GENERALIZED C-R EQUATIONS ON MANIFOLDS

Vladimír Souček
Dep. Mathematical Analysis
Charles University
Sokolovská 83, 186 00 Praha
Czechoslovakia

ABSTRACT. The paper presents a generalization of the classical ∂ and $\bar{\partial}$ operators from complex analysis on Riemann surfaces to vector-valued differential forms on conformal n-dimensional manifolds. An abstract scheme for such generalization is based on the splitting of the vector-valued de Rham sequence. The possible generalizations are classified by couples of irreducible CO(n)-modules and by a choice of a connection on the associated vector bundle. Various generalizations of C-R equations, studied by different authors during last 50 years, are discussed and it is shown how they fit into the scheme. A special attention is paid to the most interesting case of dimension 4 and to the connection of the described systems of equations with equations in mathematical physics.

1. INTRODUCTION

The classical complex analysis (in the plane and even more on manifolds) is so rich and beautiful part of mathematics that there were many attempts to look for a similar theory in higher dimensions. The generalizations went to many directions. They usually consist of a system of first order linear PDE with constant coefficients ([2] , [14] , [15] , [20] , [22] , [34] , [37] , [38] , [39]), sometimes more general elliptic systems (with variable coefficients or with nonlinear 0-order terms) are considered ([13] and references therein), or even a fully nonlinear system of self-dual Yang-Mills field equations was suggested as a generalization of C-R equations to higher dimensions ([23]). Some generalizations to maps defined on manifolds were also presented ([1] , [15] , [21] , [32]).

Any of these generalizations has its own merit and it is difficult to decide what a proper generalization of C-R equations should be. It depends clearly on the point of view,

J. S. R. Chisholm and A. K. Common (eds.), Clifford Algebras and Their Applications in Mathematical Physics, 201–217.

on the choice what aspect of the classical complex analysis wants to be preserved under the generalization.

Not all generalizations, mentioned above, are included in the scheme presented in the paper. The guiding principles for the generalizations studied here can be stated as follows.

Firstly, to open possibilities to look for generalizations of the fascinating field of complex analysis on manifolds, the wanted scheme for the generalization should be necessarily formulated not for functions on domains in R_n, but for maps defined on an appropriate type of n-dimensional manifolds.

Secondly, the classical C-R equations in the complex plane are substituted by $\bar\partial$ operator, acting on complex valued differential forms. The holomorphic functions form the kernel of $\bar\partial$ operator on 0-forms and the whole standard split de Rham sequence

$$\begin{array}{ccccc} & & \mathcal{E}^{1,0} & & \\ & \overset{\partial}{\nearrow} & & \overset{\bar\partial}{\searrow} & \\ \mathcal{E}^{0} & & \oplus & & \mathcal{E}^{2} \\ & \underset{\bar\partial}{\searrow} & & \underset{\partial}{\nearrow} & \\ & & \mathcal{E}^{0,1} & & \end{array} \qquad (1)$$

forms the inseparable picture. It would be highly desirable to find a generalization of the diagram (1) to higher dimensions.

Thirdly, 'a higher dimensional generalization' should mean that the equations reduce (at least in flat cases) back to the classical Cauchy-Riemann equations for dimension $n=2$.

The general scheme, satisfying the principles stated above, is described in the paper (following [7]). Basic idea is a certain kind of invariance under an appropriate group. The generalization of the domains of definitions of our maps is based on the fact that complex 1-dimensional manifolds (Riemann surfaces) coincide with (real) 2-dimensional manifolds with conformal structure. In higher dimensions, our maps will be defined on n-dimensional manifolds with the structure group $\tilde{G} = CO(n)$ (resp. the universal covering group G of $\tilde{G}$). Target spaces of maps will be irreducible finite dimensional G-modules over R (note that for $n=2$ such modules look like $R_2 = C$).

The basic idea of the scheme was inspired by the paper by Stein and Weiss ([38]) on generalized C-R equations and by the paper on conformally invariant first order operators by Fegan ([11]). The essence of the approach is simple, we shall describe it now in the flat case.

Let us denote by $\mathcal{E}^{\infty}(V)$ the space of smooth maps of a domain $\Omega \subset R_n$ to the vector space V and let us denote $\Lambda^j = \Lambda^j(R_n^*)$. For $n=2$ we take the de Rham sequence

$$\mathcal{E}^\infty(\Lambda^0) \xrightarrow{d} \mathcal{E}^\infty(\Lambda^1) \xrightarrow{d} \mathcal{E}^\infty(\Lambda^2) \qquad (2)$$

and we shall consider its complexification (i.e. maps into $\Lambda_c^j = \Lambda^j \otimes_R C$). The splitting

$$\Lambda_c^1 = \Lambda^{1,0} \oplus \Lambda^{0,1} \qquad (3)$$

leads directly to the split de Rham sequence (1). Now, any (nontrivial) SO(2)-module V looks like $R_2 \cong C$ and the tensor product $\Lambda^1 \otimes_R V$ splits (as SO(2)-module) into two pieces $F_1 \oplus F_2$, both isomorphic to R_n. The equations coming from this splitting are equivalent to C-R equations.

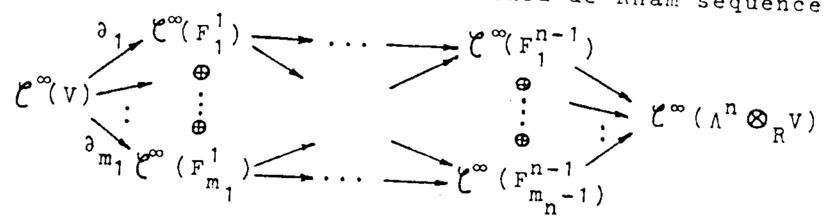
Let us take a G-module V (over R) and let us tensor the de Rham sequence

$$\mathcal{E}^\infty(\Lambda^0) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^\infty(\Lambda^n) \qquad (4)$$

with V (over R). The products $\Lambda^j \otimes_R V$ can be decomposed into irreducible pieces (as G-modules), say

$$\Lambda_V^j = \Lambda^j \otimes_R V = F_1^j \oplus \ldots \oplus F_{m_j}^j \qquad (5)$$

It induces the splitting of V-valued de Rham sequence



(6)

which is proposed to be the proper generalization of (1) to higher dimensions. Every individual operator in (6) for every V is in such a way a generalization of ∂ or $\bar\partial$ operator from (1). Note that the structure of the diagram (6) can be much more complicated than that of (1) and that it depends generally on the choice of V. Moreover, the operators ∂ and $\bar\partial$ in (1) were quite similar one to another, while the individual operators in (6) will be in general quite different in character and as to properties of their solutions.

The best way to classify the individual operators in (6) is by the highest weights of the corresponding spaces F_1^j and F_1^{j+1} (note that the same operator can appear in the diagram (6) for different modules V and on different places and that every one can appear in the first column of operators).

The aim of this review is to discuss how various generalizations of C-R equations, studied by different authors during last 50 years, fit into the general scheme. So the general definition of the split de Rham sequence for vector-valued forms is introduced first (§.2.) and then various generalizations of C-R equations and their relations to the suggested scheme are discussed (§.3. and §.4.).

No attempt was made to make the list of examples, results and references complete (in fact, it is almost impossible, for example, for spinor fields on space-times). They were chosen with respect to the knowledge and interests of the author.

2. VECTOR-VALUED DIFFERENTIAL FORMS

Let us consider the conformal group $\tilde{G} = CO(n)$ and its universal covering group G. Let V be an irreducible, finite dimensional G-module over R. The fundamental representation of $CO(n)$ on R_n induces the structure of G-module on $\Lambda^j = \Lambda^j(R_n^*)$ and the tensor product $\Lambda^j \otimes_R V$ splits (as G-module) into irreducible pieces:

$$\Lambda^j \otimes_R V = F_1^j \oplus \dots \oplus F_{m_j}^j , \quad j=1,\dots,n-1 \qquad (7)$$

Let M be a (real) oriented n-dimensional manifold with conformal structure, i.e. we have the corresponding principal $CO(n)$-bundle $\tilde{P}$. Suppose that $\tilde{P}$ lifts to a principal G-bundle P, i.e. that we have a (fibre bundle) homomorphism $f : P \longrightarrow \tilde{P}$. We shall denote by $\underline{V}$, $\underline{F}_1^j$ and $\underline{\Lambda}_j$ the vector bundles associated to the G-modules V, F_1^j and $\Lambda^j(R_n^*)$.

Finally, let us choose a covariant derivative $\nabla : \Gamma(\underline{V}) \longrightarrow \Gamma(\underline{\Lambda}^1 \otimes \underline{V})$ and extend it in the usual way to

$$\nabla : \Gamma(\underline{\Lambda}^j \otimes \underline{V}) \longrightarrow \Gamma(\underline{\Lambda}^{j+1} \otimes \underline{V}), j=1,\dots,n-1.$$

Then the sequence $\Gamma(\underline{V}) \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \Gamma(\underline{\Lambda}^n \otimes \underline{V})$ for V-valued differential forms splits as

$$\Gamma(\underline{V}) \xrightarrow{\partial_1, \dots, \partial_{m_1}} \begin{matrix}\Gamma(\underline{F}^1_1)\\ \oplus\\ \vdots\\ \oplus\\ \Gamma(\underline{F}^1_{m_1})\end{matrix} \longrightarrow \cdots \longrightarrow \begin{matrix}\Gamma(\underline{F}^{n-1}_1)\\ \oplus\\ \vdots\\ \oplus\\ \Gamma(\underline{F}^{n-1}_{m_{n-1}})\end{matrix} \longrightarrow \Gamma(\underline{\Lambda}^n \otimes \underline{V}) \qquad (8)$$

Remarks. 1. In flat spaces the definition simplifies a lot. It is not necessary to introduce the principle fibre bundles $\tilde{P}$ and P, all bundles being trivial. There is the distinguished connection $\nabla = d$ on $R_n \times V$. The sections in $\Gamma(\underline{\Lambda}^j \otimes \underline{V})$ are simply smooth maps in $\mathcal{C}^\infty(\Lambda^j \otimes_R V)$ and the diagram (8) coincide with (6).

To give a G-module V, it is necessary to choose a Spin(n)-module V and to specify the conformal weight. But the conformal weight does not influence the splitting (7), so the scheme (8) depends in the flat cases only on the choice of the Spin(n)-module V.

2. It is possible (and useful) to use the same procedure also for the case of the group $G = CO(p,q)$. The corresponding operators cannot be, of course, considered as generalizations of C-R equations. But, the representations of Spin(p,q) being the same as those for Spin(p+q), the corresponding operators are 'analytic continuation' of those for CO(p+q) case. This is especially interesting in dimension 4, where the operators in CO(4) case are analytic continuations to Euclidean spacetime of operators used in mathematical physics. It is also possible to relate properties of solutions of corresponding systems of equations (see e.g. 3.7).

3. Another variation of the described scheme can be often found, which works with complexifications of all vector bundles. The only changes needed are to consider complex valued forms and for a G-module V over C to consider the splitting of the tensor product $\Lambda^j_c \otimes_C V$. Basically, there is only a little difference between these two versions (for a more careful discussion see [8]), but the analogy with the classical case $n=2$ is closer for the version given here.

3. EXAMPLES (FLAT SPACES)

In all examples, presented in this paragraph, we shall consider only maps on R_n (flat space), so (with respect to Remark 1) we shall use the simpler diagram (6). Note that in these examples the scheme (8) gives the generalization of the corresponding equations to conformal manifolds.

3.1 Regular spinor fields

Let us consider the case, when the module V is one of the basic spinor modules (over R). We shall include the more general case of the group $Spin(p,q)$. The Clifford algebra $\mathcal{C}_{p,q}$ (corresponding to the quadratic form with p pluses and q minuses) is very useful for the description of these modules. For $q \geq 1$ we have the inclusion $Spin(p,q) \subset \subset \mathcal{C}^{+}_{p,q} \cong \mathcal{C}_{p,q-1}$ and any minimal left ideal in $\mathcal{C}_{p,q-1}$ is the basic $Spin(p,q)$-module (for more details see [20])

The tensor product splits in this case into two pieces

$$\Lambda^1 \otimes_R V = F_1 \oplus F_2 \ ,$$

where one summand on the right hand side (say F_1) is again a basic spinor module ([8]). So the first column in the diagram (6) looks like

$$\mathcal{E}^{\infty}(\Lambda^1 \otimes_R V) \begin{array}{l} \xrightarrow{\partial_1} \mathcal{E}^{\infty}(F_1) \\ \xrightarrow[\partial_2]{} \mathcal{E}^{\infty}(F_2) \end{array}$$

Choosing an orthonormal basis $e_1,\dots,e_{p+q}$ in R_{p+q} and denoting $a_1=\dots=a_p=-a_{p+1}=\dots=-a_{p+q}=1$, we can write the coordinate description of these operators:

$$\partial_1 : f \longmapsto \sum_k dx_k \otimes e_k \Big(\sum_j a_j e_j \frac{\partial f}{\partial x_j} \Big) \frac{1}{n} \qquad (9)$$

$$\partial_2 : f \longmapsto \sum_k dx_k \otimes \Big\{ \frac{\partial f}{\partial x_k} - e_k \Big(\sum_j a_j e_j \frac{\partial f}{\partial x_j} \Big) \frac{1}{n} \Big\} \ ,$$

where the multiplication $e_k e_j v$ means the action of $e_k e_j$ on $v \in V$ ([8]). In the Riemannian case all a_j disappear.

The equation $\partial_1 f = 0$ is (after a simple redefinition $e_i \longrightarrow -e_i$, $i=p+1,\dots,p+q$) just the condition for regular spinor field presented by Lounesto ([20]) in general $Spin(p,q)$ case. The case $p=o$ (or $q=0$) is the most interesting and the most common from all generalization of C-R equations. The operator is usually called Dirac operator. There are many more results known for the Dirac operator than for any other considered operator. In the paper [20] the generalized Cauchy integral formula is proved (in the elliptic case) for the Dirac operator. We shall discuss it again in Examples 3.2,3.4,3.7,3.8,4.1,4.2,4.3,4.5, where further results will be discussed.

3.2 Clifford analysis

Let us consider the Clifford algebra $\mathcal{C}_n = \mathcal{C}_{o,n}$ with the standard grading $\mathcal{C}_n = \mathcal{C}_n^+ + \mathcal{C}_n^-$. The Clifford algebra $\mathcal{C}_n$ is the (reducible) Spin(n)-module due to the inclusion $\mathrm{Spin}(n) \subset \mathcal{C}_n^+$ and, say, left multiplication. We can decompose it into irreducible pieces $\mathcal{C}_n = V_1 \oplus .. \oplus V_k$, where all V_j are basic Spin(n)-modules. As in 3.1, all tensor products $V_j \otimes_R \Lambda^1$ decomposes into $F_{j,1} \oplus F_{j,2}$, where all $F_{j,1}$ are again basic spinor modules. Adding all pieces together, we shall obtain two operators for maps $\Psi : R_n \longrightarrow \mathcal{C}_n$

$$\mathcal{C}^\infty(V_1 \oplus \ldots \oplus V_k) \cong \mathcal{C}^\infty(\mathcal{C}_n) \begin{array}{l} \xrightarrow{\partial_1} \mathcal{C}^\infty(F_{1,1}\oplus..\oplus F_{k,1}) \cong \mathcal{C}^\infty(\mathcal{C}_n) \\ \xrightarrow[\partial_2]{} \mathcal{C}^\infty(F_{1,2}\oplus..\oplus F_{k,2}) \end{array}$$

To describe the coordinate form of the operator ∂_1, let us choose an orthonormal basis $e_1,\ldots,e_n$ of $\mathcal{C}_n$. Applying the results of the section 3.1 to every piece V_j, the equation $\partial_1 \Psi = 0$ for $\mathcal{C}_n$-valued maps looks like

$$\sum_1^n e_j \frac{\partial \Psi}{\partial x_j} = 0 \quad . \tag{10}$$

This is the equation, studied by Delanghe ([9]). Note that multiplication by e_j need not preserve the individual pieces V_j, but preserves, of course, the whole algebra $\mathcal{C}_n$.

Multiplying the equation (10) by $(-e_1)$ we shall obtain the equivalent equation

$$\sum_1^n f_j \frac{\partial \Psi}{\partial x_j} = 0 \tag{11}$$

where $f_1 = 1$ and $f_j = -e_1 e_j$, $j=2,\ldots,n$ are generators of the algebra $\mathcal{C}_n^+ \cong \mathcal{C}_{n-1}$. Restricting our attention to maps $\Psi : R_n \longrightarrow \mathcal{C}_{n-1} = \mathcal{C}_n^+$, we shall get the equation for monogenic functions in Clifford analysis.

The study of solutions of the equation (11) is well advanced. A remarkable amount of results is known already for monogenic functions (see [2]), they will be described in more details in the lecture by Prof. Delanghe. Note that even if the system of equations studied in Clifford analysis is reducible (it consists of several copies of the equation for regular spinor maps), there are definitive advantages in notation (use of Clifford numbers) and concepts, which makes this setting of the problem valuable.

3.3 Fueter's regular functions

Fueter and his coworkers started to study quaternionic analysis 50 years ago. Their regular functions are special cases of monogenic functions in Clifford analysis for the case $n=3$, where maps $f : R_4 = H \longrightarrow \mathcal{C}_3 = H \oplus H$ are considered. To obtain Fueter's equation for regular functions, it is sufficient to split the value $f = (f_1, f_2)$ and consider only, say, f_1. A nice and modern account of basic results in quaternionic analysis can be found in [39].

3.4 Generalized C-R equations of Stein and Weiss

In this case we shall use the complexified form of the procedure, described in Remark 3 (section 2). For any Spin(n)-module V (over C) we shall split the tensor product $V \otimes_C \Lambda^1_c$ into two pieces. There is the exceptional irreducible piece in the product, called Cartan (or Jung) product of V and Λ^1_c. It is characterized by the fact that its highest weight is the sum of the highest weights of V and Λ^1_c. Let us denote it by F_1, so $V \otimes_C \Lambda^1_c = F_1 \oplus F_2$, where F_2 is (not necessarily irreducible) Spin(n)-module. Denoting π_2 the projection to F_2, we can write the equations of Stein and Weiss as

$$(\pi_2 \circ d)\ \Psi = 0\ .$$

It was proved in [38] that solutions of such equations are (componentwise) harmonic functions and that the modulus of the maps to the power p is subharmonic function for $p \geq (n-2)/(n-1)$. The most interesting special cases are:

i) V is the basic spinor module (see 3.1), the case $n=3$ was studied by Moisil and Theodoresco ([22])

ii) $V = \Lambda^r_c$, then the tensor product $\Lambda^r_c \otimes_C \Lambda^1_c$ splits into 3 pieces (see [38]): the Cartan product F_1, $F_2 = \Lambda^{r+1}_c$, and $F_3 = \Lambda^{r-1}_c$; the equations defined using the projections F_2, resp. F_3 are just the operators d, resp. δ, the equation coming from the projection onto $F_2 \oplus F_3$ being the Hodge operator $d + \delta$. Both these cases are well studied on manifolds.

3.5 Massless fields

More complete discussion can be given for dimension $n=4$. At the same time it is the most interesting case because of its close connection to mathematical physics. All bundles and tensor products considered in this section will be complex. There is the isomorphism $Spin(4) \cong Sp(1) \times Sp(1)$. The two basic spinor modules V_+ and V_- can be realized e.g. by left multiplication by one of the factors on $V = H$. All irreducible $Spin(4)$-modules can be realized as submodules of tensor products of the two basic modules V^+ and V^- and can be classified by a couple of nonnegative integers (j,k):

$$V^{j,k} = S^j V_+ \otimes S^k V_- ,$$

where S^j denotes symmetrized tensor product.

It can be shown ([11], [17], [8]) that

$$V^{j,k} \otimes_C \Lambda^1_c = V^{j+1,k+1} \oplus V^{j+1,k-1} \oplus V^{j-1,k+1} \oplus V^{j-1,k-1}$$

(if $j=0$ or $k=0$, then there are only two pieces in the decomposition).

The four (resp. two) differential operators obtained in such a way are 'analytic continuation' of operators for massless fields, described by Gårding ([12]). Many of these equations are used often in mathematical physics, sometimes they are considered on complexified Minkowski space. The most important cases, when $j=0$ (or $k=0$) leads to the equations, which are usually called massless field equations and twistor (or Killing) equations. The description of these operators on manifolds is given in 4.2. A lot of results are known for them (see e.g. [17], [24], [27], [28]).

3.6 Quaternionic valued differential forms on R_4

Here we shall describe the whole split de Rham sequence (6) in the special case $n=4$ and $V = V_+$ (see 3.5). Only real modules will be considered here.

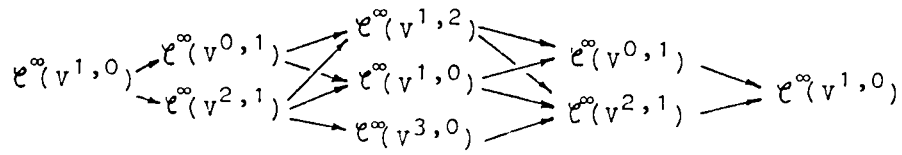
The spaces $V^{j,k}$, considered as modules over R, are irreducible only for $j+k$ odd. For $j+k$ even there is a real subspace $rV^{j,k} \subset V^{j,k}$ such that $rV^{j,k} \otimes_R C = V^{j,k}$ (see [18]) The exterior powers Λ^j have the following description in the classification:

$$\Lambda^1 = rV^{1,1} , \; \Lambda^2 = rV^{2,0} \oplus rV^{0,2} , \; \Lambda^3 = \Lambda^1 ,$$

which then leads to the splittings

$$V^{1,0} \otimes_R \Lambda^1 = V^{0,1} \oplus V^{1,2} , \; V^{1,0} \otimes_R \Lambda^2 = V^{3,0} \oplus V^{1,0} \oplus V^{1,2}$$

(the splitting for 3-forms being the same as for 1-forms). The diagram (6) hence has the form



It can be compared with the splitting of quaternionic valued forms, described in [34] using quaternionic coordinates. The similar splitting can be written for other spinor modules, too.

3.7 Integral formulae and Leray residue

If the equations coming from the diagram (6) are reasonable generalizations of C-R equations, it should be possible to write a generalized Cauchy integral formula for their solutions. It is possible to do it in many cases (e.g. if another first order operator can be found such that the composition of both gives the Laplace operator, see [6]). Generalized Cauchy integral formulae were discussed in many papers (e.g. [2],[5],[6],[9],[10],[13],[14],[15],[20],[35],[37]). All these integral formulae are of elliptic type, i.e. the value at the point P is expressed using values on a sphere around P .

It was mentioned already that the equations coming from the diagram (6) have both elliptic and hyperbolic versions. The integral formulae for hyperbolic equations have quite different character, but even if these two types of integral formulae are very different indeed, there is very simple and nice principle how to deduce one from another using the Leray residue ([19],[29]). Leray's extension of the classical residue theorem can be described as follows.

Let X be a complex manifold of a (complex) dimension n , let S be a submanifold of X of codimension 1 . For every (p-1)-cycle γ in S we shall denote by $\delta\gamma$ the Leray cobord of γ . It is, roughly speaking, the boundary of a tubular neighborhood around γ , so $\delta\gamma$ is a p-cycle in $X\setminus S$.

Theorem (Leray):

Let τ be a smooth, closed p-form on $X\setminus S$, then there is a (p-1)-form Res τ closed on S , such that

$$\int_{\delta\gamma} \tau = 2\pi i \int_{\gamma} \operatorname{Res} \tau$$

holds for every (p-1)-cycle γ in S .
The theorem holds for vector valued forms, too, and can be used in the study of integral formulae in the following way.

Suppose that the map $f : R_n \to V$ satisfies equations for which Cauchy integral formula holds. So suppose that there is a (n-1)-form ω_P, depending on P such that

$$f(P) = \int_{S_{n-1}} \omega_P \tag{12}$$

where the point P lies inside $S_{n-1} \subset R_n$.

Suppose further that the form ω_P is the restriction of a closed form on $C_n \setminus N$, where N is the complex null cone in C_n with the vertex in P.

Then, defining the index of the point P with respect to $(n-1)$-dimensional cycle $\Sigma \subset C_n \setminus N$ properly, the formula

$$\mathrm{Ind}_\Sigma\, P . f(P) = \int_\Sigma \omega_P \tag{13}$$

holds for every cycle $\Sigma \subset C_n \setminus N$ ([6]).

But then the Leray residue theorem tells us that

$$\mathrm{Ind}_{\delta\gamma}\, P . f(P) = 2\pi i \int_\gamma \mathrm{Res}\, \omega_P \tag{14}$$

for every $(n-2)$-cycle γ in N. Taking the cycle γ especially inside the intersection of N with the Minkowski slice through P gives then the corresponding integral formula for solutions of the hyperbolic system. In C_n we have both possibilities (either to express $f(P)$ using $\Sigma \subset C_n \setminus N$ or using $\gamma \subset N$).

Using the procedure it is possible e.g. to deduce Riesz's integral formula for solutions of the wave equation ([30]) from the standard integral formula for harmonic functions ([10]) or to deduce integral formulae, due to Penrose for spin n/2 massless fields on Minkowski space ([26],[27]) from the Cauchy integral formulae in hypercomplex analysis ([36]). A new integral formula for the Dirac operator on Minkowski space M_n was deduces by Bureš from the standard Cauchy integral formula in Clifford analysis ([5]).

3.8 Vector valued forms and cohomology

To illustrate the usefulness of the diagram (6) let us consider the following well known fact from the classical complex analysis. The number of holes in a domain $\Omega \subset C$ can be found using only properties of holomorphic functions on Ω. Holomorphic functions without primitives can exist on Ω and the dimension of the vector space

$$H(\Omega) \;/\; \frac{d}{dz} H(\Omega)$$

is equal to the number of holes in Ω.

Let us express it in more modern language.

Let us consider a part of the diagram (1)

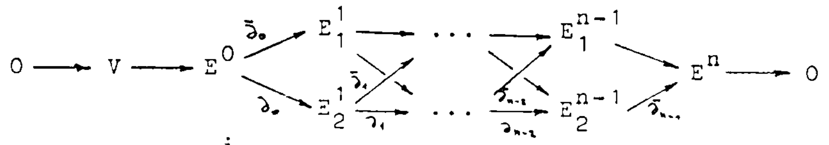
$$0 \longrightarrow C \longrightarrow \mathcal{E}^\infty(\Omega,\Lambda^0) \xrightarrow{\ \partial\ } \mathcal{E}^\infty(\Omega,\Lambda^{1,0})$$

and its subcomplex $0 \longrightarrow C \longrightarrow M^0 \xrightarrow{\ \partial\ } M^1$,
where $M^0 = \{f\in\mathcal{E}^\infty(\Omega,\Lambda^0) \mid \bar\partial f=0\}$, $M^1 = \{f\in\mathcal{E}^\infty(\Omega,\Lambda^{1,0}) \mid \bar\partial f=0\}$.
Then $H^1(\Omega,C) = M^1 / \partial(M^0)$.

All this can be generalized to R_n using the diagram (6). To give an example, let V be the basic spinor module (over R). Then we shall consider the spaces

$$E^j = \mathcal{E}^\infty(\Omega,\Lambda^j\otimes_R V)$$

of V-valued differential forms on $\Omega\subset R_n$. It may be proved by induction that one of irreducible pieces in the decomposition of $\Lambda^j\otimes_R V$, which we denote by F_1^j, is isomorphic to a basic Spin(n)-module. Let us denote by F_2^j the complementary invariant subspace, then we have the splitting $E^j = E_1^j \oplus E_2^j$, where $E_1^j = \mathcal{E}^\infty(\Omega, F_1^j)$, $E_2^j = \mathcal{E}^\infty(\Omega,F_2^j)$. So we shall obtain the diagram



Let us denote $M^j = \operatorname{Ker}\bar\partial_j$, then the homology of the sequence

$$0 \longrightarrow V \longrightarrow M^0 \xrightarrow{\ \partial_0\ } M_1^1 \xrightarrow{\ \partial_1\ } \cdots \xrightarrow{\ \partial_{n-2}\ } M^{n-1} \longrightarrow 0$$

coincide with $H^j(\Omega, V)$ for $j=o,\dots,n-1$ (see [31],[33]).

4. EXAMPLES (MANIFOLDS)

4.1 Dirac operator

It is the only case where important and deep global results are already known. The Dirac operator plays, for example, the important role in the Atiyah-Singer index theorem ([25]).

To describe the Dirac operator using the general procedure of the section 2 (for complex bundles, see Remark 3), three choices have to be made - a Spin(n)-module V, the conformal weight w and a connection ∇ . We shall consider only the case of even dimension, $n=2m$.

Let us take $w = (n-1)/2$. Consider a minimal left ideal V in the complex Clifford algebra $\mathcal{C}_n^c$ with generators $e_1,\dots,e_n$. Let us choose a Riemannian metric in the given conformal class. It will induce the connection ∇ on the Spin bundle V.

The tensor product $\Lambda^1_c \otimes_C V$ again splits into two pieces F_1, F_2; F_1 being the basic spinor module. The space F_1 is often identified with V using the map, given in coordinates by

$$v \in V \longmapsto \frac{1}{n}\{\sum dx_k \otimes e_k \cdot v\} \in \Lambda^1_c \otimes_C V \ .$$

Then the projection of $\Lambda^1_c \otimes_C V$ onto $F_1 = V$ is given by (compare with (9))

$$m : \quad \sum dx_j \otimes v_j \longrightarrow \sum e_j \cdot v_j \in V \ .$$

Under such identification the Dirac operator looks like

$$\Gamma(V) \xrightarrow{\nabla} \Gamma(\Lambda^1_c \otimes_C V) \xrightarrow{m} \Gamma(V)$$

and its symbol is given by Clifford multiplication.

Solutions of the Dirac equation are called harmonic spinors by Hitchin ([16]). The module V splits into two irreducible spinor modules V^+ and V^-, which leads to operators D^+, D^- and to spaces of positive (negative) harmonic spinors. The special value chosen for w is quite important, because in this case the Dirac operator does not depend on the choice of the Riemannian connection (see 4.2). It was proved in [16] that the dimension of the space of harmonic spinors is conformally invariant, but that it cannot be expressed in terms of topological invariants of the manifold.

In the case of conformally flat manifolds it is possible to give a nice coordinate description of the spinor bundle and of the Dirac operator. It was shown by Ahlfors ([0]) that conformal maps in higher dimensions can be expressed compactly using Clifford numbers in a very close analogy with the complex case. This notation can be used to give a simple formula for transition functions of the spinor bundle. The condition that solutions transform into solutions under conformal transformations (with a weight) picks out again the conformal weight $w = (n-1)/2$ as the only possibility (for details see [4] ,[21]).

4.2 Conformally invariant operators

Let us consider again complex bundles. Let us take any irreducible Spin(n)-module V and any conformal weight w. Let us consider further any irreducible piece W in $\Lambda^1_c \otimes_C V$.

A natural possibility for the choice of ∇ is to consider a Riemannian metric inside the given conformal structure and to take the associated connection ∇ on V. It will give us the corresponding operator ∂ in the first column of the diagram (8). The operators ∂ depend generally on the choice of the Riemannian metric, but there is an exceptional, unique case, when all operators ∂ for all possible Riemannian metrics in the given conformal class coincide. It was proved by Fegan ([11]) (and by algebraic methods by Hitchin ([17])) that for every choice of V, W there is exactly one conformal weight w such that the operator ∂ is conformally invariant (in the sense described above).

Another very interesting subject is discussed in the paper by Hitchin ([17]). The twistor theory, created by R. Penrose, is nowadays very rich and extended theory. The theory is firmly rooted in physics (namely in general relativity) and its evolution has led to a lot of deep and important mathematical results for nonlinear equations. It was a part of the Penrose's twistor programme to study massless fields using the correspondence between Minkowski and twistor spaces. This transformation is studied in [17] for massless fields (i.e. $V = V^{j,o}$ in 3.5) and for the Laplace equation on self-dual 4-dimensional manifolds.

4.3 Generalized spherical C-R operator

Let M be n-dimensional Riemannian manifold with an exterior structure, given by the Weingarten map. Take the Clifford algebra $\mathcal{C}_n$ for the G-module V. A special connection was defined on the associated vector bundle using the Riemannian connection and the Weingarten map in [32]. The corresponding piece in the decomposition was used there as the generalization of the spherical C-R operator.

4.4 Kähler equation

Let us take the space Λ^r for the G-module V. Then (see the section 3.4) the tensor product $\Lambda^r \otimes_R \Lambda^1$ splits into 3 pieces F_1, F_2, F_3 and $F_2 \cong \Lambda^{r+1}$, $F_3 \cong \Lambda^{r-1}$. The corresponding operators ∂_2 and ∂_3 are just d and δ.

Now, if we take $V = \Lambda^* = \Lambda^o \oplus \ldots \oplus \Lambda^n$, then we can apply the procedure piece by piece and we shall end with the operators d and δ on Λ^*. The values of these operators are, strictly speaking, in $\Lambda^* \otimes_R \Lambda^1$, but we can identify them with Λ^*. We can even consider their sum $d + \delta$, but then the result differs in both cases. The operator $d + \delta$ with values in Λ splits (locally) in a quite different way and it coincides (after the usual identification of Λ^* with $\mathcal{C}_n$) with the basic operator in Clifford analysis (so it splits into the sum of the Dirac operators for spinor valued fields).

The operator d+δ in the second sense is usually called the Dirac operator and its physical interpretation was studied in [1].

ACKNOWLEDGEMENT

In the paper the common work with J. Bureš is reported ([7]).

REFERENCES

[0] L.V.Ahlfors: 'Möbius transformations and Clifford numbers'. See elsewhere in this volume.

[1] I.Benn, R.Tucker: 'Fermions without spinors', *Com.Math.Physics*,89,3,1983, 341-362

[2] F.Brackx, R.Delanghe, F.Sommen: *Clifford analysis*, Res. Notes in Math. 76, Pitman,1982

[3] R.Delanghe, F.Brackx: 'Hypercomplex function theory and Hilbert modules with reproducing kernel', *Proc.Lond.Math.Soc.*,37, 1978, 545-576

[4] J.Bureš: 'Hypercomplex analysis on conformally flat manifolds', to be published

[5] J.Bureš: 'Integral formulas for left regular spinor-valued functions in Clifford analysis', to appear

[6] J.Bureš, V.Souček: 'Generalized hypercomplex analysis and its integral formulas', to appear in *Complex Variables*

[7] J.Bureš, V.Souček: 'On generalized Cauchy-Riemann equations on manifolds', Proc. of the 12th Winter School, Srni *Suppl.Rend.Circolo Mat.Palermo*,II,6,1984, 31-42

[8] J.Bureš, V.Souček: 'Regular spinor valued mappings', to appear in *Seminarii di geometria*, Bologna

[9] R.Delanghe: 'On regular-analytic functions with values in a Clifford algebra', *Math.Ann.*,185,1970,91-111

[10] M.Dodson, V.Souček: 'Leray residue applied to solutions of the Laplace and wave equations', to appear in *Seminarii di geometria*, Bologna

[11] H.D.Fegan: 'Conformally invariant first order differential operators', *Q.Jour.Math.*,27,1976,371-378

[12] L.Gårding: 'Relativistic wave equations for zero restmass', *Proc.Camb.Phil.Soc.*,41,1945,49-56

[13] R.P.Gilbert, J.L.Buchanan: *First order elliptic systems*, Math. in Science and Eng. 163, Academic Press, 1983

[14] H.Haefeli: 'Hyperkomplexe Differentiale', *Comment.Math.Helv.*, 20,1947,382-420

[15] D.Hestenes, G.Sobczyk: *Clifford algebra to geometric calculus*, D.Reidel Publ. Comp., Dordrecht, 1984

[16] N.J.Hitchin: 'Harmonic spinors' *Adv.Math.*, 14,1974,1-55

[17] N.J.Hitchin: 'Linear field equations on self-dual spaces, *Proc.R.Soc.London* A 370,1980,173-191

[18] D.Husemoller: *Fibre bundles*, McGraw Hill, 1966

[19] J.Leray: 'Le calcul differentiel et integral sur une variete analytique complex, Probleme de CauchyIII *Bull.Soc.math.Fr.*, 87,1959,81-180

[20] P.Lounesto: 'Spinor-valued regular functions in hypercomplex analysis', thesis, Helsinky University of Techn. Report-HTKK-MAT-A154, 1979

[21] M.Markl: 'Regular functions over conformal quaternionic manifolds', *Com.Math.Univ.Carolinae*, 22,1981,579-583

[22] G.C.Moisil, N.Theodoresco: Fonctions holomorphes dans l'espace" *Mathematica V*, 1931,142-159

[23] W.Nahm: Self-dual magnetic monopoles and generalizations of holomorphic functions', Proc. of the 12th Winter School, Srni *Suppl.Rend.Circ.Mat.Palermo*, II,6,1984,233-242

[24] P. van Nieuwenhuizen, N.P.Warner: 'Integrability conditions for Killing spinors', *Com.Math.Physics*, 1984, 277-284

[25] R.S.Palais: *Seminar on the Atiyah-Singer index theorem*, Princeton Univ. Press, 1965

[26] R.Penrose: 'Null hypersurface initial data for classical fields of arbitrary spin and for general relativity', *Gen.Rel.Grav.*, 12,1980225-264

[27] R.Penrose, W.Rindler: *Spinors and space-time I*, Camb.Univ.Press, 1984

[28] R.Penrose, M.Walker: 'On quadratic first integrals of the geodesic equations for type 22 spacetimes', *Com.Math.Physics*, 18,1970,265-274

[29] F.Pham: *Introduction a l etude topologique des singularites de Landau,*
Paris, Gauthier.Villars, 1967

[30] M.Riesz: 'A geometric solution of the wave equation in space-time of even dimension',
Comm.Pure Appl.Math., XIII, 1960, 329-351

[31] F.Sommen: Monogenic differential forms and homology theory',
Proc.Royal Irish Acad., 84A, 2, 1984, 87-109

[32] F.Sommen: 'Monogenic functions on surfaces'
to be published

[33] F.Sommen, V.Souček: 'Hypercomplex differential forms applied to the de Rham and the Dolbeault complex',
to appear in *Seminarii di geometria*, Bologna

[34] V.Souček: 'H-valued differential forms on H',
Proc. of the 11th Winter School, Zelezna Ruda
Suppl.Rend.Circ.Mat.Palermo, II, 3, 1984, 293-299

[35] V.Souček: 'Complex-quaternionic analysis applied to spin-½ massless fields',
Complex Variables, 1, 1983, 327-346

[36] V.Souček: 'Boundary value type and initial value type integral formulae for massless fields',
Twistor Newsletters, 14, 1982,

[37] V.Souček: 'Holomorphicity in quaternionic analysis'
Seminarii di geometria, 1982-83,
Istituto di Geometria, Universita Bologna, 147-171

[38] E.M.Stein, G.Weiss: 'Generalization of the Cauchy-Riemann equations and representations of the rotation group',
Amer.J.Math., 90, 1968, 163-196

[39] A.Sudbery: 'Quaternionic analysis',
Math.Proc.Camb.Phil.Soc., 85, 1979, 199-225