INTEGRAL FORMULAE FOR SPINOR FIELDS

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A scheme for the group theoretic classification of possible generalized Cauchy-Riemann equations on manifolds was described in [1]. The aim of the paper is to present some possibilities how to distinguish those of them, which are really in a reasonable sense generalizations of C-R equations. The simplest criterion, which we use here, is the possibility to find an analogue of Cauchy integral formula for solutions of corresponding equations. We shall treat here only real, flat case, mainly in dimension 4 and we shall restrict ourselves to the group $G = \mathrm{Spin}(n)$.

Let us first recall the way how differential operators are defined using representations of G. Consider an irreducible representation E_μ of the group G with the highest weight μ. Let $C^\infty(V)$ denote smooth maps from a fixed domain $\Omega \subset R_n$ into the vector space V. The tensor products $\Lambda^j \otimes E_\mu$ (where $\Lambda^j = \Lambda^j R_n^*$) split into irreducible pieces as the representation spaces for G, say $\Lambda^j \otimes E_\mu = \sum_{k=1}^{m_j} F^j_{\lambda_k}$, where λ_k's are the heighest weights of $F^j_{\lambda_k}$.

Then the vector-valued de Rham sequence on Ω

$$C^\infty(\Lambda^0 \otimes E_\mu) \xrightarrow{d} C^\infty(\Lambda^1 \otimes E_\mu) \xrightarrow{d} \dots \xrightarrow{d} C^\infty(\Lambda^n \otimes E_\mu)$$

splits into the diagram

$$\begin{array}{ccccccc} & \overset{\partial_1}{\nearrow} & C^\infty(F^1_{\lambda_1}) & \longrightarrow \dots \longrightarrow & C^\infty(F^{n-1}_{\lambda_1}) & \overset{\tilde{\partial}_1}{\searrow} & \\ & & \oplus & & \oplus & & \\ C^\infty(E_\mu) & & \vdots & & \vdots & & C^\infty(\Lambda^n \otimes E_\mu) \\ & & \oplus & & \oplus & & \\ & \underset{\partial_{m_1}}{\searrow} & C^\infty(F^1_{\lambda_{m_1}}) & \longrightarrow \dots \longrightarrow & C^\infty(F^{n-1}_{\lambda_{m_{n-1}}}) & \underset{\tilde{\partial}_{m_1}}{\nearrow} & \end{array} \qquad (1)$$

The operators ∂_j are defined as the composition of the exterior derivative d with the projection to corresponding pieces in the diagram (1).

Any operator among $\partial_1, \dots, \partial_{m_1}$ or perhaps a sum of them can be considered as a possible candidate for higher dimensional analogue of C-R equations for E_μ -valued maps on Ω . We want to discuss now some possibilities how to find a generalized Cauchy integral formula for solutions of such operators in some cases. It is clear that the necessary conditions for it is the inequality $\dim E_\mu \le \dim F_{\nu_j}$ (for $\partial_j: E_\mu \to F_{\nu_j}$) ,because there would be more components than equations and some components could be choosen arbitrarily. To explain the procedure which we want to use, let us consider first the following two cases.

1. Let $\mathcal{C}_n$ be the Clifford algebra of $\mathbb{R}_n$ (with the standard negative definite quadratic form) with generators $e_1, \dots, e_n$. For left regular mappings $f: R_{n+1} \to \mathcal{C}_n$ (i.e for solutions of the equation $\partial f = 0$, $\partial = (\partial/\partial x_0 + \sum_1^n e_i \, \partial/\partial x_i)$ the following integral formula holds

$$f(P) = \frac{1}{\omega_n} \int_{S_n} \frac{(Q-P)^+}{|Q-P|^{n+1}} \, DQ \; f(Q) \quad ,$$

where ω_n is the volume of the unit n-sphere, DQ is the standard $\mathcal{C}_n$- valued n-form on $R_{n+1} \subset \mathcal{C}_n$ and $^+$ denote the main antiinvolution in $\mathcal{C}_n$ (e.g. $e_j^+ = -e_j$, $e_0^+ = e_0$) . More details can be found in [3] .

2. Let V_1, V_2 be real vector spaces, both considered as subspaces of $V = V_1 \oplus V_2$. Let us consider a linear map $\bar{\Phi}: R_{n+1} \to L(V,V)$ such that

$$\bar{\Phi}(x)\,(V_1) \subset V_2 \; , \quad \bar{\Phi}(x)\,(V_2) \subset V_1$$

for all $x \in R_{n+1}$. Let us denote further

$$\Phi(x) = \bar{\Phi}(x) / V_1 \quad , \quad \tilde{\Phi}(x) = \bar{\Phi}(x) / V_2$$

It was shown in [2] that if the conditions

$$\begin{aligned} &(i) \; \tilde{\Phi}(e_i) \cdot \Phi(e_i) = -\tilde{\Phi}(e_0)\,\Phi(e_0) = -id \\ &(ii) \; \tilde{\Phi}(e_i) \cdot \Phi(e_0) = \tilde{\Phi}(e_0) \cdot \Phi(e_i) \\ &(iii) \; \tilde{\Phi}(e_i) \cdot \Phi(e_j) + \tilde{\Phi}(e_j) \cdot \Phi(e_i) = 0 \end{aligned} \qquad (2)$$

hold on V_1 ,then for solutions of the equation

$$D_\Phi(f) = \sum_0^n \Phi(e_i) \, \frac{\partial f}{\partial x_i} = 0$$

we have the following integral formula

$$f(P) = \frac{1}{\omega_n} \int_{S_n} \tilde{\Phi}\left(\frac{(Q - P)}{|Q - P|}n+1\right) \quad \Phi(DQ)\ f(Q)$$

(for details see [7]).

The conditions (2) means just the same as the condition that

$$\left(\sum_0^n \tilde{\Phi}(e_i^+) \frac{\partial}{\partial x_i}\right)\left(\sum_0^n \Phi(e_i)\frac{\partial}{\partial x_i}\right) \quad f = \Delta f$$

i.e that the operator $D_{\Phi}(f)$ can be complemented by a suitable **first** order operator to give together the Laplace operator.

So the case 2 tells us that to obtain a generalized Cauchy integral formula in such a way ,it is sufficient for an operator ∂_j from (1) to find a complementary operator $\tilde{\partial}_j$ such that $\tilde{\partial}_j \circ \partial_j$ is just the Laplace operator componentwise for E_μ -valued maps. We'd like ,moreover ,to find the complementary operator in a way, which could be used in future in general situation,i.e. on manifolds. There is one natural possibility,which could be succesful. The Hodge star operator $*$ on forms give the isomorphism of Λ^j with Λ^{n-j} (also as Spin(n) - modules) ,so the decomposition of $\Lambda^j \otimes E_\mu$ and $\Lambda^{n-j} \otimes E_\mu$ are the same (i.e.isomorphic). The corresponding pieces in the decomposition are mapped one onto another by the $*$-operator suggesting possibility to take as a complementing operator for ∂_j the corresponding operator $\tilde{\partial}_j$ given by the scheme (1) from $F_\mu \subset \Lambda^{n-1} \otimes E_\mu$ into $\Lambda^n \otimes E_\mu$.

We shall present now some examples showing that the described procedure can,but need not ,give the desired result.We shall concentrate our attention to the simplest but the representative and, in the connection with mathematical physics,the most interesting case of dimension 4 .

So let us consider the case of R_4,G = Spin(4). There are two basic spinor representations,say $S_A, S_{A'}$ of Spin(4) ,where both S_A and $S_{A'}$ are complex 2-dimensional vector spaces and the representationon them are coming from the identification Spin(4) $\simeq$ SU(2) x SU(2). Every other irreducible representation of Spin(4) can be obtained by taking tensor products $S_{\underbrace{A....E}_{m}\ \underbrace{A'....F'}_{n}} \simeq S_{m,n}$

of these two representations ,symmetrized in primed and unprimed indices. So all ireducible representations of G can be classified by the couple (m,n) of nonnegative integers.(There is a simple relation of this classification to the classification by highest weights, but we shall not need it here

A lot of information on the 4-dimensional case can be found in the paper by Garding [4] on 'square roots' of wave equation on Minkowski space. He is studying there the first order systems of equations with the property that every component of a solution satisfies the wave equation. The notion of 'minimal set' of equation was introduced in [4]. It is a first order system of equations with the described property which is in a sense minimal within the set of all such systems of equations (for details see[4]).

The same discussion can be applied also in our Riemannian case (Euclidean space $\mathbb{R}_4$ instead of Minkowski space), the component of minimal set will satisfy now the Laplace equation. It can be checked that the procedure using in [4] to find minimal sets of equations is just the same as the procedure described above (which use the complementing operator $\tilde{\partial}$ to ∂).

The result of the discussions is the following.

If we take $E_\mu = S_{m,n}$ $(m \leq n)$, then we have the splitting

$$\Lambda^1 \otimes S_{m,n} \simeq S_{m+1,n+1} \oplus S_{m-1,n+1} \oplus S_{m+1,n-1} \oplus S_{m-1,n-1}$$

resp. (for the case $m = 0$)

$$\Lambda^1 \otimes S_{0,n} \simeq S_{0,n+1} \oplus S_{0,n-1}$$

The minimal set of equations is either any of two operators for the case $m = 0$, or sum of any two operators in the general case $m > 0$. So for any first order system forming a minimal set of equations we can use described procedure and to find the generalized Cauchy integral formula. Such systems hence can be considered quite well to be generalizations of C-R equations to higher dimensions. The most interesting cases as for applications in mathematical physics are two cases with $m = 0$.

Example 1: $E_\mu \simeq S_{0,n} \simeq S_{A',\dots,F'}$

(i) The operator $\partial_1 : S_{0,n} \to S_{0,n-1}$ can be written in the spinor notation (see[6]) as $\nabla^{AA'} \Phi_{A',\dots,F'} = 0$ and is usually called the massless field equation. The corresponding generalized Cauchy integral formula was described in [2],

(ii) The operator $\partial_2 : S_{0,n} \to S_{0,n+1}$ looks in the spinor notation like $\nabla^A_{(A'} \Phi_{B',\dots,G')} = 0$ (where the round brackets denote the symmetrization) and is usually called the twistor equation (esp.

for $n = 1$) or Killing equation.

Returning back to the necessary condition for dimensions of E and F, it is easy to check that the condition is satisfied not only for minimal set of equations but also for further two cases which will be discussed in more details in the following two examples (recall that $\dim S_{m,n} = (m+1)(n+1)$).

Example 2 :

The operator $\partial_3 : S_{m,n} \to S_{m+1,n+1}$ has the spinor form

$$\nabla^{(A}_{(A'} \Phi^{B.\dots.F)}_{B'.\dots.G')} = 0$$

and was used by Penrose and Walker [7] as the equation for Killing spinors. For $m = n = 1$ it is just the equation for conformal Killing vectors $D_{(\mu} v_{\nu)} = \phi . g_{\mu\nu}$ (see [5]). This equation is quite interesting example because it does not form a minimal set, but it can be shown by direct computation that if we complement it using the procedure described above, the resulting operator of the second orderwill not be the Laplace operator, but a more general elliptic operator. It means that even if the procedure leading to generalized Cauchy integral formula cannot be used here, there is still a hope to find a more general version of it. So the considered system of equations can still be a reasonable generalization of C-R equations.

Example 3:

The operator $\partial_4 : S_{m,n} \to S_{m+1,n-1}$ ($m < n$) has the spinor form

$$\nabla^{(A}{}_{A'} \Phi^{B.\dots.E)A'.\dots.F'} = 0$$

and it can be shown that the second order operator obtained by the described procedure is not elliptic one, so this procedure never can lead to a generalized Cauchy integral formula.

So summarizing the above discussion, it was shown that the simple counting of dimensions (the necessary condition) is not sufficient to distinguish proper generalization of C-R equations and that further study and another means are necessary to treat some cases for which the natural procedure presented above doesn't lead to generalized Cauchy integral formula.

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