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Residues in Clifford analysis

1. Introduction

The residue theorem in complex analysis forms a highlight of the standard calculus of one complex variable. Its generalization for monogenic functions in Clifford analysis is well known (see [3]). As in complex analysis, the residue is not, in fact, defined for a left (or right) monogenic function, but for a Clifford-valued $(m-1)$-form of the type $fd\sigma$ or $d\sigma g$, where f (resp. g) is a left (resp. right) monogenic function and $d\sigma$ is the standard Clifford-valued form $d\sigma = \sum(-1)^{j+1}e_j d\hat{x}_j$.

This type of residue was extended to $(m-1)$-forms of the type $\omega = fd\sigma g$, where f and g are, respectively, right and left monogenic on a domain in R^m (see [17, 18]).

In both cases described, the residue was considered for pointwise singularities and it was shown that it is possible to compute it using coefficients in the Taylor and Laurent expansion around singular points, as it is standard in complex analysis.

It is known from the first papers by Fueter on quaternionic analysis that the singularity set of a monogenic function need not be a point, but can have an arbitrary dimension. So it is natural to try to find a notion of residue for higher dimensional singularities as well and to prove the corresponding residue theorem.

Once this is done, still another generalization is possible as for the type of forms considered. The residue studied so far was introduced for monogenic functions or monogenic $(m-1)$-forms or for their product. As the theory of monogenic functions was recently extended to so called monogenic differential forms ([15, 4]) of any degree, it is natural to try to formulate the residue theory for such forms as well.

A main tool used in the paper to define residue for higher dimensional singularities is the Leray-Norguet theory of residue for a closed differential form having a singularity on a submanifold of any codimension. The Leray residue was originally introduced in a complex situation (singularities considered were on a complex hypersurface, see ([9]) and most of its generalizations were formulated in the complex case. The real version, announced by Norguet in [5], is not treated in literature in full details and was not systematically used. It is exactly this version of the residue theory which is needed in Clifford analysis.

We are describing the Leray-Norguet residue and cobord in detail in Sect.4, giving full proofs for the results announced in the short communication by Norguet (see [5]). It is very convenient to do so using the language of currents (see [10]). For the convenience of the reader, the results from the theory of currents needed are shortly reviewed in Sect.2 and a few facts on tubular neighborhoods are collected in Sect.3.

The general theory of residues for monogenic differential forms with higher dimensional singularities is described then in Sect.5 and the general Residue Theorem is proved.

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A big advantage of the residue in complex analysis is that it can be often computed using Taylor and Laurent series. In Clifford analysis, the corresponding notions for monogenic functions with singularities on surfaces were not yet developed.

In sections 6 and 7, we are introducing generalized Taylor and Laurent series in the case where singularities are on a compact subset of $R^q \subset R^m$ and we are proving their basic properties.

Finally, in Sect.8 we discuss examples and show how to compute residues using either Taylor and Laurent series or by direct calculations. Using the Cauchy transform it is possible to show that any possible residue on a compact surface really appears as a residue of a suitable monogenic form.

2. Direct and Inverse Images of Currents; Integration along Fibers

The definition of the Leray residue and of the Leray cobord, given in Sect.3, is based on the one hand on the notion of integration along fibers for differential forms and on the other hand on the notion of the inverse image of a chain. A systematic way how to define them is to use the language of currents. This language will also be useful in some examples discussed in the Sect.8, so we shall summarize here some basic facts on direct and inverse images of currents on manifolds. Details can be found in the book by L. Schwartz ([10], Chap.IX). The integration along fibers can also be found in the book by Bott and Tu ([2],p.61-65), they are considering there the case of vector bundles, which is quite similar.

Even if the theory explained below can be built up for general, nonoriented manifolds (see [2, 10]), we shall consider only oriented manifolds in our paper. The reason for it is to keep the technical details at low level and to make basic ideas involved clear and more understandable. So, without further explicit repetition, we suppose all manifolds to be smooth and oriented.

2.1. Currents on manifolds

Let us consider a manifold M of dimension m and let us denote by $\mathcal{E}(M, \Lambda^k)$ (resp. $\mathcal{D}(M, \Lambda^k)$) the space of smooth k-forms on M (resp. smooth forms with a compact support). The space $\mathcal{E}(M, \Lambda^0) = \mathcal{E}(M)$ (similarly for $\mathcal{D}$) is the space of smooth functions.

The space of $(m-k)$-currents is then defined as the dual $\mathcal{D}(M, \Lambda^k)'$. The analogous space $\mathcal{E}(M, \Lambda^k)'$ is the space of compactly supported currents. So the spaces of m-currents are the standard spaces of (compactly supported) distributions.

Example.

Let $F = \sum_A F_A dx_{\{1,\dots,m\}\setminus A}$ be an $(m-k)$-form; then

$$T_F[\varphi] = \langle \varphi, F \rangle := \int_M \varphi \wedge F$$

$$_F T[\varphi] = \langle F, \varphi \rangle := \int_M F \wedge \varphi$$

are $(m-k)$-currents.

An easy generalization of the example above leads to the notion of a distributional $(m-k)$-form. The space $\mathcal{D}'(M, \Lambda^{m-k})$ of all such generalized forms is defined as the completion of the space $\mathcal{E}(M, \Lambda^{m-k})$ with respect to the topology given by the set of seminorms $p_\varphi F = |\int_M \varphi \wedge F|, \varphi \in \mathcal{D}(M, \Lambda^k)$. Locally, on a neighborhood U with coordinates x_i, an element $F \in \mathcal{D}'(M, \Lambda^{m-k})$ can be represented by a distribution-valued $(m-k)$-form $\sum_A F_A dx_{\{1,\dots,m\}\setminus A}, F_A \in \mathcal{D}'(U)$.

Note that currents and distributional forms are two basically different concepts. But there are two natural isomorphisms between $\mathcal{D}'(M, \Lambda^{m-k})$ and $\mathcal{D}(M, \Lambda^k)'$.

The first one is given by a symbolic description

$$F \mapsto T_F,\ T_F[\varphi] := \int_M \varphi \wedge F,$$

which means (again symbolically) that in a coordinate neighborhood U with $\varphi = \sum_A \varphi_A dx_A$, $F = \sum_A F_A dx_{\{1,\dots,m\}\setminus A}$ that

$$T_F[\varphi] \quad = \quad \sum sgn\, A \cdot \int_M \varphi_A F_A dx_{\{1,\dots,m\}}.$$

It leads to the (local) definition of the corresponding map as

$$T_F[\varphi] := \sum_A sgn\, A \cdot F_A[\varphi_A].$$

The map is extended then by linearity using a partition of unity on M.

The second map is given by $F \mapsto {}_F T$

$${}_F T[\varphi] = \int_M F \wedge \varphi = (-1)^{k(m-k)} \int_M \varphi \wedge F := (-1)^{k(m-k)} T_F[\varphi].$$

Examples.

1. Let C be a k-chain in M. Then a k-current T_C is given by

$$T_C := \int_C \varphi.$$

As we know, there are two distributional $(m-k)$-forms δ_C^r and δ_C^l symbolically defined by

$$T_C[\varphi] = \int_M \varphi \wedge \delta_C^r = \int_M \delta_C^l \wedge \varphi \qquad .$$

so that, in fact,

$$\delta_C^r \equiv \delta_C := \sum_A \delta_{C,A} dx_{\{1,\dots,m\}\setminus A}$$

with $\delta_{C,A} \in \mathcal{E}'(U)$ given by

$$\delta_{C,A}[\varphi] = sgn\, A \cdot T_C[\varphi dx_A] = sgn\, A \int_C \varphi dx_A$$

and $\delta_C^l = (-1)^{k(m-k)} \delta_C$.

2. Take $y \in M$ fixed and let $\varphi = \sum_A \varphi_A dx_A$. Put

$$I_{y,A}[\varphi] = \varphi_A(y),\ I_{y,A} \in \mathcal{E}(M,\Lambda^k)'.$$

The distributional forms associated to $I_{y,A}$ are given by

$$I_{y,A}[\varphi] = \int_M \varphi \wedge \delta_{y,A} = \int_M \delta^l_{y,A} \wedge \varphi.$$

From this it is clear that

$$\delta_{y,A} = sgn\, A \cdot \delta(x-y) dx_{\{1,\ldots,m\}\setminus A}$$

$$\delta^l_{y,A} = sgn\,(\{1,\ldots,m\} \setminus A)\delta(x-y)dx_{\{1,\ldots,m\}\setminus A}.$$

3. Let $t = \sum_j t_j \partial_{x_j} \in T_y M$; then for $\varphi \in \mathcal{E}(M,\Lambda^1)$ we put

$$\begin{aligned} t[\varphi] &= t \rfloor \varphi(y) = \sum_j t_j \varphi_j(y) \\ &= \int_M \varphi_j t_j \delta(x-y) dx_{\{1,\ldots,m\}} \\ &= \int_M \varphi \wedge \delta(x-y) \sum_j (-1)^{j+1} t_j d\hat{x}_j \\ &= \int_M \varphi \wedge \delta(x-y) t \rfloor dx_{\{1,\ldots,m\}} \\ &= (-1)^{m-1} \int_M [\delta(x-y) t \rfloor dx_{\{1,\ldots,m\}}] \wedge \varphi. \end{aligned}$$

4. Let $t_1,\ldots,t_k \in T_y M$. Then we may put

$$t_1 \wedge \ldots \wedge t_k[\varphi] = \varphi(y)(t_1,\ldots,t_k).$$

Note that for $A = \{\alpha_1,\ldots,\alpha_k\}$

$$\partial_{x_{\alpha_1}} \wedge \ldots \wedge \partial_{x_{\alpha_k}}[\varphi] = \varphi_A(y) = I_{y,A}[\varphi].$$

2.2. Operations on currents

The boundary of a current. Let $T \in \mathcal{D}(M,\Lambda^k)'$ be an $(m-k)$-current; then we put

$$\partial T[\varphi] := T[d\varphi],\ \varphi \in \mathcal{D}(M,\Lambda^k). \tag{1}$$

The operator ∂ generalizes the boundary operator for chains. Indeed, let $C_k(M)$ be the space of k-chains; then the boundary map

$$\partial : C_k(M) \mapsto C_{k-1}(M)$$

is well defined and we have Stokes' theorem:

$$\int_C d\varphi = \int_{\partial C} \varphi,$$

from which it follows that

$$\partial T_C[\varphi] = T_C[d\varphi] = T_{\partial C}[\varphi].$$

Furthermore, $C_k(M)$ is dense in $\mathcal{D}(M, \Lambda^k)'$ and so the above boundary operator is the only possible extension of the operator ∂ defined on chains.

The derivative of a distributional form. The operator $d = \sum_j dx_j \partial_j$ is defined on $\mathcal{D}'(M, \Lambda^{m-k})$ by the extension of the outer differential defined on the dense subspace $\mathcal{E}(M, \Lambda^{m-k})$. For $F \in \mathcal{E}(M, \Lambda^{m-k}), \varphi \in \mathcal{D}(M, \Lambda^{k-1})$ we have

$$\int_M \varphi \wedge dF = (-1)^{k-1} \int_M d\varphi \wedge F.$$

From this it is also clear that

$$T_{dF}[\varphi] \doteq (-1)^{k-1} T_F[d\varphi] = (-1)^{k-1} \partial T_F[\varphi].$$

On the other hand,

$$\int_M dF \wedge \varphi = (-1)^{m-k} \int_M F \wedge d\varphi,$$

from which we get ${}_{dF}T = (-1)^{m-k} \partial_F T$.

This lead to a definition of the differential of a current:

$$dT[\varphi] := (-1)^{m-k} T[d\varphi] = (-1)^{m-k} \partial T[\varphi],$$

$$Td[\varphi] = (-1)^{k-1} \partial T[\varphi],$$

so that also $T_{(dF)} = T_F d$ and ${}_{(dF)}T = d_F T$.

The generalized de Rham theorem. The derivative d on the space of distributional forms defines the differential complexes $(\mathcal{D}'(M, \Lambda^k), d)$, resp. $(\mathcal{E}'(M, \Lambda^k), d)$. The standard de Rham complexes of differential forms (resp. forms with compact supports) are clearly their subcomplexes. It is shown in [10] that the homologies of the corresponding complexes coincide. It means that the standard cohomology groups $H^k(M)$, resp. $H^k_c(M)$, are described by the complex of distributional forms as well as by ordinary forms.

The correspondence $F \mapsto T_F, \mathcal{D}'(M, \Lambda^k) \mapsto \mathcal{D}(M, \Lambda^{m-k})'$, identifies distributional forms with currents and it translates (up to a sign) d onto ∂. So the homology of the complex $(\mathcal{D}(M, \Lambda^{m-k})', \partial)$ gives again $H^k(M)$. By the Poincaré duality, it means that the complex $(\mathcal{D}(M, \Lambda^l)', \partial)$ and its subcomplex of standard chains with the usual boundary operator give both the same homology groups $H_l(M)$.

It is sometimes useful to represent elements of cohomology or homology groups by currents. We shall do so in Sect.8.

The product with a smooth form. Suppose that $F \in \mathcal{E}(M, \Lambda^{m-k})$ and $\alpha \in \mathcal{E}(M, \Lambda^l)$. Then we have for $\varphi \in \mathcal{D}(M, \Lambda^{k-l})$

$$T_{\alpha\wedge F}[\varphi] = \int_M \varphi \wedge \alpha \wedge F = T_F[\varphi \wedge \alpha],$$

$$_{F\wedge\alpha}T[\varphi] = \int_M F \wedge \alpha \wedge \varphi = {}_F T[\alpha \wedge \varphi].$$

We may now define for a current $T \in \mathcal{D}(M, \Lambda^k)'$

$$\alpha \wedge T[\varphi] := T[\varphi \wedge \alpha],$$

$$T \wedge \alpha[\varphi] := T[\alpha \wedge \varphi],$$

where $\varphi \in \mathcal{D}(M, \Lambda^{k-l})$.

We have then the obvious rules

$$T_{\alpha\wedge F} = \alpha \wedge T_F, \, {}_{F\wedge\alpha}T = {}_F T \wedge \alpha.$$

The direct image of currents. Suppose that $\Phi : M \to N$ is a smooth proper map between two manifolds, $\dim M = m$, $\dim N = n$. For $T \in \mathcal{D}(M, \Lambda^k)'$, we define

$$\Phi_* T[\varphi] := T[\Phi^* \varphi], \varphi \in \mathcal{D}(M, \Lambda^k),$$

where $\Phi^* \varphi$ is the standard inverse image of the form φ. As an immediate consequence, we get that for $\alpha \in \mathcal{E}(N, \Lambda^l)$

$$\begin{aligned} (\alpha \wedge \Phi_* T)[\varphi] &= \Phi_* T[\varphi \wedge \alpha] \\ &= T[\Phi^* \varphi \wedge \Phi^* \alpha] \\ &= \Phi_*(\Phi^* \alpha \wedge T)[\varphi]. \end{aligned}$$

So

$$\Phi_*(\Phi^* \alpha \wedge T) = \alpha \wedge \Phi_* T. \tag{2}$$

It is easy to deduce in the same way that

$$\partial(\Phi_* T) = \Phi_*(\partial T) \tag{3}$$

$$d(\Phi_* T) = (-1)^{m-n} \Phi_*(dT) \tag{4}$$

$$(\Phi_* T)\, d = \Phi_*(T\, d). \tag{5}$$

Example.

If C is a k-chain in M, then the direct image $\Phi_* T_C$ is always defined as a current by the formula

$$\Phi_* T_C[\varphi] = \int_C \Phi_* \varphi$$

(whether or not Φ is injective).

Integration along fibers. Let $\Phi : M \to N$ be a fibration such that both M and N are oriented manifolds and that all fibers $\Phi^{-1}(u)$, $u \in N$, are diffeormorphic to a given compact manifold W. Let us denote $\dim M = m$, $\dim N = n$. Then there is a unique orientation induced on the fibers, i.e. we can choose an orientation on W such that there is a locally finite covering $\{U_\gamma\}$ of M for which

$$\Phi^{-1}(U_\gamma) \simeq U_\gamma \times W,$$

where on the right hand side we consider the canonical orientation of the Cartesian product.

On one piece $U_\gamma \times W$, any differential form of degree $k \geq n$ can be written as a finite sum of summands having the form

$$g\alpha \wedge \beta,\ g \in \mathcal{E}(U_\gamma \times W),\ \alpha \in \mathcal{E}(W, \Lambda^p),\ \beta \in \mathcal{E}(U_\gamma, \Lambda^q),\ p + q = k.$$

(To be more precise, we mean that α is an inverse image of a form on W under the corresponding projection, similarly for β.)

Taking into account the convention that the integral of a form over a manifold is defined to be zero if the degree of the form does not coincide with the dimension of the manifold, we define the integral over fibers as the map $\tilde{\Phi}_*$ given by

$$\tilde{\Phi}_*(g\alpha \wedge \beta) := \left(\int_W g\alpha \right) \beta$$

and we extend it by linearity to any form on $U_\gamma \times W$. Using the partition of unity subordinated to the covering $\{U_\gamma\}$, the map $\tilde{\Phi}_*$ can be extended by linearity to any form of degree k on M.

To prove that the definition of the integral over fibers is independent of all choices made, it is sufficient to show that it coincides with the direct image in the sense of currents which is, of course, uniquely defined. More precisely, we show that the current given by $\tilde{\Phi}_*(\varphi)$, $\varphi \in \mathcal{E}(M, \Lambda^k)$ is the direct image of the current given by φ, i.e. that ${}_{(\tilde{\Phi}_*(\varphi))}T = \Phi_*({}_{\varphi}T)$. In view of linearity, it can be checked in local coordinates. Now for a function g on $U \times W$ and for forms α on W and β, ψ on U having suitable degrees we have, using the projection $\pi : U \times W \to W$,

$$\Phi_* \left({}_{(g\pi^*\alpha \wedge \Phi^*\beta)}T \right) [\psi] = {}_{(g\pi^*\alpha \wedge \Phi^*\beta)}T[\Phi^*\psi] = \int_{U \times W} g\pi^*\alpha \wedge \Phi^*\beta \wedge \Phi^*\psi$$

$${}_{(\tilde{\Phi}_*(g\pi^*\alpha \wedge \Phi^*\beta))}T[\psi] = \int_U \left[\int_W g\pi^*\alpha \right] \wedge \beta \wedge \psi,$$

which proves the claim.

We shall not distinguish in what follows between the maps Φ_* and $\tilde{\Phi}_*$ and we shall denote them both by Φ_*.

It is immediately clear from the definition that the integration along fibers is transitive, i.e. that

$$(\Phi \circ \Psi)_* = \Phi_* \circ \Psi_*. \tag{6}$$

The inverse image of currents. If $\Phi : M \to N$ is a fibration, as above, then the integration over fibers for differential form can be used to define, by duality, the inverse image $\Phi^*(T)$ of an $(m-k)$-current T by the formula

$$\Phi^*(T)[\varphi] = T[\Phi_*(\varphi)], \; \varphi \in \mathcal{D}(M, \Lambda^k).$$

It follows immediately from the definition that Φ^* commutes (up to a sign) with the boundary map ∂.

3. Tubular Neighborhoods

Some properties of tubular neighborhoods of submanifolds will be needed in Sect.4. They are described in detail in the book by Hirsch ([6]). We shall use the properties proved there in a slightly changed situation, namely we shall assume all vector bundles and manifolds to be oriented.

Suppose that X is a manifold and $S \subset X$ is a submanifold such that $\dim X = m$, $\dim S = n$.

An oriented tubular neighborhood of S in X is a pair (f, ξ), where $\xi = (\pi, E, S)$ is an oriented vector bundle over S and $f : E \to X$ is an imbedding such that:

1. f preserves the orientations
2. $f|S = 1_S$ (S being identified with the zero section of E);
3. $f(E)$ is an open neighborhood of S in X.

An oriented closed tubular neighborhood of radius $\varepsilon > 0$ of S in X is the image $f(D_\varepsilon)$ given by a tubular neighborhood (f, ξ), where $\xi = (\pi, E, S, ||.||)$ is an oriented orthogonal vector bundle (i.e. there is a norm $||.||$ coming from a positive definite scalar product on fibers, depending smoothly on a point) and $D_\varepsilon = \{x \in E| \, ||x|| \le \varepsilon\}$ is the disk subbundle of ξ with radius ε.

The main theorem on the existence of a tubular neighborhood, proved in [6],p.116, is:

Theorem 3.1 *Let S be an oriented submanifold of an oriented manifold X. Then there exists an oriented tubular neighborhood of S in X.*

It is proved in [6] that it is always possible to choose an orthogonal structure on a fibre bundle, so closed tubular neighborhoods exist as well.

An important information concerning different tubular neighborhoods is contained in the following theorem.

Theorem 3.2 *Let $S \subset X$ be a submanifold, let both S and M be oriented. Let $(f_i, \xi_i, ||.||_i, \varepsilon_i)$, $i = 0, 1$, be two oriented closed tubular neighborhoods of S in X. Then these two tubular neighborhoods are isotopic by an isotopy of oriented tubular neighborhoods $F_t : E_0 \to X$, $0 \le t \le 1$, with $F_1(D_{\varepsilon_0}) = f_1(D_{\varepsilon_1})$. Here the isotopy of oriented tubular neighborhoods means a homotopy $F : E_0 \times I \to X$ such that:*

1. *the related map* $\hat{F} : E_0 \times I \to X \times I, \hat{F}(x,t) = (F_t(x), t)$ *is an imbedding;*

2. *S is left invariant by the isotopy;*

3. $F_0 = f_0$;

4. $F_1(E_0) = f_1(E_1)$;

5. $f_1^{-1}F_1 : E_0 \to E_1$ *is an isomorphism of oriented vector bundles.*

Both these theorems are stated (and proved) in [6] without any assumption concerning orientations, but the proofs can be modified so to respect given orientations; the fibers are always oriented in a compatible way with orientations of S and X.

4. The Leray-Norguet Residue and Cobord

The standard operation of inverse image of forms (resp. direct image of chains) under a map is used for the definition of the corresponding induced maps on the cohomology (resp. homology) groups.

The less usual operations of the integration along fibers and inverse image of currents will be used now to define Leray-Norguet residue and cobord maps. We are following closely the approach announced in [5].

Lemma 4.1 *Let S be a submanifold of a manifold X and let (f,ξ) be an oriented tubular neighborhood of S in X. Furthermore, let an orthogonal structure on ξ be chosen and let us denote by U the image $f(D_1)$ of the disk subbundle of radius 1. Then the map $\pi : \partial U \to S$ induced by f also induces the maps*

$$\pi_* : \mathcal{E}(\partial U, \Lambda^p) \to \mathcal{E}(S, \Lambda^{p-(m-n-1)})$$

and

$$\pi_* : H^p(\partial U) \to H^{p-(m-n-1)}(S).$$

Proof.

The first map is the integration along fibers, the second one is well-defined, because the map π_* commutes (up to a sign) with d. ∎

Lemma 4.2 *Under the same assumptions as above, the map π induces the map*

$$\pi^* : \mathcal{E}(S, \Lambda^{p-(m-n-1)})' \to \mathcal{E}(\partial U, \Lambda^p)'$$

and

$$\pi^* : H_q(S) \to H_{q+(m-n-1)}(\partial U).$$

Proof.

The first map is the inverse image of currents, commuting with the boundary map. So it descends to the homology groups (represented here by the differential complex of currents). ∎

Theorem 4.1 *Let S be a submanifold of X, $\dim X = m, \dim S = n$. Let $U = f(D_1)$ be an oriented closed tubular neighborhood given by a choice of $(f, \xi, ||.||)$. Let $\pi : \partial U \to S$ be the induced projection and denote by i the inclusion $\partial U \subset X \setminus S$.*

Then the map

$$\mathrm{Res} := \pi_* \circ i^* : H^P(X \setminus S) \to H^{p-(m-n-1)}(S)$$

is independent of all choices made.

Proof.

Let $(f_i, \xi_i, ||.||_i)$, $i = 0, 1$, be two possible choices. A consequence of the Theorem 3.2 is that there is an isomorphism $\Phi : \xi_0 \to \xi_1$ between the two bundles which implies that it is possible to take $f_1' := f_1 \circ \Phi$ instead of f_1 and to define both tubular neighborhoods using the same bundle ξ.

So let us suppose from the beginning that $\xi_0 = \xi_1 = \xi$, $\xi = \{\pi : E \to S\}$. Putting $U_i = f_i(D_1)$, we have maps $\pi_i : \partial U_i \to S$ defining then the maps $(\pi_i)_*$.

Now, the maps f_i being isomorphisms of D_1 onto U_i,

$$(\pi_i)_*\omega = \pi_* \circ (f_i)^*\omega.$$

The isotopy theorem (Th.3.2) tells us that the maps f_i are homotopic with each other which implies that the forms $(f_i)^*\omega$ are in the same class of cohomology. But then the same is true also for their integral along the fibers of the maps $\pi_* \circ (f_i)^*\omega$. ∎

Definition 4.1 *The map*

$$\mathrm{Res} := \pi_* \circ i^* : H^p(X \setminus S) \to H^{p-(m-n-1)}(S)$$

is henceforth called the Leray-Norguet residue.

Theorem 4.2 *Under the same assumptions as above, the map*

$$\delta := i_* \circ \pi^* : H_q(S) \to H_{q+(m-n-1)}(X \setminus S)$$

is independent of choices made.

Proof. The maps Res and δ are (for a given choice of a tubular neighborhood) clearly dual to each other. So the independence of δ on choices made is a consequence of that of Res. ∎

Definition 4.2 *The map*

$$\delta := i_* \circ \pi^* : H_q(S) \to H_{q+(m-n-1)}(X \setminus S)$$

is henceforth called the Leray-Norguet cobord.

In the original paper ([9]), the Leray residue and the Leray cobord were introduced independently of each other. Their connection was then given by the following theorem.

Theorem 4.3 (Leray residue theorem) *For any closed $(p+m-n-1)$-form ω on $X \setminus S$ and for any p-dimensional cocycle C on S, we have the formula*

$$\delta C[\omega] = C[\operatorname{Res}\omega].$$

Notice that when both C and δC are cocycles, the usual form of the Leray residue theorem is

$$\int_{\delta C} \omega = \int_C \operatorname{Res}\omega.$$

The definition of the Leray-Norguet cobord map being somewhat abstract, it is useful, also for computations, to have a geometrical picture in mind.

So let us consider the simple situation in which the fiber bundle ξ, giving the tubular neighborhood of S in X, is a trivial fibre bundle $S \times V$. Let $f : D_1 \to U$ then be the corresponding closed tubular neighborhood.

Consider now a p-dimensional cocycle C given, for simplicity, by a smooth map $C : I_p \to S$, where I_p is the p-dimensional interval. Let us choose any map $b : I_{m-n-1} \to S_1$ representing the unit sphere $S_1 \subset V$ as a cocycle. Applying the definition of the Leray-Norguet residue and cobord, it is easy to see that for every form ω we have

$$\int_{f\circ[C\times b]} \omega = \int_C \operatorname{Res}\omega,$$

whence the map $f \circ [C \times b] : I_{p+m-n-1} \to X \setminus S$ represents the Leray-Norguet cobord δC.

Hence, geometrically speaking, the Leray-Norguet cobord map consists of replacing every point of a chain C by a $(m-n-1)$-dimensional sphere having its center in the given point and going to a transversal direction (given by a choice of the tubular neighborhood and its projection).

Finally notice that we shall use the residue theorem above in a slightly more general situation, namely in the case of vector-valued differential forms. To this end, let V be a fixed finite dimensional real vector space. The Leray-Norguet residue for V-valued forms is then defined by the same methods as above and it belongs to the cohomology space $H^{p-(m-n-1)}(S, V)$ with coefficients in V. Note that by choosing fixed basis for V, everything is reduced to componentwise computation of the residue.

5. The Residue Theorem

In this section, we consider monogenic differential forms having higher dimensional singularities, we introduce the notion of residue for a singularity on a q-dimensional surface and we prove the corresponding Residue Theorem. It is also shown that in the case of pointwise singularities this general notion of residue coincides with the one given by Zöll in [17].

5.1. Residues as numbers

Let us consider first the simplest case of monogenic $(m-1)$-forms, i.e. the case of a form $\omega = d\sigma\, g$, where g is a left monogenic function (the space of all left, resp. right, monogenic functions on $\Omega \subset R^m$ is denoted by $\mathcal{M}_l$, resp. $\mathcal{M}_r$). We know that ω is a closed form.

Definition 5.1 *Let Ω be a domain in R^m and let $\Sigma \subset \Omega$ be a compact (oriented) submanifold of dimension q, $q = 0, \ldots, m-2$.*

If ω is a (left) monogenic $(m-1)$-form in $\Omega \setminus \Sigma$ and $\operatorname{Res}\omega \in H^q(\Sigma)$ is its Leray-Norguet residue, then the number

$$res_\Sigma \omega := \int_\Sigma \operatorname{Res}\omega$$

will be called the residue of the form ω at the submanifold Σ.

The residue $res_\Sigma\omega$ does not differ too much from the Leray-Norguet residue $\operatorname{Res}\omega$ itself; it is just the evaluation of the corresponding cohomology class on a generator of the top dimensional homology of Σ, represented by Σ itself. Because of the fact that the vector space $H^q(\Sigma)$ is one-dimensional, the information carried by both notions is the same.

Now, as in complex analysis, the importance of the residue is due to the fact that the result of an integration of ω over a boundary $\partial\Omega$ of a "big" domain Ω is computable in terms of a local information on the behaviour of ω near the singularities, encoded in the value of the residue. This is exactly the information offered in the Residue Theorem.

Theorem 5.1 (Residue Theorem) *Let us suppose that Ω is a domain in R^m and $\Omega' \subset\subset \Omega$ is a relatively compact subdomain with a smooth boundary (oriented by its outer normal).*

Furthermore, suppose that Σ_i, $i \in I$, is a finite family of pairwise disjoint compact submanifolds of Ω', the dimensions of which can vary from 0 up to $m-2$ and let $\omega = d\sigma\, f$, $f \in \mathcal{M}_l(\Omega \setminus (\cup_{i\in I}\Sigma_i))$.

Then

$$\int_{\partial\Omega'} \omega = \sum_{i\in I} res_{\Sigma_i}\omega.$$

Proof.

For every $i \in I$, we shall choose an (oriented) tubular neighborhood U_i of Σ_i in Ω' in such a way that their closures are pairwise disjoint and we shall consider their boundaries ∂U_i oriented by the outer normal (in every fiber). Then the boundary of the domain $\Omega'' = \Omega' \setminus (\cup_{i\in I}\bar{U}_i)$ (together with the orientation given by the outer normal) will split as a union of $\partial\Omega'$ (oriented by the outer normal) and all ∂U_i, $i \in I$ (with the opposite orientation to that described above).

By the Cauchy theorem applied to the domain Ω'', we get

$$\int_{\partial\Omega'} \omega = \sum_{i\in I} \int_{\partial U_i} \omega.$$

But the Leray theorem tell us that

$$\int_{\partial U_i} \omega = \int_{\Sigma_i} \text{Res } \omega = res_{\Sigma} \omega.$$

∎

5.2. Residues as functionals

Let us discuss now the more general case of differential forms of the type $\omega = f d\sigma g$ with f and g monogenic. In complex analysis, such a case is not studied because it is just a special case of the ordinary Residue Theorem (the product of holomorphic functions being again a holomorphic function). No analogue of that property is available in Clifford analysis, so it is natural to discuss the case of the form ω mentioned above. The forms of such a type are met in Clifford analysis quite often.

The Cauchy theorem tells us that ω is a closed form. So if ω has a singularity on a submanifold Σ, the corresponding residue

$$res\,\omega = \int_{\delta\Sigma} \omega = \int_{\Sigma} \text{Res } \omega$$

is well-defined and it carries an invariant topological information concerning the form ω.

A good way how to look upon it is to consider f to be a "test function". The knowledge of the residue for all f yields a comprehensive information on the behaviour of the monogenic form $d\sigma\, g$ near the singularity. It can be seen easily by analogy with the complex case that it amounts to have information about all terms in the Laurent expansion around the singularity (if anything like a generalized Laurent expansion is available).

This way of looking to residue was first introduced by F. Sommen in [13] in the discussion of monogenic differential forms. Following ideas described there, we come to the following defintion.

Definition 5.2 *Let Σ be a compact q-dimensional manifold in a domain $\Omega \subset R^m$, where $q = 0, \ldots, m-2$. Introducing the space of right monogenic functions on Σ as the direct limit*

$$\mathcal{M}_r(\Sigma) := \lim\text{ind}_{U \supset \Sigma} \mathcal{M}_r(U),$$

we shall suppose that $\omega = d\sigma\, g$ is an $(m-1)$-form with $g \in \mathcal{M}_l(\Omega \setminus \Sigma)$.

Then the functional $Rs_{\Sigma}\omega \in \mathcal{M}_r(\Sigma)'$ given by

$$Rs_{\Sigma}\omega[f] := \int_{\Sigma} \text{Res}_{\,\Sigma}(f\omega),\ f \in \mathcal{M}_r(\Sigma)$$

will be called the grand residue of ω at Σ.

The grand residue can be used to compute integrals of the form $f\omega$ over surfaces of dimension $m-1$ for any monogenic function f.

Theorem 5.2 *Let $\Omega' \subset\subset \Omega$ be a relatively compact subdomain with a smooth boundary (oriented by its outer normal) and let Σ_i, $i \in I$, be a finite family of pairwise disjoint* *compact submanifolds of Ω', the dimensions of which can vary from* 0 *to* $m-2$. *Suppose further that* $\omega = d\sigma\, g\ g \in \mathcal{M}_l(\Omega \setminus (\cup_{i\in I}\Sigma_i))$ *and that* $f \in \mathcal{M}_r(\Sigma)$. *Then*

$$\int_{\partial\Omega'} f\omega = \sum_{i\in I}(Rs_{\Sigma_i}\omega)[f].$$

The interpretation of the residue as a functional allows further generalization of the residue to the case, where the singularity set is a compact set K without a better structure (i.e., for example, without a structure of a manifold). In such a more general situation all information is kept in a functional from the space $\mathcal{M}'_r(K)$. For a certain type of monogenic q-forms, a study of such residue can be found in [13].

5.3. Residues for monogenic q-forms

The definition of the residue in the sense of functionals can be applied to monogenic differential forms of degrees other than $m-1$.

There are several different notions of monogenic q-forms in literature (see [12, 13, 14, 15, 16]). We shall not review these various definitions here, but we would like to explain the general scheme how to introduce the residue for such forms.

The basic idea is to find two complementary spaces $\mathcal{M}^1_r$ and $\mathcal{M}^2_l$ of monogenic forms together with a bilinear map $*$ mapping two forms $\tau \in \mathcal{M}^1_r$ and $\omega \in \mathcal{M}^2_l$ into a form $\tau * \omega$ of degree $m-1$ such that the Cauchy theorem holds, i.e. that the product $\tau * \omega$ is always closed.

Then again the residue $Rs_\Sigma\omega$ on a surface Σ can be defined for a monogenic form $\omega \in \mathcal{M}^2_l(\Omega \setminus \Sigma)$ as a functional on $\mathcal{M}^1_r(\Sigma)$ given by

$$Rs_\Sigma\omega[\tau] = \int_\Sigma \mathrm{Res}\,_\Sigma(\tau * \omega).$$

It is straightforward to formulate the corresponding Residue Theorem in such situation in the same way as above. We cannot, however, introduce the ordinary residue (without a test form) in a general case, because monogenic forms are not necessarily closed.

In [13], a pairing was defined between forms of degree $m-q$ and q; in [15] the pairing between forms of degree $m-q$ and $q-1$ was just the wedge product.

5.4. Pointwise singularities

The residue for $(m-1)$-forms $\omega = fd\sigma g$ with f and g monogenic was discussed for pointwise singularities in [17]. We would like to show how this case fits into the scheme described above.

This is just the case, when the dimension of the manifold Σ is zero. In such a case, the Leray-Norguet residue is just a number and an integral of it over a point is just equal to this number. The Leray cobord of a point p is represented by any sphere S containing p in its interior and the integral along fibers of an $(m-1)$-form ω is the number given by the integral $\int_S \omega$.

So if $\omega = f d\sigma g$, $f \in \mathcal{M}_r(\Omega \setminus \{p\})$, $g \in \mathcal{M}_l(\Omega \setminus \{p\})$, then (if $S \subset \Omega$ is small enough)

$$res_{\{p\}}\omega = \int_S \omega.$$

It was shown in [17] that if c_β, resp. d_β, are Taylor coefficients of f, resp. g, in the point p and if similarly $\tilde{c}_\beta$, resp. $\tilde{d}_\beta$ are Laurent coefficients of f, resp. g, in p, then

$$res_{\{p\}}\omega = \sum_\beta (\tilde{d}_\beta c_\beta + d_\beta \tilde{c}_\beta),$$

(in fact, this property was used there as a definition of the residue, here it is a theorem).

6. Generalized Taylor Series

Suppose now that R^m is split into the sum $R^m = R^p \oplus R^q$. In this section, we want to obtain a suitable generalization of the Taylor series for monogenic functions, namely a generalization such that $\vec{y} \in R^q$ is considered as a parameter and that we have a Taylor decomposition in the variable $\vec{x} \in R^p$. We shall use some basic facts on the space of spherical monogenics, for details see [3].

The first idea coming into mind is to use functions of the form $P_k(\vec{x})A(\vec{y})$ as analogues of Taylor polynomials, where P_k is an element of the space $M_{+,k}$ of all inner monogenics of degree k and $A(\vec{y})$ is analytic, and to express any monogenic function around R^q as a series in such polynomials. This does not work, however, since polynomials of the described type are usually not monogenic on R^m. Nevertheless, there is a possibility to correct such a function by adding a series of functions of the type $\vec{x}^{\,l} P_k(\vec{x})A_l(\vec{y})$, $l > 0$, in such a way that the resulting function is monogenic in R^m. As we shall see, there is exactly one way to do so. Note that all functions added to the original function $P_k(\vec{x})A(\vec{y})$ are also homogeneous polynomials in $\rho = |\vec{x}|$, but of higher order. As a consequence, the leading term of the sum "near" the axis R^q is the original function we started with. So such a sum will serve well as a good analogue of a Taylor polynomial in our situation. We shall show below that every monogenic function in a symmetric domain around R^q can be decomposed into functions of the described type and that these components are uniquely characterized by the corresponding "Taylor coefficients" $A(\vec{y})$, thus giving rise to a generalized Cauchy-Kowalewska extension property.

6.1. Generalized Cauchy-Kowalewska extension

Let us consider an SO(p)-invariant domain $\tilde{\Omega} \subset R^m$ and let us assume that the intersections of $\tilde{\Omega}$ with all subspaces parallel to R^p are convex (and hence, if nonvoid, they contain a point in R^q). Putting $\Omega = \tilde{\Omega} \cap R^q$, we have

Theorem 6.1. *Let $P_k \in M_{+,k}$ be an inner monogenic of degree k and let $A_0(\vec{y})$ be a (Clifford-valued) analytic function in Ω. Then there exists a unique sequence $\{A_l(\vec{y}))\}_{l>0}$ of analytic functions such that the series*

$$f(\vec{x},\vec{y}) := \sum_{l=0}^{\infty} \vec{x}^{\,l} P_k(\vec{x})A_l(\vec{y})$$

is convergent in a neighborhood $U \subset R^m$ of the domain Ω and its sum f is monogenic in U. The function $A_0(\vec{y})$ is determined by the relation

$$P_k(\vec{\xi})A_0(\vec{y}) = \lim_{\rho\to 0}\frac{1}{\rho^k}f(\vec{x},\vec{y}),\ \vec{x} = \rho\vec{\xi}.$$

Furthermore, the sum f is formally given by the expression

$$f(\vec{x},\vec{y}) = \Gamma\left(k+\frac{p}{2}\right)\left(\frac{\rho\sqrt{\Delta_{\vec{y}}}}{2}\right)^{-(k+\frac{p}{2})}$$

$$\left[\frac{\rho\sqrt{\Delta_{\vec{y}}}}{2}J_{k+\frac{p}{2}-1}\left(\rho\sqrt{\Delta_{\vec{y}}}\right)+\frac{\vec{x}\,\partial_{\vec{y}}}{2}J_{k+\frac{p}{2}}\left(\rho\sqrt{\Delta_{\vec{y}}}\right)\right](P_k(\vec{x})A_0(\vec{y})),$$

where $\sqrt{\Delta_{\vec{y}}}$ denotes the square root of the Laplacian $\Delta_{\vec{y}}$ (of which only even powers occur in the resulting series).

Proof.

Let us state first the following basic identities:

$$\partial_{\vec{y}}\,\vec{x}^{\,l} = (-1)^l\vec{x}^{\,l}\partial_{\vec{y}},$$

where $\partial_{\vec{y}}$ and $\vec{x}^{\,l}$ are both considered as operators on functions, and

$$\partial_{\vec{x}}(\vec{x}^{\,2l}P_k(\vec{x})) = -2l\vec{x}^{\,2l-1}P_k(\vec{x}),$$

$$\partial_{\vec{x}}(\vec{x}^{\,2l+1}P_k(\vec{x})) = -(2l+2k+p)\vec{x}^{\,2l}P_k(\vec{x}).$$

The last two formulae can be found in [11]; they express the fact that the space $M_{+,k}$ of inner monogenics of order k and its shifts of the form $\vec{x}^{\,l}\cdot M_{+,k}$ are characterized exactly as subspaces of Clifford-valued polynomials of a given order such that the operations $\partial_{\vec{x}}$ and multiplication by $\vec{x}$ are inverses of each other (modulo a suitable constant).

If we apply now the operator $\partial_{\vec{x}} + \partial_{\vec{y}}$ to the (formal) sum f, we get, using the formulae above:

$$\begin{aligned}(2l+2k+p)P_k(\vec{x})A_{2l+1}(\vec{y}) &= \partial_{\vec{y}}[P_k(\vec{x})A_{2l}(\vec{y})],\\ (2l+2)P_k(\vec{x})A_{2l+2}(\vec{y}) &= -\partial_{\vec{y}}[P_k(\vec{x})A_{2l+1}(\vec{y})].\end{aligned}$$

Hence the functions $A_l(\vec{y})$ are uniquely determined by the formulae

$$\begin{aligned}P_k(\vec{x})A_{2l}(\vec{y}) &= \frac{(-1)^l\partial_{\vec{y}}^{2l}[P_k(\vec{x})A_0(\vec{y})]}{2l(2l-2)\dots 2(2l+2k+p-2)\dots(2k+p)} =\\ &= \frac{(-1)^l\Gamma(k+\frac{p}{2})\partial_{\vec{y}}^{2l}[P_k(\vec{x})A_0(\vec{y})]}{2^{2l}\,l!\,\Gamma(l+k+\frac{p}{2})},\end{aligned}$$

and

$$\begin{aligned}P_k(\vec{x})A_{2l+1}(\vec{y}) &= \frac{(-1)^l\partial_{\vec{y}}^{2l+1}[P_k(\vec{x})A_0(\vec{y})]}{2l(2l-2)\dots 2(2l+2k+p)\dots(2k+p)} =\\ &= \frac{(-1)^l\Gamma(k+\frac{p}{2})\partial_{\vec{y}}^{2l+1}[P_k(\vec{x})A_0(\vec{y})]}{2^{2l+1}\,l!\,\Gamma(l+k+\frac{p}{2}+1)}.\end{aligned}$$

The last formula of the theorem follows from the expansion (for an indeterminate variable α)

$$\sum_{l=0}^{\infty} \frac{\alpha^{2l}}{2^{2l}\, l!\, \Gamma(l+k+\frac{p}{2})} = \Gamma\left(\frac{p}{2}\right)\left(\frac{\alpha}{2}\right)^{-k-\frac{p}{2}+1} J_{k+\frac{p}{2}-1}(\alpha).$$

Finally, we have to show that the series converges in a neighborhood of Ω in R^m. To this end, it is sufficient first to note that a function $f \in C_\infty(\Omega)$ is real analytic iff for every compact set $K \subset \Omega$ there exists $R_f(K) > 0$ such that the series

$$\sum_{n=0}^{\infty} \frac{M_n(f,K)}{(2n)!} z^n, \; M_n(f,K) = \sup_{\vec{y}\in K} |\Delta_{\vec{y}}^n f(\vec{y})|,$$

has the radius of convergence equal to $R_f(K)$ (see [1]) and then to use Stirlings inequality to prove that, up to factors of slow growth in k, we have

$$2^{2l}\, l!\, \Gamma(l+k+\frac{p}{2}) \approx 2^{2l}\,(l!)^2 \approx \frac{(2l)^{2l}}{e^{2l}} \approx (2l)!.$$

∎

Definition 6.1 *Let $k \in N$ be fixed and let Ω be a domain in R^q. Then $\mathcal{T}_k(\Omega)$ stands for the space of all functions of the form*

$$f = \sum_{l=0}^{\infty} \vec{x}^{\,l} A_l(\vec{x}, \vec{y})$$

such that

1. *for any $\vec{y}$ fixed, $A_l(\vec{x}, \vec{y}), \vec{x} = \rho\vec{\xi}$, belongs to the space $M_{+,k}$ of inner monogenics*
2. *the sum converges in a neighborhood of Ω and f is monogenic there.*

Functions belonging to $\mathcal{T}_k(\Omega)$ will be called generalized Taylor functions of order k.

The spaces $\mathcal{T}_k(\Omega)$ are direct generalizations of the one dimensional spaces

$$\{a_k z^k | a_k \in C\}, k \geq 0,$$

generated by z^k in complex analysis. We shall show now that every element of $\mathcal{T}_k(\Omega)$ can be characterized by an $M_{+,k}$-valued function of $\vec{y}$ which is playing the role of a Taylor coefficient.

Theorem 6.2

1. *For every $f \in \mathcal{T}_k(\Omega)$ and every $\vec{y} \in \Omega$, the limit*

$$\lim_{\rho\to 0} \frac{1}{\rho^k} f(\rho\vec{\xi}, \vec{y})$$

exists. Moreover, putting

$$T_k(f)(r\vec{\xi},\vec{y}) := r^k \cdot \lim_{\rho\to 0} \frac{1}{\rho^k} f(\rho\vec{\xi},\vec{y})$$

we have that for every $\vec{y} \in \Omega$ fixed, $T_k(f)(\vec{x},\vec{y})$ is inner monogenic, whence $T_k(f)$ can be considered to be an $M_{+,k}$-valued analytic function on Ω.

2. *For every $M_{+,k}$-valued analytic function A on Ω, there exists a unique element $f \in \mathcal{T}_k$ such that*

$$T_k(f) = A.$$

Proof. Suppose first that $f \in \mathcal{T}_k(\Omega)$ is given by

$$f = \sum_{l=0}^{\infty} \vec{x}^{\,l} A_l(\vec{x},\vec{y}),$$

where $A_l(\vec{x},\vec{y})$ are $M_{+,k}$-valued functions on Ω. Then, because of homogeneity in ρ,

$$\lim_{\rho\to 0} \frac{1}{\rho^k} f(\rho\vec{\xi},\vec{y}) = A_0(\vec{\xi},\vec{y}),$$

whence $T_k(f) = A_0$.

Suppose next that an $M_{+,k}$-valued function $A(\vec{x},\vec{y})$ is given. If $\{P_{k,\alpha}\}_{\alpha\in B_k}$ is any basis for the finite dimensional space $M_{+,k}$, then $A(\vec{x},\vec{y})$ can be uniquely written in the form

$$A(\vec{x},\vec{y}) = \sum_{B_k} P_{k,\alpha}(\vec{\xi})t_{k,\alpha}(\vec{y}),$$

where $t_{k,\alpha}(\vec{y})$ are analytic functions on Ω. If $f_{k,\alpha}(\vec{x},\vec{y})$ are the monogenic functions corresponding to $t_{k,\alpha}(\vec{y})$ by Theorem 6.1, then their sum $f = \sum_{B_k} f_{k,\alpha}$ clearly belongs to $\mathcal{T}_k(\Omega)$ and $T_k(f) = A$. The unicity follows from the fact that if

$$f = \sum_{l=0}^{\infty} \vec{x}^{\,l} A_l(\vec{x},\vec{y}) \in \mathcal{T}_k(\Omega)$$

with $A_0 \equiv 0$, then the condition of monogenicity implies immediately (by induction) that all other functions $A_l, l > 0$ are trivial. ∎

Definition 6.2 *For any $f \in \mathcal{T}_k(\Omega)$ given, the associated function*

$$T_k(f)(r\vec{\xi},\vec{y}) := r^k \cdot \lim_{\rho\to 0} \frac{1}{\rho^k} f(\rho\vec{\xi},\vec{y})$$

is henceforth called the generalized Taylor coefficient of f of order k.

6.2. Generalized Taylor series

Let us recall (see [7]) that any function on the sphere S^{p-1} admits an expansion

$$f(\vec{\xi}) = \sum_{k=0}^{\infty} \Pi_k f(\vec{\xi}),$$

where

$$\Pi_k f(\vec{\xi}) = P_k f(\vec{\xi}) - \vec{\xi} P_k(\vec{\xi} f)(\vec{\xi}),$$

P_k being the projection onto the space $\mathcal{M}_{+,k}$ of inner spherical monogenics. The series converges uniformly.

For any function $f(\vec{x}, \vec{y})$ on $\tilde{\Omega}$ and for any fixed pair $(\rho, \vec{y})$ such that $(\vec{x}, \vec{y}) \in \tilde{\Omega}$, $\vec{x} = \rho\vec{\xi}$, we can thus decompose f as a function of $\vec{\xi}$ by the procedure above.

Definition 6.3 *The projection operator Π_k will be extended to the space of smooth Clifford-valued functions on $\tilde{\Omega}$ by the formula*

$$[\Pi_k f](\rho\vec{\xi}, \vec{y}) = \Pi_k[f(\rho, \vec{\xi}, \vec{y})],$$

(Π_k being applied on the right hand side for $(\rho, \vec{y})$ fixed).

The projections Π_k commute with $\partial_{\vec{x}} + \partial_{\vec{y}}$, (see [7]), hence if f is monogenic in $\tilde{\Omega}$, then the functions $\Pi_k f$ are monogenic.

The image of the space $M_l(\tilde{\Omega})$ of (left) monogenic functions under the map Π_k will be denoted by $\mathcal{T}_{l,k}(\tilde{\Omega})$.

Theorem 6.3 (Generalized Taylor theorem)

1. *Suppose that f is a monogenic function in $\tilde{\Omega}$. Then f can be written in a unique way as*

$$f = \sum_{k=0}^{\infty} f_k, f_k \in \mathcal{T}_k(\tilde{\Omega}),$$

the series converges uniformly on compact sets in $\tilde{\Omega}$. The functions f_k are given by the formula

$$f_k = P_k f(\rho, \vec{\xi}, \vec{y}) - \vec{\xi} P_k(\vec{\xi} f)(\rho, \vec{\xi}, \vec{y}).$$

2. *The functions f_k are generalized Taylor coefficients of order k, i.e. $f_k \in \mathcal{T}_{l,k}(\Omega)$ and the generalized Taylor coefficients of f_k are given by given by*

$$T_k(f_k)(r\vec{\xi}, \vec{y}) := r^k \cdot \lim_{\rho \to 0} \frac{1}{\rho^k} P_k f(\rho, \vec{\xi}, \vec{y}).$$

The function f is uniquely determined by its set of generalized Taylor coefficients.

Proof.

The first part of the theorem follows immediately from the properties of the projection operator Π_k. The uniform convergence on compact sets was proved in [7].

As for the second part, the spaces $\mathcal{M}_{+,k}$ are finite dimensional, hence any function $\Pi_k f$ can be written as a (finite) sum of functions of the form

$$P_k(\vec{\xi})A(\rho,\vec{y}) + \vec{\xi}\,P_k(\vec{\xi})B(\rho,\vec{y}), \tag{7}$$

where $P_k \in \mathcal{M}_{+,k}$ and A, B are analytic. But a function of the form (7) is analytic in points belonging to Ω iff it can be written in the form

$$P_k(\vec{x})a(\rho^2,\vec{y}) + \vec{x}\,P_k(\vec{x})b(\rho^2,\vec{y}),$$

where a and b are (for fixed $\vec{y}$) analytic near $\rho^2 = 0$ and $P_k(\vec{x}) = \rho^k P_k(\vec{\xi}) \in M_{+,k}$. This implies that such functions belong to $\mathcal{T}_k(\Omega)$.

The uniqueness of the Taylor series follows from the fact that the spaces $\mathcal{T}_k(\Omega)$ are linearly independent. In the limit, only the part $P_k(\vec{x})a(\rho^2,\vec{y})$ survives, which proves the rest of the theorem. ∎

Definition 6.4 *Let $f = \sum_k f_k$, $f_k \in \mathcal{T}_k(\Omega)$, be monogenic in $\tilde{\Omega}$ and let $T_k(f_k)$ be the generalized Taylor coefficient of f_k. Then $T_k(f_k)$ is called the generalized Taylor coefficient of f of order k and will be denoted by $T_k(f)$.*

If a basis $\{P_{k,\alpha}\}_{\alpha\in B_k}$ for the space $M_{+,k}$ is given, then the generalized Taylor theorem can be written in a form which is closer to the standard Taylor theorem, the variable $\vec{y}$ playing the role of a parameter.

Theorem 6.4 *Let f be a monogenic function in $\tilde{\Omega}$. Then the generalized Taylor coefficient of f of order k can be decomposed in a unique way as*

$$T_k(f)(\vec{\xi},\vec{y}) = \sum_{\alpha\in B_k} P_{k,\alpha}(\vec{\xi})T_{k,\alpha}(f)(\vec{y}),$$

where $T_{k,\alpha}(f)(\vec{y})$ are real analytic functions on Ω.

If the generalized Taylor functions, corresponding to $P_{k,\alpha}(\vec{\xi})T_{k,\alpha}(f)(\vec{y})$, are denoted by $T_{k,\alpha}$, then

$$f = \sum_k \sum_{\alpha\in B_k} T_{k,\alpha}$$

in a neighborhood of Ω.

7. Generalized Laurent Series

Knowing what the generalized Taylor series for a monogenic function is, we now wish to define generalized Laurent series. The basic idea behind it is to use the duality between left and right monogenic functions g and f given by

$$\int_{\partial\Omega} f\,d\sigma\,g.$$

Moreover, the notion of a generalized Laurent coefficient is introduced which appears to be an $M_{r,+,k}$-valued analytic functional (the subscript r referring to right monogenicity). It will be convenient to identify below the spaces $M_{r,+,k}$ and $\mathcal{M}_{r,+,k}$ (the isomorphism being the restriction to the unit sphere).

7.1. Taylor and Laurent part of the series

Before proceeding further, let us first show how any monogenic function in $\tilde{\Omega} \setminus K$, $K \subset R^q$ being compact, can be written as a sum of its Taylor and Laurent parts.

In what follows we are assuming that $R^m = R^p \times R^q$, $\tilde{\Omega} \subset R^m$ is an SO(p)-invariant domain such that the intersections of $\tilde{\Omega}$ with all subspaces parallel to R^p are convex, $\Omega = \tilde{\Omega} \cap R^q$ and $K \subset \Omega$ is compact. Calling $M_{l,0}(R^m \setminus K)$ the space of left monogenic functions in $R^m \setminus K$ which vanish at infinity, we have

Theorem 7.1

$$M_l(\tilde{\Omega} \setminus K) = M_{l,0}(R^m \setminus K) \oplus M_l(\tilde{\Omega}),$$

i.e. each $f \in M_l(\tilde{\Omega} \setminus K)$ *admits a unique decomposition of the form* $f = Tf + Lf$, *where* $Tf \in M_{l,0}(R^m \setminus K)$ *and* $Lf \in M_l(\tilde{\Omega})$.

Proof. Suppose first that $\tilde{\Omega}$ is a bounded set. We can exhaust $\tilde{\Omega}$ by a sequence $\tilde{\Omega}_n$ of open domains with a piecewise smooth boundary such that $\tilde{\Omega}_n \subset\subset \tilde{\Omega}_{n+1} \subset \tilde{\Omega}$. Indeed, taking a suitable sequence $\epsilon_n > 0, \epsilon \downarrow 0$, we choose for every n a finite cover of the compact set $\partial\tilde{\Omega}$ by balls $B(p_i, \epsilon_n), i \in A$, of radius ϵ_n and we define a neighborhood of $\partial\tilde{\Omega}$ by

$$U_n = \bigcup_{i \in A} B(p_i, \epsilon_n).$$

Analogously, we may define a neighborhood of K by

$$U'_n = \bigcup_{i \in A'} B(p'_i, \epsilon_n).$$

Then we shall define

$$Tf := \int_{\partial U'_n} E(\vec{x} - \vec{y}) d\sigma_{\vec{y}} f$$

on $R^m \setminus U'_n$ and

$$Lf = \int_{\partial U_n \cap \tilde{\Omega}} E(\vec{x} - \vec{y}) d\sigma_{\vec{y}} f$$

on $\tilde{\Omega} \setminus U_n$. The first integral defines, in fact, a sequence of monogenic functions, vanishing at ∞, which coincide on the intersection of domains of definition, so it gives a monogenic function on $R^m \setminus K$, vanishing at ∞. The function Lf is defined, in the same way, on $\tilde{\Omega}$. The fact that $f = Tf + Lf$ follows then from the Cauchy theorem.

In the case that $\tilde{\Omega}$ is not bounded, we shall exhaust it first by an increasing sequence of bounded domains and then we shall apply the above procedure. ∎

7.2. Generalized Laurent series

The generalized Laurent series will be defined now by using the duality between the space $M_{l,0}(R^m \setminus K)$ and the space $M_r(K)$ given by

$$\int_{\partial U} f\, d\sigma\, g,$$

where U is a suitable neighborhood of K.

The spaces $M_r(K)$ are defined as the inductive limits

$$M_r(K) := \lim\mathrm{ind}_{U \supset K} M_r(U).$$

The standard duality theory tells us (see [3]) that we have the following

Theorem 7.2 *Let $K \subset R^m$ be compact. Then*

$$M_{l,0}(R^m \setminus K) \simeq M_r(K)'$$

as topological vector spaces (the topology of uniform convergence on compact subsets is taken on $M_{l,0}(R^m \setminus K)$ and the strong topology is taken on $M_r(K)'$). The corresponding isomorphism is given by the Cauchy transform

$$\hat{T}(\vec{x}\,) = \frac{-1}{A_m}\langle T_{\vec{u}}\,, \frac{\vec{x}\, - \vec{u}}{|\vec{x}\, - \vec{u}\,|^m}\rangle,\ T \in M_r(K)', \vec{x}\, \in R^m \setminus K,$$

where A_m is the area of the $(m-1)$-dimensional unit sphere.

The map $B\, :\, M_{l,0}(R^m \setminus K) \to M_r(K)'$ inverse to the Cauchy transform can be described as follows:

$$B(f)[g] = \int_{\partial U} g\, d\sigma\, f,\ f \in M_{l,0}(R^m \setminus K),\ g \in M_r(K),$$

where U is a suitable neighborhood of K. The notation T_f instead of $B(f)$ is often used.

The spaces $M_{l,0}(R^m \setminus K)$ and $M_r(K)$ thus form a so called dual pair of topological linear spaces (see [8]), which means by definition that the duality is nondegenerate in both variables.

The splitting of monogenic functions on open domains into generalized Taylor functions was described in the last section. By duality, it leads to a splitting of $M_{l,0}(R^m \setminus K)$.

Let us denote by $\mathcal{T}_{r,k}(K)$ the inductive limit of the spaces $\mathcal{T}_{r,k}(U), U \supset K$ (the subscript r referring to right monogenic functions). Then Theorem 6.3 implies that

$$M_r(K) = \bigoplus_0^\infty \mathcal{T}_{r,k}(K). \tag{8}$$

The spaces $\mathcal{T}_{r,k}(K)$ are closed subspaces, hence the dual splitting is induced on the dual space:

$$M_r(K)' = \bigoplus_0^\infty \mathcal{T}'_{r,k}(K), \tag{9}$$

where the spaces $\mathcal{T}'_{r,k}(K)$ are orthogonal to $\oplus_{k'\neq k}\mathcal{T}_{r,k'}(K)$. More precisely, the direct sum in (9) means that each element of $M_r(K)'$ can be written in a unique way as the sum of elements in $\mathcal{T}'_{r,k}(K)$, where the sum is taken in the weak topology. The Cauchy transform then translates the splitting to the space $\mathcal{L}_{l,k}(R^m \setminus K)$. It leads to the following definition.

Definition 7.1 *The space $\mathcal{L}_{l,k}(R^m \setminus K)$ of (left) generalized Laurent functions of order k on $R^m \setminus K$ is defined by*

$$\mathcal{L}_{l,k}(R^m \setminus K) := \{f \in M_{l,0}(R^m \setminus K) | B(f)[g] = 0 \text{ for all } g \in \mathcal{T}_{l,k'}, k' \neq k\}.$$

As an immediate consequence of the definition we get

Theorem 7.3 (Generalized Laurent theorem) *We have*

$$M_{l,0}(R^m \setminus K) = \bigoplus_{0}^{\infty} \mathcal{L}_{l,k}(R^m \setminus K)$$

and

$$M_l(\tilde{\Omega} \setminus K) = \left[\bigoplus_{0}^{\infty} \mathcal{T}_{l,k}(\tilde{\Omega})\right] \oplus \left[\bigoplus_{0}^{\infty} \mathcal{L}_{l,k}(R^m \setminus K)\right],$$

both series being uniformly convergent on compact subsets.

Proof. The Cauchy transform gives (topological) isomorphism

$$\mathcal{L}_{l,k}(R^m \setminus K) \simeq \mathcal{T}'_{r,k}(K).$$

The assertion follows from the fact that the space $M'_r(K)$ is a Frechet-Montel space ([8]), whence the weak and strong sequential convergence coincide. ∎

7.3. Generalized Laurent coefficients

We have seen that the generalized Taylor functions can be characterized by their generalized Taylor coefficients. We would like to do the same with generalized Laurent functions.

Let us denote by $\mathcal{A}(U, M_{r,+,k})$ the space of all $M_{r,+,k}$-valued analytic functions on a domain U and by $\mathcal{A}(K, M_{r,+,k})$ their inductive limit over open neighborhoods of K, similarly for $\mathcal{T}_{r,k}(K)$. The Theorem 6.2 implies then that the spaces $\mathcal{A}(K, M_{r,+,k})$ and $\mathcal{T}_{r,k}(K)$ are isomorphic. By duality, we then have the following diagram:

$$\begin{array}{ccc}
\mathcal{T}_{r,k}(K) & \xrightarrow{T_k} & \mathcal{A}(K, M_{r,+,k}) \\
\updownarrow & & \updownarrow \\
\mathcal{T}'_{r,k}(K) & \xleftarrow{T'_k} & \mathcal{A}'(K, M_{r,+,k}) \\
\uparrow B & \nearrow L_k & \\
\mathcal{L}_{l,k}(R^m \setminus K) & &
\end{array}$$

Here B denotes the inverse to the Cauchy transform. The map T_k is an isomorphism, so the same is true for T'_k. The map L_k is then defined as the composition of $(T_k^{-1})'$ and B, hence it follows immediately that it is an isomorphism as well.

Definition 7.2 *For each $f \in \mathcal{L}_{l,k}(R^m \setminus K)$, the functional*

$$L_k(f) \in \mathcal{A}'(K, M_{r,+,k})$$

is called the generalized Laurent coefficient of f of order k.

Note that the function $f \in \mathcal{L}_{l,k}(R^m \setminus K)$ is uniquely determined by its Laurent coefficients $L_k(f)$. It is desirable to have a possibility to compute the Laurent coefficients of functions effectively.

Let us recall that the generalized Taylor coefficients of a generalized Taylor function f of order k were defined as the limit

$$T_k(f)(r\vec{\xi}, \vec{y}) = r^k \cdot \lim_{\rho \to 0} \frac{1}{\rho^k} P_k f(\rho\vec{\xi}, \vec{y}),$$

the limit being pointwise.

We would like to have a similar formula for the computation of the generalized Laurent coefficients. Pointwise limits similar to that described above cannot be used anymore and should be substituted by a limit taken in a suitable topology on the corresponding function space. It is necessary first to describe how the space $\mathcal{A}(\overline{U}, \mathcal{M}_{l,+,k})$ of $\mathcal{M}_{l,+,k}$-valued functions on $\overline{U}, U \subset R^m$ open is imbedded into the space $\mathcal{A}(\overline{U}, \mathcal{M}_{r,+,k})'$. The (finite dimensional) spaces $\mathcal{M}_{l,+,k}$ and $\mathcal{M}_{r,+,k}$ are dual to each other, the corresponding duality being

$$\langle f, g \rangle = \int_{S^{p-1}} f(\vec{\xi}) g(\vec{\xi}) dS, \; f \in \mathcal{M}_{r,+,k}, \; g \in \mathcal{M}_{l,+,k},$$

where dS stands for the standard surface element on the sphere.

For $f \in \mathcal{A}(\overline{U}, \mathcal{M}_{r,+,k})$, $g \in \mathcal{A}(\overline{U}, \mathcal{M}_{l,+,k})$, the duality is given by

$$\langle f, g \rangle_U = \int_U \langle f(\vec{y}), g(\vec{y}) \rangle d\vec{y}.$$

The spaces $M_{r,+,k}$ and $\mathcal{M}_{r,+,k}$ are usually identified (the isomorphism being the restriction to the unit sphere).

Let us study now in more details the action of the Laurent coefficient $L_k(f)$ of a function $f \in \mathcal{L}_{l,k}(R^m \setminus K)$ on an element $\varphi \in \mathcal{A}(K, \mathcal{M}_{r,+,k})$.

By definition,

$$\langle L_k(f), \varphi \rangle = \int_{\partial V} \phi \, d\sigma \, f, \tag{10}$$

where V is a suitable neighborhood of K in R^m and $T_k(\phi) = \varphi, \phi \in \mathcal{T}_{r,k}(K)$.

It is possible to choose V in the form $V = B_\rho \times U$, where B_ρ is a ball in R^p of radius ρ and U is a neighborhood of K in R^q. We can suppose that $V \subset\subset \tilde{\Omega}$ and that ϕ is defined on a neighborhood of $\overline{V}$. Denoting the restriction of ϕ to $\partial B_\rho \times U$ by $\phi_\rho(\vec{\xi}, \vec{y})$, we know that we can write it as a sum

$$\phi_\rho = \phi_\rho^1 - \phi_\rho^2 \vec{\xi},$$

where ϕ_ρ^i are in $\mathcal{A}(U, \mathcal{M}_{r,+,k})$ and that $T_k(\phi) = \varphi$ is given by

$$T_k(\phi)(\vec{\xi}, \vec{y}) = \lim_{\rho \to 0} \frac{1}{\rho^k} \phi_\rho^1(\vec{\xi}, \vec{y}). \tag{11}$$

The restriction f_ρ of the function f to $\partial B_\rho \times U$ can be decomposed in the same way into the sum

$$f_\rho = f_\rho^1 - \vec{\xi} f_\rho^2,$$

where f_ρ^i are in $\mathcal{A}(U, \mathcal{M}_{l,+,k})$. Moreover,

$$\lim_{\rho \to 0} \frac{1}{\rho^k} \phi_\rho^2(\vec{\xi}, \vec{y}) = 0. \tag{12}$$

Under the duality (10), only two of the four pieces survive and because of (12), only the piece

$$\int_{\partial B_\rho \times U} \phi_\rho^1 \, d\sigma \, \vec{\xi} \, f_\rho^2$$

is expected to be important for us.

In analogy with the case of generalized Taylor functions, an expression for the generalized Laurent coefficient $L_k(f)$ of a monogenic function f can be guessed:

$$L_k(f)(\vec{\xi}, \vec{y}) = \lim_{\rho \to 0} \rho^{k+p-1} f_\rho^2(\vec{\xi}, \vec{y}), \; f_\rho^2(\vec{\xi}, \vec{y}) = P_k(\vec{\xi} f_\rho)(\vec{\xi}, \vec{y}).$$

It is far from being clear, however, if such a limit exists and in what sense it should be taken. We are going to show now that for every neighborhood U of K in Ω the limit exists in the weak topology of the space $\mathcal{A}'(U, \mathcal{M}_{r,+,k})$ and that the limit is equal to the corresponding Laurent coefficient. The proof of this fact will be done in several steps.

Theorem 7.4 Let us suppose that $f \in \mathcal{L}_{l,k}(R^m \setminus K)$ and that $f_\rho = f_\rho^1 - \vec{\xi} f_\rho^2$, where f_ρ^i are in $\mathcal{A}(R^q, \mathcal{M}_{l,+,k})$.

If the limits

$$f_0^i = \lim_{\rho \to 0} \rho^{k+p-1} f_\rho^i, \; i = 1, 2$$

exist in the weak topology of the space $\mathcal{A}'(\overline{U}, \mathcal{M}_{r,+,k})$ for every open neighborhood U of K in R^q, then the limit f_0^2 does not depend on U and

$$L_k(f)(\vec{\xi}, \vec{y}) = f_0^2(\vec{\xi}, \vec{y}).$$

Proof.

Let $f_0^2|_U$ denote the limit of $\rho^{k+p-1} f_\rho^2$ in the weak topology of the space $\mathcal{A}'(\overline{U}, \mathcal{M}_{r,+,k})$. It is sufficient to show that $\langle L_k(f), \varphi \rangle = \langle f_0^2|_U, \varphi \rangle_U$ for all neighborhoods U of K and for all $\varphi \in \mathcal{A}(\overline{U}, \mathcal{M}_{r,+,k})$.

So let us consider a neighborhood U and a fixed element $\varphi \in \mathcal{A}(\overline{U}, \mathcal{M}_{r,+,k})$. Let us define again $\phi = T_k^{-1}(\varphi) \in \mathcal{T}_{r,k}(\overline{U})$. Then the open sets

$$V_n := B(0, 1/n) \times U$$

contain K and the element $\langle L_k(f), \varphi\rangle$ can be computed by

$$\langle L_k(f), \varphi\rangle = \int_{\partial V_n} \phi \, d\sigma \, f$$

for all $n \geq n_0$, n_o big enough.

Recalling that the restrictions of f and ϕ to the boundary of V_n define elements $f^i_{1/n}$ and $\phi^i_{1/n}$ in $\mathcal{A}(\overline{U}, \mathcal{M}_{l,+,k})$ and $\mathcal{A}(\overline{U}, \mathcal{M}_{r,+,k})$, respectively, and that $d\sigma = \rho^{p-1}\vec{\xi}\, dSd\vec{y}$ on $S_{1/n} \times U$, we get

$$\begin{aligned}\langle L_k(f), \varphi\rangle &= \lim_{n\to\infty} \int_U \rho^{p-1}\langle \phi^1_{1/n}(\vec{y}), f^2_{1/n}(\vec{y})\rangle d\vec{y} + \\ &\quad + \lim_{n\to\infty} \int_U \rho^{p-1}\langle \phi^2_{1/n}(\vec{y}), f^1_{1/n}(\vec{y})\rangle d\vec{y}\,.\end{aligned}$$

But $\rho^{k+p-1} f^i_{1/n} \to f^i_0|_U$ and $\rho^{-k}\phi^1_{1/n} \to \varphi$, $\rho^{-k}\phi^2_{1/n} \to 0$, whence

$$\langle L_k(f), \varphi\rangle = \lim_{n\to\infty} \langle \rho^{-k}\phi^1_{1/n}, \rho^{k+p-1} f^2_{1/n}\rangle_U = \langle f^2_0|_U, \varphi\rangle.$$

■

Using the theorem just proved, we shall be able to compute some important examples of generalized Laurent coefficients.

Examples.

1. Let us consider first the Cauchy kernel

$$E(\vec{x} + \vec{y}) = -\frac{1}{A_m}\frac{\vec{x} + \vec{y}}{|\vec{x} + \vec{y}|^m}.$$

It is easy to see that E belongs to $\mathcal{L}_{l,0}(R^m \setminus \{0\})$ and its parts E^i in the decomposition $E = E^1 - \vec{\xi} E^2$ are given by

$$E^1 = -\frac{1}{A_m}\frac{\vec{y}}{[\rho^2 + |\vec{y}|^2]^{m/2}}, \; E^2 = \frac{1}{A_m}\frac{\rho}{[\rho^2 + |\vec{y}|^2]^{m/2}}.$$

We shall show that for every ball $U \subset R^q$ with the center in the origin the limit

$$\lim_{\rho\to 0} \rho^{p-1}(E^1_\rho - \vec{\xi} E^2_\rho) = \delta(\vec{y})$$

exists in the weak topology on the space $\mathcal{A}'(\overline{U}, \mathcal{M}_{l,+,k})$, whence

$$L_0(E) = \delta(\vec{y}).$$

Indeed, for each φ analytic in a neighborhood of $\overline{U}$ we have

$$\begin{aligned}
\lim_{\rho\to 0}\rho^{p-1}\langle(E^1_\rho-\vec{\xi}\,E^2_\rho),\varphi\rangle_U &= \lim_{\rho\to 0}\int_U\frac{\rho^{p-1}}{A_m}\int_{S^{p-1}}\frac{\varphi(\vec{y})(\rho-\vec{y})}{[\rho^2+|\vec{y}|^2]^{m/2}}dSd\vec{y}\\
&= \lim_{\rho\to 0}\int_U\frac{A_p\rho^{p-1}(\rho-\vec{y})}{A_m[\rho^2+|\vec{y}|^2]^{m/2}}\varphi(\vec{y})d\vec{y}\\
&= \lim_{\rho\to 0}\int_{\frac{1}{\rho}\cdot U}\frac{A_p\varphi(\rho\vec{u})(1-\vec{u})}{A_m[1+|\vec{u}|^2]^{m/2}}d\vec{u}\\
&= \frac{A_p}{A_m}\int_{R^q}\frac{\varphi(0)}{[1+|\vec{u}|^2]^{m/2}}d\vec{u}\\
&= \frac{A_pA_q}{A_m}\int_0^\infty\frac{s^{q-1}}{(1+s^2)^{m/2}}ds\cdot\varphi(0)\\
&= \varphi(0).
\end{aligned}$$

2. If $F\in\mathcal{A}'(K)$ is an analytic functional on a compact subset K in R^q, then we can define its Cauchy transform $C(F)$ by the action of the functional F on E

$$C(F)(\vec{x}+\vec{y})=F_{\vec{v}}[E(\vec{x}+\vec{y}-\vec{v})],\ \vec{x}+\vec{y}\in R^m\setminus K.$$

The Cauchy transform $C(F)$ is a (left) monogenic function in the domain $R^m\setminus K$. Moreover, $L_0(C(F))=F$, i.e. the Cauchy transform is the inverse of L_0.

Indeed, for each φ analytic in a neighborhood of $\overline{U}$ we have

$$\begin{aligned}
&\lim_{\rho\to 0}\rho^{p-1}\int_U F_{\vec{v}}[E_\rho(\rho\vec{\xi}+\vec{y}-\vec{v})]\varphi(\vec{y})d\vec{y}=\\
&= \lim_{\rho\to 0}F_{\vec{v}}[\rho^{p-1}\int_U E_\rho(\rho\vec{\xi}+\vec{y}-\vec{v})]\varphi(\vec{y})d\vec{y}]\\
&= \lim_{\rho\to 0}F_{\vec{v}}[\int_{\frac{1}{\rho}\cdot U}\frac{A_p}{A_m}\frac{\varphi(\rho\vec{u}+\vec{v})(1-\vec{u})}{[1+|\vec{u}|^2]^{m/2}}d\vec{u}]\\
&= F_{\vec{v}}[\varphi(\vec{v})].
\end{aligned}$$

3. Let $P_k(\vec{x})$ be a (left) inner spherical monogenic of degree k in R^p. Then it is an inner spherical monogenic of degree k in R^m as well and its inverse in R^m

$$\begin{aligned}
K_k(\vec{x},\vec{y}) &:= \frac{-1}{A_m}\frac{\vec{x}+\vec{y}}{|\vec{x}+\vec{y}|^m}P_k\left(\frac{\vec{x}}{|\vec{x}+\vec{y}|^2}\right)\\
&= \frac{-1}{A_m}\frac{\vec{x}+\vec{y}}{|\vec{x}+\vec{y}|^{2k+m}}P_k(\vec{x})
\end{aligned}$$

is monogenic in $R^m\setminus\{0\}$ and vanishes at infinity. It is clear that it is a generalized Laurent function of degree k and its two parts K_k^i are given by

$$K^1_{k,\rho}=-\frac{1}{A_m}\frac{\rho^k\vec{y}\,P_k(\vec{\xi})}{[\rho^2+|\vec{y}|^2]^{k+m/2}},\ K^2_{k,\rho}=\frac{1}{A_m}\frac{\rho^{k+1}P_k(\vec{\xi})}{[\rho^2+|\vec{y}|^2]^{k+m/2}}.$$

We shall compute the Laurent coefficient $L_k(K_k)$. Let us consider again a test function $\varphi \in \mathcal{A}(\overline{U}, \mathcal{M}_{r,+,k})$, then

$$\begin{aligned}
&\lim_{\rho\to 0} \rho^{k+p-1} \langle K_{k,\rho}, \varphi\rangle_U \\
&= \lim_{\rho\to 0} \int_U \frac{\rho^{2k+p-1}}{A_m} \int_{S^{p-1}} \frac{\varphi(\vec{\xi},\vec{y})P_k(\vec{\xi})(\rho-\vec{y})}{[\rho^2+|\vec{y}|^2]^{k+m/2}} dS d\vec{y} \\
&= \lim_{\rho\to 0} \int_{\frac{1}{\rho}\cdot U} \int_{S^{p-1}} \frac{\varphi(\vec{\xi},\rho\vec{u})P_k(\vec{\xi})(1-\vec{u})}{A_m[1+|\vec{u}|^2]^{k+m/2}} dS d\vec{u} \\
&= \frac{A_q}{A_m} \int_0^\infty \frac{s^{q-1}}{(1+s^2)^{k+m/2}} ds \int_{S^{p-1}} \varphi(\vec{\xi},0)P_k(\vec{\xi})dS \\
&= \frac{\Gamma(k+\frac{p}{2})\Gamma(\frac{m}{2})}{\Gamma(\frac{p}{2})\Gamma(k+\frac{m}{2})} \frac{1}{A_p} \int_{S^{p-1}} \varphi(\vec{\xi},0)P_k(\vec{\xi})dS.
\end{aligned}$$

The final answer is hence

$$L_k(K_k) = \frac{\Gamma(k+\frac{p}{2})\Gamma(\frac{m}{2})}{\Gamma(\frac{p}{2})\Gamma(k+\frac{m}{2})} \delta(\vec{y})P_k(\vec{\xi}).$$

The computation in the last example leads us to the following definition.

Definition 7.3 *Let K be a compact subset of R^q and let $F \in \mathcal{A}'(K)$. Let us choose a (left) monogenic $P_k(\vec{x})$ of degree k in R^q.*

Then the Cauchy transform $C_k(F)$ of order k of F is defined by the convolution

$$C_k(F)(\vec{x}+\vec{y}) = F_{\vec{v}}[K_k(\vec{x}+\vec{y}-\vec{v})],\ \vec{x}+\vec{y} \in R^m \setminus K,$$

where

$$K_k(\vec{x}+\vec{y}-\vec{v}) = \frac{-1}{A_m} \frac{\vec{x}+\vec{y}-\vec{v}}{|\vec{x}+\vec{y}-\vec{v}|^{2k+m}} P_k(\vec{x})$$

(note that the kernel K_k depends on the choice of P_k).

The Cauchy transform (of order k) $C_k(F)$ is a monogenic function in $R^m \setminus K$. As in Example 2, the computation done in Example 3 yields, by convolution, that all weak limits of the corresponding restrictions exist and that the generalized Laurent coefficient is given by

$$L_k(C_k(F)) = \frac{\Gamma(k+\frac{p}{2})\Gamma(\frac{m}{2})}{\Gamma(\frac{p}{2})\Gamma(k+\frac{m}{2})} F \otimes P_k(\vec{\xi}),$$

where $L_k(C_k(F)) \in \mathcal{A}'(K) \otimes \mathcal{M}_{l,+,k} \simeq \mathcal{A}'(K, \mathcal{M}_{r,+,k})$. So the Cauchy transform of order k describes the inverse of the map L_k. Now we are finally able to prove that the Laurent coefficients can be computed by the corresponding limits in all cases.

Theorem 7.5 *Let us suppose that f belongs to $\mathcal{L}_{l,k}(R^m \setminus K)$ and that $f_\rho = f_\rho^1 - \vec{\xi} f_\rho^2$, where f_ρ^i are in $\mathcal{A}(R^q, \mathcal{M}_{l,+,k})$. Then*

$$f_0^i|_U := \lim_{\rho \to 0} \rho^{k+p-1} f_\rho^i,\ i = 1, 2$$

exists in the weak topology of the space $\mathcal{A}'(\overline{U}, \mathcal{M}_{r,+,k})$ for every open neighborhood U of K in R^q and

$$L_k(f)(\vec{\xi}, \vec{y}) = f_0^2|_U(\vec{\xi}, \vec{y}).$$

Proof.

Let us take any $f \in \mathcal{L}_{l,k}(R^m \setminus K)$ and let us choose a basis $\{P_{k,\alpha}\}_{\alpha \in B_k}$ for the space $\mathcal{M}_{l,+,k}$. Then $L_k(f) \in \mathcal{A}'(K, \mathcal{M}_{r,+,k}) \simeq \mathcal{A}'(K) \otimes \mathcal{M}_{l,+,k}$ can be written as a sum

$$L_k(f) = \sum_{\alpha \in B_k} c_\alpha P_{k,\alpha},\ c_\alpha \in \mathcal{A}'(K).$$

If we denote the Cauchy transform of order k corresponding to the monogenic $P_{k,\alpha}$ by $C_{k,\alpha}$, we can define

$$\tilde{f} = \left[\frac{\Gamma(k + \frac{p}{2}) \Gamma(\frac{m}{2})}{\Gamma(\frac{p}{2}) \Gamma(k + \frac{m}{2})} \right]^{-1} \sum_{\alpha \in B_k} C_{k,\alpha}(c_\alpha).$$

Then $L_k(\tilde{f}) = \sum_{\alpha \in B_k} c_\alpha P_{k,\alpha} = L_k(f)$, whence $\tilde{f} = f$. So we can use all information on the limits computed in the examples, which finishes the proof. ∎

8. Computation of Residues

In this section, we shall describe a few basic results showing how residues can be computed. It is clear that the technique for calculations of residues should be developed in more details in future.

8.1. Residues of indicatrices

If Σ is a compact submanifold of R^m of dimension q, then any q-form ω on Σ defines, by convolution with the Cauchy kernel, a monogenic function $\hat{\omega}$ outside Σ. We shall call it, by tradition, the indicatrix of ω. As could be expected, the Leray-Norguet residue of $\hat{\omega}$ is, as a cohomology class, represented by ω (see Theorem 8.1 below).

It shows at the same time that the theory of residues is rich enough, it gives a possibility to construct for any class of cohomology a suitable monogenic function f such that the residue of $d\sigma f$ is represented by ω.

More generally, let us recall that the cohomology groups of Σ can also be represented by closed distributional forms in $\mathcal{E}'(\Sigma, \Lambda^q)$. Then the corresponding indicatrix $\hat{\omega}$ can be defined by the same formula

$$\hat{\omega}(\vec{x}) = \int_{\Sigma_{\vec{y}}} E(\vec{x} - \vec{y}) \omega(\vec{y}),$$

and we have again that $\mathrm{Res}_{\Sigma}(d\sigma\hat{\omega})$ is represented by the distributional form ω in $H^q(\Sigma)$.

As a typical example, the residue $\mathrm{Res}_{\Sigma}E_{\vec{y}}$ of the Cauchy kernel in a point $\vec{y} \in \Sigma$ is represented in $H^q(\Sigma)$ by the destributional q-form $\delta_{\vec{y}}$.

We would like to stress now that even much more is true, if we consider the residue as a functional. We know that the grand residue $Rs_{\Sigma}(\omega)$ is a functional in $\mathcal{M}_r'(\Sigma)$, acting on test functions φ which are restrictions of monongenic functions on a neighborhood of Σ. For some monogenic forms, the grand residue can well belong to the subspace $\mathcal{E}'(\Sigma, \Lambda^q)$ of distributinal q-forms. It means that the action of $Rs_{\Sigma}(d\sigma\hat{\omega})$ can be extended to the space of all smooth functions. It is true in the case described above. Note that as a special case we have $Rs_{\Sigma}E_{\vec{y}} = \delta_{\vec{y}}$.

Theorem 8.1 *Let Σ be a compact submanifold of R^m of dimension q and let $\omega \in \mathcal{E}'(\Sigma, \Lambda^q)$. Let us define the indicatrix $\hat{\omega}$ by*

$$\hat{\omega}(\vec{x}) = \int_{\Sigma_{\vec{y}}} E(\vec{x} - \vec{y})\omega(\vec{y}),$$

where E is the Cauchy kernel.

Then

$$Rs_{\Sigma}(d\sigma\,\hat{\omega}) = \omega.$$

Proof. To prove that $Rs_{\Sigma}(d\sigma\,\hat{\omega})$ is equal to ω, it is sufficient to prove that for all $f \in \mathcal{M}_r(\Sigma)$

$$\int_{\delta\Sigma} f d\sigma\hat{\omega} = \int_{\Sigma} f\,\omega.$$

But

$$\begin{aligned} \int_{\delta\Sigma} f d\sigma\hat{\omega} &= \int_{\delta\Sigma_{\vec{x}}} f(\vec{x})d\sigma_{\vec{x}} \int_{\Sigma_{\vec{y}}} E(\vec{x} - \vec{y})\omega(\vec{y}), \\ &= \int_{\Sigma_{\vec{y}}} \left[\int_{\delta\Sigma_{\vec{x}}} f(\vec{x})d\sigma(\vec{x})E(\vec{x} - \vec{y})\right]\omega(\vec{y}), \\ &= \int_{\Sigma_{\vec{x}}} f(\vec{y})\omega(\vec{y}). \end{aligned}$$

■

8.2. Computation of residues by Laurent series

The theory developed above on Taylor and Laurent coefficients can be used to computed residues of monogenic differential forms.

Theorem 8.2 *Let Σ be a compact submanifold of dimension q in $\tilde{\Omega} \subset R^m$ and let $\omega \in \mathcal{E}^q(\Sigma)$. Let $K \subset \Omega$ be a compact subset such that there is an open subset Ω of an affine subspace A_q of dimension q containing K. Then for all functions $f \in M_r(\tilde{\Omega} \setminus K)$ and $g \in M_l(\tilde{\Omega} \setminus K)$, we have*

$$res_{\Sigma}[f\,d\sigma\,g] = \sum_{k=0}^{\infty}(L_k(f)[T_k(g)] + L_k(g)[T_k(f)]),$$

where $L_k(f) \in \mathcal{A}'(K, \mathcal{M}_{l,+,k})$, $L_k(g) \in \mathcal{A}'(K, \mathcal{M}_{r,+,k})$, $T_k(f) \in \mathcal{A}(K, \mathcal{M}_{r,+,k})$ *and* $T_k(g) \in \mathcal{A}(K, \mathcal{M}_{l,+,k})$.

Proof.

After a suitable Euclidean transformation of coordinates, we can supose that $\Omega \subset R^k$.

Taking into account that $K \subset \Omega$ and that f and g are monogenic in $\tilde{\Omega} \setminus K$, the Cauchy theorem implies that for a sufficiently small neighborhood U of K in R^m

$$\int_{\delta\Sigma} f\, d\sigma\, g = \int_{\partial U} f\, d\sigma\, g.$$

We can write now the generalized Laurent series for f and g and to use the orthogonality relations for spherical monogenics in dimension $m-k$, (i.e. for values of f and g) to get the result. Note that if the set K above is a point, then we can imbed it into an affine space of dimension 0 and the theorem above coincide with the one given by Zöll (see [17]). Note also that it is necessary to have a flat piece of dimension q around K to apply the decomposition into generalized Laurent series. It is a quite interesting problem to ttry to formulate generalized Laurent series for non-flat manifolds. We shall not discuss this difficult problem here.

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