{AA'})$.

It follows immediately that $\det(z_{AA'})$ is preserved under the action of an element of $SL(2, C)$. This fact means that the corresponding linear transformation of z_μ preserves the Minkowski norm, hence it is the Lorentz transformation. It can be checked that the corresponding mappings of $SL(2, C)$ onto the (proper) Lorentz group is 2:1 universal covering.

It is easy to show that $\det z_{AA'} = 0$ if and only if there are spinors ω_A, $\pi_{A'}$ such that $z_{AA'} = \omega_A \pi_{A'}$. Note that such a matrix $z_{AA'}$ is Hermitian (i.e. belongs to the null cone in real Minkowski space) if and only if $\omega_A = \bar{\pi}_A$.

(II.4) The Weyl equation.

If we define $\nabla^{AA'} := \partial/\partial z_{AA'}$, we can relate this differential operator to $\partial_{z_j} := \partial/\partial z_j$ by

$$\nabla^{AA'} = \begin{bmatrix} \nabla^{00'}, & \nabla^{01'} \\ \nabla^{10'}, & \nabla^{11'} \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} \partial_{z_0} + \partial_{z_3} , & \partial_{z_1} - i\partial_{z_2} \\ \partial_{z_1} + i\partial_{z_2} , & \partial_{z_0} - \partial_{z_3} \end{bmatrix}$$

The Weyl equation (one form of it) is the equation $\nabla^{AA'}\Phi_{A'} = 0$, where $\Phi_{A'} = \Phi_{A'}(z_\mu)$ is a spinor field on a subset of complex Minkowski space. In elementary particle physics such an equation is used to describe neutrino. (The other 3 version of the Weyl equation could be written as $\nabla_{AA'}\phi^{A'}=0$, $\nabla^{AA'}\Phi_A=0$, $\nabla_{AA'}\phi^{A}=0$.) The Weyl equation is only one of the series of so called massless field equations (the other outstanding members of the family are the wave equation, Maxwell equation for elmg field and the equation for the linearized gravitation - see [ATT]).

Remark:

Note that the equivalent form of the Weyl equation is

$$\nabla^{A}{}_{[A'}\Phi_{B']} = 0 \quad ,$$

where [,] means the antisymmetrization.

(II.5) The twistor equation.

It can be written in the form (this form is usualy called the dual twistor equation)

$$\nabla^{A}{}_{(A'}\Phi_{B')} = 0 \quad ,$$

where (,) means the symmetrization.

Hence the Weyl and the twistor operators are just the projections onto antisymmetric, resp. symmetric parts of $\nabla^{A}{}_{A'}\Phi_{B'}$.

(II.6) Complex quaternions in spinor language.

The algebra CH of complex quaternions can be defined as $CH := C \otimes_R H$. The typical element of CH can be written as $Q = Q_0 + i_1Q_1 + i_2Q_2 + i_3Q_3$; $Q_0,\dots,Q_3 \in C$. Let us define further $Q^{\dagger} := Q_0 - i_1Q_1 - i_2Q_2 - i_3Q_3$, $|Q|^2 := Q.Q^{\dagger} = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2$.

It is no more true that $|Q|^2 = 0$ only for $Q = 0$. There are zero divisors in CH. The inverse Q^{-1} exists in and only if $|Q|^2 \neq 0$ and then $Q^{-1} = Q^{\dagger}/|Q|^2$.

As a complex vector space CH can be identified with C_4. We shall restrict our attention here to holomorphic mappings from CH to CH (as mappings from C_4 to C_4). Nothing is lost in such a way, because the solutions of the equations of (real) quaternionic analysis are real-analytic and after the complexification they are going to be holomorphic.

To get a connection with mathematical physics, we have to represent the algebra CH by 2x2 complex matrices. The basic identification can be written as follows:

$$Q = Q_0 + i_1Q_1 + i_2Q_2 + i_3Q_3 \in CH \longleftrightarrow Q_{AA'} = \begin{bmatrix} Q_0 - iQ_3, & -Q_2 - iQ_1 \\ Q_2 - iQ_1, & Q_0 + iQ_3 \end{bmatrix} \in S_A \otimes S_{A'} \longleftrightarrow$$

$$\longleftrightarrow \quad z_\mu = [Q_0, -iQ_1, -iQ_2, -iQ_3] \in CM$$

Under this identification we have:

$Q^{\dagger}$ corresponds to $Q^{AA'}$ and to z^μ ;

$|Q|^2$ corresponds to $\det(Q_{AA'}) = \frac{1}{2}(Q_{AA'}, Q^{AA'})$ and to $z_\mu z^\mu$.

To translate differential operators let us note first that

$\partial_{Q_0} := \partial/\partial Q_0 = \partial_{z_0}$, $\partial_{Q_1} := \partial/\partial Q_1 = -i\partial_{z_1}$, etc.

Hence

$$\nabla^{AA'} = \frac{1}{2}\begin{bmatrix} \partial_{Q_0} + i\partial_{Q_3}, & \partial_{Q_2} + i\partial_{Q_1} \\ -\partial_{Q_2} + i\partial_{Q_1}, & \partial_{Q_0} - i\partial_{Q_3} \end{bmatrix}$$

By substituting the corresponding 2x2 matrices instead of i_1, i_2, i_3 in the definition of ∂ we obtain

$$\partial = \begin{bmatrix} \partial_{Q_0} + i\partial_{Q_3}, \partial_{Q_2} + i\partial_{Q_1} \\ -\partial_{Q_2} + i\partial_{Q_1}, \partial_{Q_0} - i\partial_{Q_3} \end{bmatrix}.$$

Hence $\quad \partial \longleftrightarrow \nabla_{AA'}$, $\quad \partial^{\dagger} \longleftrightarrow \nabla^{AA'}$.

(II.7) Complexified Fueter equation.

Let F be a map from CH to CH . The complexified Fueter's equation looks like

$$\partial^{\dagger}F := (\partial_{Q_0} + i_1\partial_{Q_1} + i_2\partial_{Q_2} + i_3\partial_{Q_3})\, F = 0 ,$$

where ∂_{Q_j} are the derivatives of the holomorphic mapping F .

The function $F : \mathcal{CH} \longrightarrow \mathcal{CH}$ corresponds under the identification $F = F_0+i_1F_1+i_2F_2+i_3F_3 \longleftrightarrow \Phi_{AA'}$, $Q \longleftrightarrow z_{BB'}$ to the mapping $\Phi_{AA'} = \Phi_{AA'}(z_{BB'})$. The Fueter equation $\partial^{\dagger}F = 0$ corresponds then to the equation $\nabla^{AA'}\Phi_{BA'} = 0$. The important thing can be seen now, namely that the complexified Fueter equation can be split into two pieces. We can consider the spinor fields $\Phi_{0A'}$ and $\Phi_{1A'}$ and the complexified Fueter equation doesn't mix them together. It is easy to see in the spinor language that it is a simple consequence of the properties of multiplication of matrices. Hence the complexified Fueter equation consists of two independent copies of Weyl equation. (To have purely quaternionic formulation of one copy of Weyl equation, it is sufficient to consider values of the mapping F in an ideal in $\mathcal{CH}$.)

(II.8) <u>The differentiability condition</u>.

Its complexification can be written as

$$\partial_{Q_0}F = -i_1\partial_{Q_1}F = -i_2\partial_{Q_2}F = -i_3\partial_{Q_3}F \ .$$

In the proof of the theorem I.2 the relations (1) - (3) were described. The functions $g=f_0+i_3f_3$ and $h=f_2+i_3f_1$ correspond now to $F_0+i_3F_3$ and to $F_2+i_3F_1$, while $\partial/\partial\bar{\zeta}$ to $\partial_{Q_0}+i_3\partial_{Q_3}$ and similarly

$$\partial/\partial\zeta \longmapsto \partial_{Q_0}-i_3\partial_{Q_3} \ , \quad \partial/\partial\eta \longmapsto \partial_{Q_2}-i_3\partial_{Q_1} \ , \quad \partial/\partial\bar{\eta} \longmapsto \partial_{Q_2}+i_3\partial_{Q_1} \ .$$

The relations (1) - (3) gives us now the relations among derivatives of individual components F_μ. These relations don't change, if $(-i)$ is substituted instead of i_3 and in such a way the relations (1) - (3) are equivalent to

$$(\partial_{Q_0}-i\partial_{Q_3})(F_0+iF_3)=(\partial_{Q_0}+i\partial_{Q_3})(F_2-iF_1)=0 \Longleftrightarrow \nabla_{00'}\Phi_{00'}=\nabla_{11'}\Phi_{01'}=0 \qquad (1')$$

$$(\partial_{Q_2}+i\partial_{Q_1})(F_0-iF_3)=(\partial_{Q_2}-i\partial_{Q_1})(F_2-iF_1)=0 \Longleftrightarrow \nabla_{10'}\Phi_{00'}=\nabla_{01'}\Phi_{01'}=0 \qquad (2')$$

$$\nabla_{11'}\Phi_{00'}+\nabla_{10'}\Phi_{01'} = \nabla_{00'}\Phi_{01'}+\nabla_{01'}\Phi_{00'}=0 \qquad (3')$$

Hence all these equations together are just the detailed description of the (dual) twistor equation $\nabla_{A(A'}\Phi_{B')} = 0$ for the spinor field $\Phi_{B'} := \Phi_{0B'}$.

So again, like for the Fueter equation, the condition of differentiability for a quaternionic function can be separated after the complexification into two independent sets of equations, each one of which can be identified with the twistor equation (considered on complex Minkowski space).

(II.9) The complex Laplace equation.

As was noted in (I.4), we have $\partial\partial^{\dagger} = 1/16\Delta$ on the space of real quaternions. After the complexification, the operator $\partial\partial^{\dagger} = 1/16\ (\sum_{0}^{3}\partial_{Q_i}^{2})$ is the complex Laplacian (componentwise). On the Minkowski slice we obtain the wave operator. In the spinor language it looks like $\nabla_{AA'}\nabla^{AA'}$.

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