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LERAY RESIDUES APPLIED TO SOLUTIONS OF THE LAPLACE AND WAVE EQUATIONS

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1. Introduction.

The aim of this paper is to describe a simple way of relating integral formulae for solutions of elliptic and hyperbolic systems of partial differential equations. Using Leray's residue theory, it is shown how these integral formulae can be drived one from another.

In general, the corresponding integral formulae for these systems are quite different in character. To describe them, let us discuss the simplest examples of Laplace's equation and the wave equation briefly.

For a solution u of Laplace's equation in a domain $\Omega \subset R_n$, with $0 \in \Omega$, we have the formula

$$u(0) = \frac{1}{(n-2)\kappa_{n-1}} \int_{\partial K} \left\{ \frac{1}{|\xi|^{n-2}} \frac{\partial u}{\partial \eta}(\xi) - \frac{\partial}{\partial \eta}\left(\frac{1}{|\xi|^{n-2}}\right) u(\xi) \right\} dS \qquad (1)$$

where K is a sufficiently small sphere about 0 , $1/(n-2)\kappa_{n-1}$ is the appropriate constant and $\partial/\partial\eta$ is the normal derivative at the point ξ on ∂K .

On the other hand for the solution of the wave equation with even dimension $n = 2k$, there is the following integral formula, due to M. Riesz ([9]):

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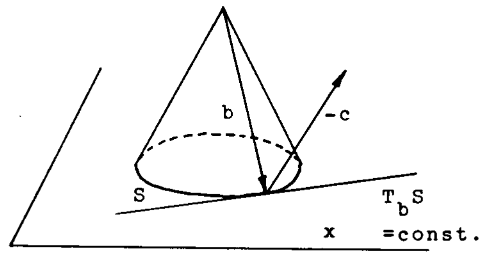

Let M_n be Minkowski space with coordinates $x = [x_0, \dots, x_{n-1}]$. Let $N = \{x \in M_n \mid (x,x) = 0,\ x_0 < 0\}$. Consider (for simplicity) the sphere $S = N \cap \{x \in M_n \mid x_0 = \text{constant}\}$ and parametrize it by $n-2$ parameters $(\lambda_1, \dots, \lambda_{n-2}) = \lambda$. At every point $b(\lambda)$ on S there are exactly two null vectors in N which are orthogonal to $T_{b(\lambda)}S$, one being just $b(\lambda)$ and the other being denoted by $c(\lambda)$, and which we shall normalize so that $(b,c) = -\frac{1}{2}$. Then in a small neighborhood of N, there are new coordinates $\sigma, \tau, \lambda_1, \dots \lambda_{n-2}$ given by

$$x = \sigma\{b(\lambda) - \tau c(\lambda)\},$$

in which the Minkowski metric has the following form:

$$dx^2 = \tau d\sigma^2 + \sigma d\sigma d\tau + \sum_{i,k=1}^{n-2} \sigma^2 \gamma_{ik} d\lambda_i d\lambda_k ,$$

where γ_{ik} does not depend on σ. Writing

$$F(\tau,\lambda) = \frac{\sqrt{\gamma(\tau,\lambda)}}{\sqrt{\gamma(0,\lambda)}} ; \quad \gamma = \det(\gamma_{ik}),$$

we have the formula

$$u(0) = (-\pi)^{1-k} \int_S \frac{\partial^{k-2}}{\partial \tau^{k-2}} \left\{\frac{1}{2}\frac{\partial F}{\partial \tau} u + F \frac{\partial u}{\partial \tau}\right\}\Big|_{\tau=0} dS \tag{2}$$

We can see now that the formulae (1) and (2) are in fact very different. First in each case we are integrating over spheres of different dimension (the dimension of S is one less than that of ∂K). Secondly, ∂K is fixed (and does not vary with the origin 0), while since S is the intersection of N with the initial data hyperplane, S varies with the origin 0. Thirdly, only value of u and its first derivatives of order $j=0,1,\dots,k-1$ are necessary. Thus these two formulae appear to be very different and do not seem to be related.

However we shall show that this is not the case and that the relationship between them is just the Leray residue formula

$$\int_{\delta\gamma} \omega = 2\pi i \int_{\gamma} \operatorname{Res} \omega \tag{3}$$

(see §.2. for details). The dimensions of $\delta\gamma$, the Leray cobord, and that of γ differ by just one (as we need) and the appearence of the higher derivatives in (2) is explained by the fact that the differential form ω in the integrand has a higher order pole on the null cone N, so that we need higher derivatives to compute the residue. The contour of integration changes with 0 in the hyperbolic case because the cycle γ belongs to N which itself depends on the vertex 0 of the cone, while in the elliptic case the cycle only goes around the intersection of the null cone with the Euclidean subspace (the intersection is just a point) and can be fixed for suitable changes of the vertex.

The same procedure can be applied to more general systems of partial differential equations (see §.3.) and can be used either to show a connection between already known integral formulae or to deduce new integral formulae from known ones in the corresponding elliptic case. A similar procedure (in coordinates and without using Leray residues) was described for the case of the wave equation in even dimensions ([2], [3]), for the case of massless fields in Minkowski space in dimension 4 for any spin ([4],[5]) and for spinor fields in basic spinor representation in even dimensions ([2]). Leray residue theory gives a simple and general formula for such a procedure and it is then sufficient to find the corresponding coordinate description of Res ω which makes the integral formula as simple as possible. No coordinate description of the ultrahyperbolic case has been published and the general integral formula presented here can be applied in this case too but this will not be treated here.

After the discussion of Leray's residue theory (extended to vector-valued forms) in §.2., we shall describe in §.3. how to use Leray's residues to corresponding integral formulae in detail. The special case of the Laplace and wave equations and the necessary computations in suitable coordinates is described in §.4.

2. Leray residues for vector-valued forms.

In this paragraph we shall recall briefly basic facts on Leray's residue theory. For applications it is necessary to extend the theory to vector-valued differential forms. We shall also prove a lemma which shows how to calculate the residue in the cases to be considered.

Let M be a complex manifold of complex dimension n. Let S be a hypersurface in M, described by a function $\tau: M \to \mathbb{C}$, i.e. $S = \tau^{-1}(0)$. Suppose that the differential $d\tau$ is nonsingular at every point of S, so that S is a submanifold of M.

Let V be a complex vector space. Denote by $E_V^*(\Omega) = E^*(\Omega) \otimes_C V$ the space of smooth V-valued differential forms on $\Omega \subset M$. The exterior differential d will be extended to E_V^* componentwise.

Defintion 2.1:

Let $\omega \in E_V^*(M\setminus S)$. We say that ω has a polar singularity of order p on S, if p is the smallest positive integer such that $\tau^p\omega$ is regular on the whole M.

Let $\omega \in E_V^k(M\setminus S)$ be a closed k-form, which has a polar singularity of order 1 on S. Then there exist (locally) forms $\sigma \in E_V^{k-1}$ and $\gamma \in E_V^k$, both regular on M, such that

$$\omega = \frac{d\tau\wedge\sigma}{\tau} + \gamma$$

on $M\setminus S$. The form $\sigma|_S \in E_V^{k-1}(S)$ is then uniquely determined by ω (see [7]).

Definition 2.2.

The form $\sigma|_S$ is called the Leray residue of ω and is denoted $\operatorname{res}\omega$.

Theorem 2.3.

Let $\omega \in E_V^k(M\setminus S)$ be a closed form, which has a polar singularity on S. Then ω is equivalent (i.e. equal modulo an exact form on $M\setminus S$) to a form $\omega' \in E_V^k(M\setminus S)$ which has a polar singularity of order 1 on S. Moreover, all forms $\operatorname{res}\omega'$ corresponding to such forms ω' belong to the same cohomology class in $H^{k-1}(S,V)$. This class of cohomology will be denoted by $\operatorname{Res}\omega$ and will be called the Leray residue of the form ω.

Proof.

It is not difficult to reduce the theorem (as well as the assertion leading to the Definition 2.2) to the standard case, proved by Leray (see [7]). Let $v_1,\dots,v_n$ be a basis for V, then

$$\omega = \sum_{i=1}^{n} \omega_i \otimes v_i, \quad \omega_i \in E^k(M\setminus S).$$

If ω has a polar singularity on S, the same is true also for every component ω_i. The exterior derivative acts componentwise, hence ω_i are closed and we can construct $\operatorname{Res}\omega_i \in H^{k-1}(S,C)$. Then $\operatorname{Res}\omega = \sum_{i=1}^{n} \operatorname{Res}\omega_i \otimes v_i$.

Remarks.

1. The Leray residue of ω is not a uniquely defined form on S, the cohomology class $\text{Res}\,\omega$ can be represented by any form belonging to it.

2. Note that the coefficients of ω need not be holomorphic, but that only the condition that ω is closed is needed. Note that the case of dimension 1 (i.e. $\dim_{\mathbf{C}} M = 1$, $S = \{p\}$ being a point in M) reduces for $\omega = f\,dz$, $z \in M$ to the standard case from complex analysis, because $d\omega = 0 \Longleftrightarrow \partial f/\partial \bar{z} = 0$. Thus in this case $\text{Res}\,\omega$ is a complex number and coincides with the standard residue of a function in complex analysis.

Remark.

To state the Leray residue theorem, we need the notion of Leray cobord. The detailed definition involves several other notions and would be too long to be explained here (it can be found in [8] or, in the original and detailed version in [7]). We shall describe here only the geometrical meaning of δ in the special case.

Let $\overline{V}$ be a closed tubular neighbourhood of S and let $\mu:\overline{V} \to S$ be a retraction, giving to $\overline{V}$ the structure of fibre bundle with typical fibre D (the unit disc in $\mathbf{C}$). Suppose further that γ is a p-dimensional cycle in S, which is also an imbedded submanifold of dimension p in S. The chain $\mu^{-1}(\gamma)$ has then the boundary $\partial(\mu^{-1}\gamma)$ which is the $(p+1)$-cycle in $M \setminus S$. It means that every point of γ is substituted by a circle going around it in $M \setminus S$. This $(p+1)$-cycle is then denoted by $\delta\gamma$ and called the Leray cobord of γ.

Theorem 2.4.

Let γ be a $(p-1)$-cycle in S and let $\omega \in E^p_V(M \setminus S)$ be a closed form, which has at most a polar singularity at S.
Then
$$\int_{\delta\gamma} \omega = 2\pi i \int_{\gamma} \text{Res}\,\omega .$$

We shall now show how to calculate Leray residue in a special case.

Theorem 2.5.

Let Ω be a domain in $\mathbf{C}_n$, let $\tau, w_1, \ldots, w_{n-1}$ be coordinates on Ω and denote $S = \{x \mid \tau(x) = 0\}$. Consider a closed form ω on $\Omega - S$ which has at most a pole of the order k on S.

Then the form ω can be expressed in the form

$$\omega = \frac{1}{\tau^k}\left[d\tau \wedge \alpha + \beta\right]$$

where α , β are regular on Ω and (written in coordinates $\tau, w_2, \dots, w_n$) contain no term with $d\tau$ and

$$\operatorname{Res} \omega = \frac{1}{(k-1)!} \left.\frac{\partial^{k-1}\alpha}{\partial \tau^{k-1}}\right|_{\tau=0} .$$

Remark.

The notation $\left.\frac{\partial^{k-1}}{\partial \tau^{k-1}}\right|_{\tau=0}$ means that the coefficients in every term of the form α are substituted by their $(k-1)$-th derivatives with respect to τ at $\tau = 0$.

Proof.

Recall that ω has a pole at most of the order k on S , so the form $\tilde{\omega} = \tau^k \cdot \omega$ is regular on S . The form $\tilde{\omega}$ can be written uniquely in the form $\tilde{\omega} = d\tau \wedge \alpha + \beta$, where α , β are regular on Ω and contain no terms with $d\tau$. So we can always write ω as

$$\omega = \frac{1}{\tau^k}\{d\tau \wedge \alpha + \beta\}.$$

To find $\operatorname{Res} \omega$, we have to find an equivalent form ω' , which has a pole of the order 1 and then apply Definition 2.2.

We shall first prove two assertions:

i) the form β has zero of the order at least 1 on M , i.e. there is a form $\tilde{\beta}$ on Ω such that $\beta = \tau\tilde{\beta}$

ii) there is a form δ on $\Omega \setminus S$ such that

$$\omega = d\delta + \tilde{\omega}, \quad \tilde{\omega} = 1/\{ (k-1)\tau^{k-1}\}.\left[d\tau \wedge \frac{\partial \alpha}{\partial \tau} + \beta'\right],$$

where β' is regular on Ω and contains no terms with $d\tau$.

Proof of i):

We have $d\omega = 0$, $\omega = 1/\tau^k.\{d\tau \wedge \alpha + \beta\}$, hence

$$(-k)\frac{d\tau \wedge \beta}{\tau^{k+1}} - \frac{d\tau \wedge d\alpha}{\tau^k} + \frac{d\beta}{\tau^k} = 0$$

$$(-k)d\tau \wedge \beta - \tau d\tau \wedge d\alpha + \tau d\beta = 0$$

and multiplying the equation by $d\tau$ we obtain $d\tau \wedge d\beta = 0$, hence the form $d\beta$ contains only terms with $d\tau$, i.e. $d\beta = d\tau \wedge \frac{\partial \beta}{\partial \tau}$ (note that β does not contain any term with $d\tau$). Coming back to the equation we obtain $(-k)\beta - \tau d\alpha + \tau \frac{\partial \beta}{\partial \tau} = 0$, hence $\beta = \tau\tilde{\beta}$, $\tilde{\beta} = 1/k.\{ \frac{\partial \beta}{\partial \tau} - d\alpha\}$.

Proof of ii):

We have $\omega = \dfrac{d\tau \wedge \alpha + \tau\widetilde{\beta}}{\tau^k}$.

Denote $\delta = 1/\{(1-k)\tau^{k-1}\}\cdot\alpha$, then $d\delta = \dfrac{1}{\tau^k}\,d\tau\wedge\alpha + \dfrac{d\alpha}{(1-k)\tau^{k-1}}$

and $\omega = d\delta + \dfrac{d\alpha + (k-1)\widetilde{\beta}}{(k-1)\tau^{k-1}} = d\delta + \dfrac{d\tau\wedge\frac{\partial\alpha}{\partial\tau} + \beta'}{(k-1)\tau^{k-1}}$,

where β' contains no terms with $d\tau$ (because the term $d\tau\wedge\frac{\partial\alpha}{\partial\tau}$ is exactly the part of $d\alpha$ containing terms with $d\tau$).

Now, applying both assertions (k-1)-times, we end up with the formula

$$\omega = d(\delta') + \frac{d\tau\wedge\frac{\partial^{k-1}\alpha}{\partial\tau^{k-1}} + \beta''}{(k-1)!\,\tau}$$

where the form β'' is regular on Ω and contains no terms with $d\tau$.

Applying the assertion i) the last time to the quotient at the right hand side, we see that $\beta'' = \tau\widetilde{\beta}''$, $\widetilde{\beta}''$ regular on Ω , hence the assertion of the lemma follows from the definition of the form Res ω .

3. Elliptic and hyperbolic integral formulae.

We describe here in more detail the connection between integral formulae for elliptic and hyperbolic (possibly ultrahyperbolic) systems, or more precisely how to deduce integral formulae in hyperbolic cases from known integral formulae in corresponding elliptic cases.

We shall discuss here (generalized) massless field equations. In dimension 4 (and in hyperbolic case, i.e. on Minkowski space) these equations are just what is called a minimal set of equations by Garding ([6]). He classified all 'minimal' systems of hyperbolic equations on Minkowski space such that every component of a solution satisfied the wave equation. How similar types of equations can be generalized to higher dimensions and how they can be classified by group representation theory was described in [4] . The equation themselves are well defined on C_n, but by restriction to Euclidean or Minkowski slice of C_n we shall obtain either elliptic or hyperbolic systems of PDE (with constant coefficients). In many cases integral formulae for corresponding elliptic systems areknown (see [5]). All of them have the common feature that the corresponding vector-valued differential forms under the integral sign are well defined on $C_n \setminus CN_o$,

where CN_o is the complex null cone, i.e. $CN_o = \{z \in C_n | \sum_1^n z_i^2 = 0\}$.
In this paragraph we want to show that in these cases we can obtain almost immediately integral formula for corresponding hyperbolic (and ultrahyperbolic) systems using Leray's residue.

So suppose that we have a system of linear partial differential equations with constant (complex) coefficients, i.e.

$$D = \sum_{|\alpha| \leq k} a_\alpha \frac{\partial^{|\alpha|}}{\partial z^\alpha}; \ \alpha = [\alpha_1, \dots, \alpha_n],$$

where a are $j \times m$ matrices, $j \geq m$ and $u : C_n \longrightarrow C_m$ is holomorphic in C_n.

Suppose that $E_n \subset C_n$ is an Euclidean subspace of C_n. Let $D.u = 0$ on C_n (for simplicity) and suppose that we have for u an integral formula of elliptic type: let Σ be a $(n-1)$-cycle in E_n then

$$\mathrm{Ind}_\Sigma\, 0.u(0) = \int_\Sigma \omega(u) \quad ,$$

where $\omega(u)$ is a (possibly vector valued) $(n-1)$-form, depending on u and $\mathrm{Ind}_\Sigma\, 0$ is the index of 0 with respect to Σ (see [5]). In all known formulae ([4]) the form $\omega(u)$ depends linearly on u and its derivatives. Suppose further that $\omega(u)$ can be extended to $C_n \setminus CN_o$ in such a way that $\omega(u)$ is closed on $C_n \setminus CN_o$. In this case the Leray residue of $\omega(u)$ is an $(n-2)$-form on $CN_o \setminus \{0\}$. From the Leray residue formula it follows that for all $(n-2)$-cycles γ in $CN_o \setminus \{0\}$ we can consider the Leray cobord (which is $(n-1)$-cycle in $C_n \setminus CN_o$) and we obtain

$$\mathrm{Ind}_{\delta\gamma} 0.u(0) = \int_{\delta\gamma} \omega(u) = \int_\gamma \mathrm{Res}\ \omega(u) \ .$$

Thus if we take $\gamma \in CN_o \cap M_n$, where M_n is Minkowski subspace of C_n, we obtain the integral formula for solutions of the corresponding hyperbolic system, while if we take γ inside the intersection of CN_o with another ultrahyperbolic slice of C_n, we would obtain the integral formula for solutions of the corresponding ultrahyperbolic system.

On this abstract level it is quite general and simple. But to gain the full advantage of the integral formula, it has to be used in the coordinate description. As a rule, the form $\mathrm{Res}\ \omega(u)$ will be quite complicated and only by using a suitable coordinates (e.g. tailored for the cycle of integration under the consideration) can we obtain a simple formula. This fact is illustrated in the next paragraph for the wave equation. Another example could be the integral formula for generalized massless fields in even dimensions (see [2],[3]), but they will not be treated here. A good coordinate description of the residue in ultrahyperbolic case is still to be found and many other systems of equations (see [4],[5]) are still to be investigated.

4. The case of the Laplace and wave equation.

In this paragraph we would like to show how the above abstract procedure can be applied in coordinates to deduce the integral formula for solutions of the wave equation, due to M. Riesz ([9]). First we shall describe the hypercomplex form of integral formula for solutions of Laplace equation and its relations to standard integral formula for harmonic functions in potential theory (see also in [3]). The complexification of the form $\omega(u)$ under the integral sign will be then written in suitable coordinates and will lead to the coordinate description of $\operatorname{Res}\,\omega(u)$.

Finally, it will be shown that if coordinates are chosen suitably with respect to a chain of integration, the form $\operatorname{Res}\,\omega(u)$ will reduce to the expression, used in the integral formula of M. Riesz ([9]).

Let us begin with the hypercomplex form of the integral formula for harmonic functions (see [2],[3]).

Consider the standard basis $e_1,\ldots,e_n$ for C_n, $n = 2k+1$ (k being nonnegative integer). The corresponding (complex) Clifford algebra $\mathfrak{C}_n^c$ has the grading (as the vector space)

$$\mathfrak{C}_n^c = (\mathfrak{C}_n^c)_0 \oplus \ldots \oplus (\mathfrak{C}_n^c)_n ,$$

where $(\mathfrak{C}_n^c)_0 \cong C$ with the basis $e_0 = 1$ and $(\mathfrak{C}_n^c)_1 \cong C_n$ with the basis $e_1,\ldots,e_n$. We shall consider holomorphic functions on $C_{n+1} = (\mathfrak{C}_n^c)_0 \oplus (\mathfrak{C}_n^c)_1$ and we shall define (for a Clifford number $Q = \sum_0^n e_i Q_i$) the squared norm $\|Q\|^2 = \sum_0^n Q_i{}^2 = QQ^+$, where

$$Q^+ = Q_0 e_0 - \sum_1^n Q_i e_i$$

We shall use two differential operaotrs

$$\partial = \sum_0^n e_i \partial/\partial Q_i \quad , \qquad \partial^+ = e_0 \partial/\partial Q_0 - \sum_1^n e_i \partial/\partial Q_i \; ,$$

noting that $\Delta^c = \partial^+\partial = \partial\partial^+ = \sum_0^n \partial^2/\partial Q_i^2$.

The complex null cone $CN_P = \{Q \in C_{n+1} \mid \|Q-P\|^2 = 0\}$ will play a privileged role in what follows. Using the special Clifford-valued n-forms

$$DQ = \sum_0^n (-1)^i e_i d\hat{Q}_i \quad , \quad DQ^+ = e_0 d\hat{Q}_0 - \sum_1^n (-1)^i e_i d\hat{Q}_i \ ,$$

where $d\hat{Q}_i = dQ_0 \wedge \ldots \wedge dQ_{i-1} \wedge dQ_{i+1} \wedge \ldots \wedge dQ_n$ and the notion of the index of a point with respect to n-dimensional cycle we can state the following hypercomplex form of the integral formula for solutions of (complex) Laplace equation (see [5]).

Definition 4.1.

Let Σ be an n-dimensional cycle in $C_{n+1} \setminus CN_P$, $P \in C_{n+1}$. Then the index of P with respect to Σ is defined by

$$\mathrm{Ind}_\Sigma P = 1/\kappa_n \int_\Sigma \frac{(Q-P)}{\|Q-P\|^{2k+2}} DQ^+ \ .$$

Remark.

It was proved in [5] that $H_n(C_{n+1} \setminus CN_P, \mathbb{Z}) \cong \mathbb{Z}$ and that $\mathrm{Ind}_\Sigma P$ is always an integer. Moreover, the sphere $S_\rho(P)$ in the Euclidean slice of C_{n+1} is the generator of $H_n(C_{n+1} \setminus CN_P, \mathbb{Z})$, so

$\Sigma \sim \mathrm{Ind}_\Sigma P . S_\rho(P)$.

Theorem 4.2 (see [5]).

Consider a domain $\Omega \subset C_{n+1}$ such that the whole segment $\overline{Q_1 Q_2}$ belongs to Ω whenever $Q_1, Q_2 \in \Omega$ and $\|Q_1 - Q_2\|^2 = 0$.

Let u be a complex function satisfying the equation $\Delta^c u = 0$ on Ω . Take $P \in \Omega$ and n-dimensional cycle Σ in $\Omega \setminus CN_P$, which is homologically trivial in Ω .

Then

$$u(P).\mathrm{Ind}_\Sigma P = 1/\kappa_n \int_\Sigma \left\{ \frac{Q-P}{\|Q-P\|^{2k+2}} . DQ^+ . u(Q) + \frac{1}{2k \cdot \|Q-P\|^{2k}} . DQ . \partial^+ u(Q) \right\} \ ,$$

where κ_n is the area of the unit sphere in R_{n+1}.

To compare the integral formula in Theorem 4.2 with the standard integral formula for harmonic functions we shall prove first the following lemma.

Lemma 4.3.

Denote $g = 1/\left[2k\|Q\|^{2k}\right]$ and

$$\phi = \sum_{\substack{i,j=0\\ i\neq j}}^{n} e_j e_i^{+}\left\{(-1)^i Q_j d\hat{Q}_i \frac{u}{\|Q\|^{2k+2}} + \frac{1}{2k\|Q\|^{2k}}(-1)^j \frac{\partial u}{\partial Q_i} d\hat{Q}_j\right\} .$$

Then
$$\phi = d\left\{\sum_{i<j}(-1)^{i+j} e_j e_i^{+} u g d\hat{Q}_{ij}\right\} ,$$

where $d\hat{Q}_{ij} = dQ_0\wedge\ldots\wedge dQ_{i-1}\wedge dQ_{i+1}\wedge\ldots\wedge dQ_{j-1}\wedge dQ_{j+1}\wedge\ldots\wedge dQ_n$.

Proof.

We have
$$\phi = \sum_{\substack{i,j=0\\ i\neq j}}^{n} e_j e_i^{+}\left\{(-1)^{i+1}\frac{\partial g}{\partial Q_j} u d\hat{Q}_i + (-1)^j g\frac{\partial u}{\partial Q_i} d\hat{Q}_j\right\} .$$

It is easy to check that $e_j e_i^{+} = -e_i e_j^{+}$; $i,j=0,\ldots,n; i\neq j$, so

$$\phi = \sum_{\substack{i,j=0\\ i\neq j}}^{n} e_j e_i^{+}\left\{(-1)^{i+1}\frac{\partial g}{\partial Q_j} u d\hat{Q}_i + (-1)^j \frac{\partial g}{\partial Q_i} u d\hat{Q}_j + (-)^j g\frac{\partial u}{\partial Q_i} d\hat{Q}_j + (-)^{i+1} g\frac{\partial u}{\partial Q_j} d\hat{Q}_i\right\} =$$

$$= d\left\{\sum_{\substack{i,j=0\\ i<j}}^{n}(-1)^{i+1} e_j e_i^{+} u g d\hat{Q}_{ij}\right\} .$$

Corollary 4.4.

The form
$$\omega(u) = 1/\kappa_n\left\{\frac{Q}{\|Q\|^{2k+2}} DQ^{+}u(Q) + \frac{1}{2k\|Q\|^{2k}} DQ\partial^{+}u(Q)\right\}$$

is equal to
$$\omega'(u) = 1/\kappa_n\left\{\frac{\sum_0^n (-1)^i Q_i d\hat{Q}_i}{\|Q\|^{2k+2}} u + \frac{\sum_0^n (-1)^i \frac{\partial u}{\partial Q_i} d\hat{Q}_i}{2k\|Q\|^{2k}}\right\}$$

modulo an exact form.

Remark.

The hypercomplex form of the differential form $\omega(u)$ in Theorem 5.2 is useful for deducing the integral formulae in an unified manner for several cases (including Laplace's equation) in hypercomplex analysis (see [5]). The Corollary 5.4 shows that $\omega(u)$ can be replaced in Theorem 5.2 by $\omega'(u)$, which is the number-valued differential form (no terms with $e_i, i=1,\ldots,n$ are contained in it). Using the connection of surface integrals of the first and the second types, it is well known that the term $\partial f/\partial\eta dS$ corresponds to the form

$$\sum_0^n (-1)^i \frac{\partial f}{\partial Q_i} dQ_i$$

for any f (e.g. also for $f = \sum_0^n Q_i^2$) , hence the integral formula (1) in the introduction is a consequence of Theorem 5.2.

Now, to compute the form Res $\omega(u)$ explicitly, we have to choose suitable coordinates. Let us consider new coordinates $(\sigma,\tau,\lambda_1,\ldots,\lambda_{n-1})$ given by the map

$$\vec{z} = \sigma(\vec{b} + \tau\vec{c}) \ ; \ \vec{z},\vec{b},\vec{c} \in C_{n+1}; \sigma,\tau \in C \ \lambda = (\lambda_1,\ldots,\lambda_{n-1}) \in C_{n-1},$$

where $\vec{b} = \vec{b}(\lambda_1,\ldots,\lambda_{n-1})$, $\vec{c} = \vec{c}(\lambda_1,\ldots,\lambda_{n-1})$ are holomorphic and such that $(\vec{b},\vec{b}) = (\vec{c},\vec{c}) = 0$ and $(\vec{b},\vec{c}) = -\frac{1}{2}$ for all λ .

The scalar product $(\vec{b},\vec{c})$ is given by $\sum_0^n b_i c_i$.

It means that $\vec{b}$ describes a section of CN_o , while $\vec{c}$ describes a transversal direction at the corresponding point of CN_o . We shall suppose that the parametrization gives us (for suitable range of parametrs) new coordinates on an open subset $\Omega \subset C_{n+1}$. Then we have

Lemma 4.5.

Let us denote $F_o(\tau,\lambda) = \det\left[\vec{b},\vec{c},\frac{\partial(\vec{b}+\tau\vec{c})}{\partial\lambda_1},\ldots,\frac{\partial(\vec{b}+\tau\vec{c})}{\partial\lambda_{n-1}}\right]$,

$$\tilde{F}_o(\tau,\lambda) = \det\left[\vec{\partial}u,\vec{c},\frac{\partial(\vec{b}+\tau\vec{c})}{\partial\lambda_1},\ldots,\frac{\partial(\vec{b}+\tau\vec{c})}{\partial\lambda_{n-1}}\right] ,$$

$$F_j(\tau,\lambda) = \det\left[\vec{\partial}u,\vec{b},\vec{c},\frac{\partial(\vec{b}+\tau\vec{c})}{\partial\lambda_1},\ldots,\frac{\partial(\vec{b}+\tau\vec{c})}{\partial\lambda_{j-1}},\frac{\partial(\vec{b}+\tau\vec{c})}{\partial\lambda_{j+1}},\ldots,\frac{\partial(\vec{b}+\tau\vec{c})}{\partial\lambda_{n-1}}\right] ,$$

$$j = 1,\ldots,n-1 .$$

Then we shall obtain

$$\operatorname{Res}\omega(u) = 1/2\pi^{k+1}\left\{\frac{\partial^k}{\partial\tau^k}(u.F_o)\Big|_{\tau=0} d\lambda + \sigma/2\frac{\partial^{k-1}\tilde{F}_o}{\partial\tau^{k-1}}\Big|_{\tau=0}\cdot d\lambda + \right.$$

$$\left. + \frac{1}{2}\sum_{j=1}^{n-1}\frac{\partial^{k-1}F_j}{\partial\tau^{k-1}}\Big|_{\tau=0}\cdot d\sigma\wedge d\lambda_j\right\} .$$

Proof.

First we shall express the form ω' in the new coordinates. Writing $d\lambda = d\lambda_1\wedge\ldots\wedge d\lambda_{n-1}$, $d\hat{\lambda}_j = d\lambda_1\wedge\ldots\wedge d\lambda_{j-1}\wedge d\lambda_{j+1}\wedge\ldots\wedge d\lambda_{n-1}$ and noting that $(\vec{z},\vec{z}) = \sigma^2\tau$ and $\kappa_n = 2\pi^{k+1}/k!$, we obtain

$$\frac{\sum_0^n (-1)^i Q_i d\hat{Q}_i}{\|Q\|^{2k+2}}\, u = \frac{u}{(\sigma^2\tau)^{k+1}}\left\{\det\left[\sigma(\vec{b}+\tau\vec{c}),\sigma\vec{c},\ldots,\frac{\partial(\sigma(\vec{b}+\tau\vec{c}))}{\partial\lambda_j},\ldots\right] d\tau\wedge d\lambda\right\} =$$

$$= \frac{u}{\tau^{k+1}} \det\left[\vec{b},\vec{c},\ldots,\frac{\partial\vec{z}}{\partial\lambda_j},\ldots\right] d\tau\wedge d\lambda ,$$

where all other determinants vanish, because they contain linearly dependent columns.

Similarly, if we denote $\vec{\partial} u = (\frac{\partial u}{\partial \lambda_1}, \dots, \frac{\partial u}{\partial \lambda_{n-1}})$, then

$$\frac{\sum_0^n (-1)^i \frac{\partial u}{\partial Q_i} d\hat{Q}_i}{2k(\sigma^2\tau)^k} = 1/2k(\sigma^2\tau)^k \Big\{ \det\left[\partial\vec{u}, \sigma\vec{c}, \frac{\partial \vec{z}}{\partial \lambda_1}, \dots, \frac{\partial \vec{z}}{\partial \lambda_{n-1}}\right] d\tau d\lambda + $$

$$+ \det\left[\vec{\partial} u, \vec{b}+\tau\vec{c}, \frac{\partial \vec{z}}{\partial \lambda_1}, \dots, \frac{\partial \vec{z}}{\partial \lambda_{n-1}}\right] d\sigma d\lambda +$$

$$+ \sum_{j=1}^{n-1}\left[\det\ \vec{\partial} u, \vec{b}+\tau\vec{c}, \sigma\vec{c}, \frac{\partial \vec{z}}{\partial \lambda_1}, \dots, \frac{\partial \vec{z}}{\partial \lambda_{j-1}}, \frac{\partial \vec{z}}{\partial \lambda_{j+1}}, \dots, \frac{\partial \vec{z}}{\partial \lambda_{n-1}}\right] d\sigma d\tau d\lambda_j \Big\}.$$

It is sufficient now to apply Theorem 2.5.

It is now possible to show how this complicated formula for Res $\omega(u)$ on the complex null cone will simplify further to obtain the integral formula of M. Riesz.

Consider first Minkowski subspace $M \subset C_{n+1}$,

$$M = \{Q \in C_{n+1} |\ Q_0 = w_0;\ Q_j = iw_j; j=1,\dots,n; w_j, w_0 \in R\} \ .$$

Take a cycle Σ_{n-1} of dimension $n-1$ in the (real) null cone.

Suppose that the cycle Σ_{n-1} is given by 1-1 smooth map $\vec{\beta}: O \subset R_{n-1} \longrightarrow N$. We shall consider (as in M. Riesz's integral formula) the map $\vec{\gamma}: O \longrightarrow N$ such that $(\vec{\gamma}, \vec{\gamma}) = (\vec{\gamma}, d\vec{\beta}) = 0$ with the normalization $(\vec{\beta}, \vec{\gamma}) = -\frac{1}{2}$. We shall suppose further for simplicity (this assumption can be removed) that the maps β, γ are real-analytic in O . The parametrization $\vec{w} = \sigma\{\vec{\beta}(\lambda) - \tau\vec{\gamma}(\lambda)\}$ can be extended to complex domain ($\sigma \in C$, $\tau \in C, \lambda \in \tilde{O} \subset C_{n-1}$, $\vec{\beta}, \vec{\gamma}$ being the holomorphic extension from O into an open subset $\tilde{O}$ of C_{n-1}) and suppose that it will give us (for a suitable range of parametrs) the new coordinates in an open subset of C_{n+1} . They are just the coordinates of the type used in Lemma 4.5, so we can use them to integrate Res ω over Σ_{n-1} . First we shall relate the function $F_o, \tilde{F}_o$ on Σ_{n-1} to the function F used in M. Riesz's formula.

Let us recall (see [2], [9]) that

$$F(\tau,\lambda) = \frac{\sqrt{\gamma(\tau,\lambda)}}{\sqrt{\gamma(0,\lambda)}} = \frac{\phi(\tau,\lambda)}{\phi(0,\lambda)}$$

where $\phi(\tau,\lambda) = \det\left[\vec{\beta},\vec{\gamma},\frac{\partial(\vec{\beta}+\tau\vec{\gamma})}{\partial\lambda_1},\ldots,\frac{\partial(\vec{\beta}+\tau\vec{\gamma})}{\partial\lambda_{n-1}}\right]$.

<u>Lemma 4.6.</u>

We have the following relations on Σ_{n-1}:

$$F_o(\tau,\lambda) = i^n\phi(\tau,\lambda)$$

$$F_o(\tau,\lambda) = i^n . 2\frac{\partial u}{\partial\tau}.\phi(\tau,\lambda) .$$

<u>Proof.</u>

The first relation follows immediately from the relations $Q = iw_j$, $j=1,\ldots,n$. To show the second one, suppose that the vectors β,γ, $\frac{\partial(\beta+\tau\gamma)}{\partial\lambda_j}$ are linearly independent (otherwise both sides vanish).

The orthogonality conditions for $\vec{\beta},\vec{\gamma}$ imply that both β and γ are orthogonal to all $\frac{\partial(\vec{\beta}+\tau\vec{\gamma})}{\partial\lambda_j}$. So if we express the gradient $\vec{\partial}u$ as $\vec{\partial}u = a_n\vec{\beta} + a_{n+1}\vec{\gamma} + \sum_{j=1}^{n-1} a_j\frac{\partial(\vec{\beta}+\tau\vec{\gamma})}{\partial\lambda_j}$,

we obtain that $a_n/2 = (\vec{\partial}u,\vec{\gamma}) = \partial u/\partial\tau$, whence

$$\tilde{F}_o = i^n \det\left[a_n\vec{\beta},\vec{\gamma},\frac{\partial(\vec{\beta}+\tau\vec{\gamma})}{\partial\lambda_1},\ldots,\frac{\partial(\vec{\beta}+\tau\vec{\gamma})}{\partial\lambda_{n-1}}\right] = i^n a_n\phi = 2i^n\frac{\partial u}{\partial\tau}.\phi .$$

So altogether we have

<u>Theorem 4.6.</u>

$$2\pi i \int_{\Sigma_{n-1}} \operatorname{Res}\omega(u) = \frac{(-1)^k}{\pi^k}\int_{\Sigma_{n-1}} \frac{\partial^{k-1}}{\partial\tau^{k-1}}\left\{\tfrac{1}{2}u.\frac{\partial F}{\partial\tau} + \frac{\partial u}{\partial\tau}.F\right\}\Big|_{\tau=0}. dS .$$

<u>Proof.</u>

First, all terms in the coordinate description of $\operatorname{Res}\omega$ (Lemma 4.5) containing $d\sigma$,vanish on Σ_{n-1} . So we are left only with two terms and we have

$$2\pi i \int_{\Sigma_{n-1}} \operatorname{Res}\omega = \frac{(-1)^k}{\pi^k}\int \left\{\frac{\partial^k}{\partial\tau^k}(u\phi)\Big|_{\tau=0} + \frac{\partial^{k-1}}{\partial\tau^{k-1}}\left(\frac{\partial u}{\partial\tau}\phi\right)\Big|_{\tau=0}\right\} d\lambda =$$
$$= \frac{(-1)^k}{\pi^k}\int \frac{\partial^{k-1}}{\partial\tau^{k-1}}\left[\frac{\partial u}{\partial\tau}\phi + u\frac{\partial\phi}{\partial\tau} + \frac{\partial u}{\partial\tau}\phi\right]\Big|_{\tau=0} d\lambda ,$$

but because $2\phi(0,\lambda)d\lambda = dS$, we have

$$2\pi i \int_{\Sigma_{n-1}} \operatorname{Res}\omega = \frac{(-1)^k}{\pi^k}\int \frac{\partial^{k-1}}{\partial\tau^{k-1}}\left[\frac{2\frac{\partial u}{\partial\tau}\phi + u\frac{\partial\phi}{\partial\tau}}{\phi(0,\lambda)}\right]_{\tau=0}\tfrac{1}{2}dS =$$
$$= \frac{(-1)^k}{\pi^k}\int \frac{\partial^{k-1}}{\partial\tau^{k-1}}\left[\frac{\partial u}{\partial\tau}.F + \tfrac{1}{2}u\frac{\partial F}{\partial\tau}\right]\Big|_{\tau=0} dS .$$

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