(1)

Differentiable manifolds

A/ Topological manifolds,

Def 1: A topological manifold M of dimension n is a topological space fullfilling:

1/ M is Hausdorff − $\forall p_1, p_2 \in M$, $p_1 \neq p_2$, $\exists V_1, V_2$ open neighborhoods of p_1, p_2 : $V_1 \cap V_2 = \phi$. (condition of separability of points)

2/ $\forall p \in M$ $\exists V$ open : $p \in V$ and $\varphi : V \to U \subseteq \mathbb{R}^n$ homeomorfism.

[Note. A function $f: X \to Y$, X, Y top. spaces, is continuous if $\forall$ $V \subseteq Y$ open subset is (the inverse image) $f^{-1}(V) := \{x \in X \mid f(x) \in V\}$ is open subset of X.

A function $f: X \to Y$, X, Y top spaces, is a homeomorphism if
i/ f is a bijection (1:1 and onto),
ii/ f is continuous,
iii/ the inverse f^{-1} is continuous (f is an open mapping.)

A topological space is a set X and a collection of subsets of X (called open sets), satisfying

$\mathcal{T}$ " topology, open sets {
i/ ϕ, X are open in X,
ii/ Any union of open sets is open,
iii/ The intersection of any finite number of open sets is open.

A base $\mathcal{B} \subseteq \mathcal{T}$ for a topological space $X, \mathcal{T}$ is a collection of open sets in $\mathcal{T}$ such that $\forall$ open set in $\mathcal{T}$ can be written as a union of elements in $\mathcal{B}$.]

3/ M satisfies the 2nd countability axiom, i.e., M has a countable basis for its topology. (implies the existence of partitions of unity.)

Example : $\mathbb{R}^n$, take all balls of rational radius and rational center $\Rightarrow$ base of standard topology on $\mathbb{R}^n$.

There is analogous definition of a manifold with boundary, where the boundary points are locally modelled by $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

②

B/ (U, φ) $\quad \varphi : U \to \varphi(U) \subset M$ homeomorphism

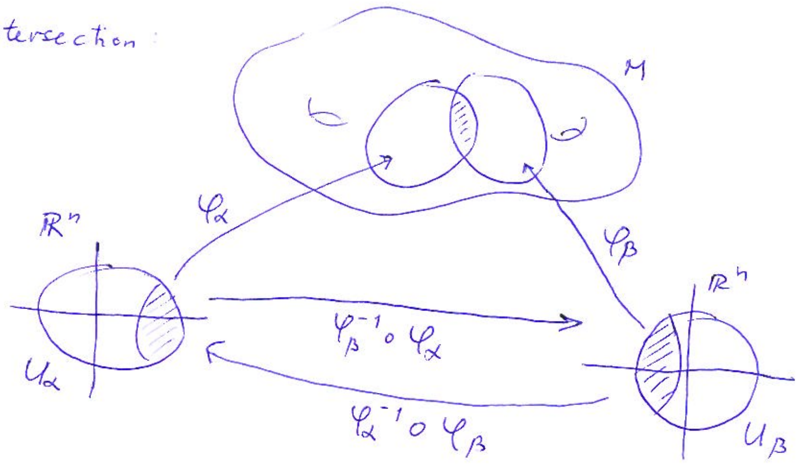
$U \subseteq \mathbb{R}^n$

(U, φ) = parametrization

φ^{-1} = coordinate system, chart (mapa)

$\varphi(U)$ = coordinate neighborhood

on the intersection:



Def 2: An n-dim differentiable (smooth) manifold M is a top. man., $\dim M = n$, and a family of parametrizations $\varphi_\alpha : U_\alpha \to M$, $U_\alpha \subseteq \mathbb{R}^n$:

1/ The coordinate neighborhoods cover M, $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$;

2/ $\forall \alpha, \beta$: $W := \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) \neq \emptyset$,

the overlap maps

$$\varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(W) \to \varphi_\beta^{-1}(W),$$
$$\varphi_\alpha^{-1} \circ \varphi_\beta : \varphi_\beta^{-1}(W) \to \varphi_\alpha^{-1}(W),$$

are C^∞.

3/ The family $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is maximal for 1/, 2/, i.e., if $\varphi_0 : U_0 \to M$ is parametrization such that $\varphi_0^{-1} \circ \varphi$, $\varphi^{-1} \circ \varphi_0$ are C^∞ $\forall \varphi$ in $\mathcal{A}$, then (U_0, φ_0) is in $\mathcal{A}$.

$\mathcal{A}$ is called (maximal, C^∞)-atlas on M (or, differentiable structure on M.)

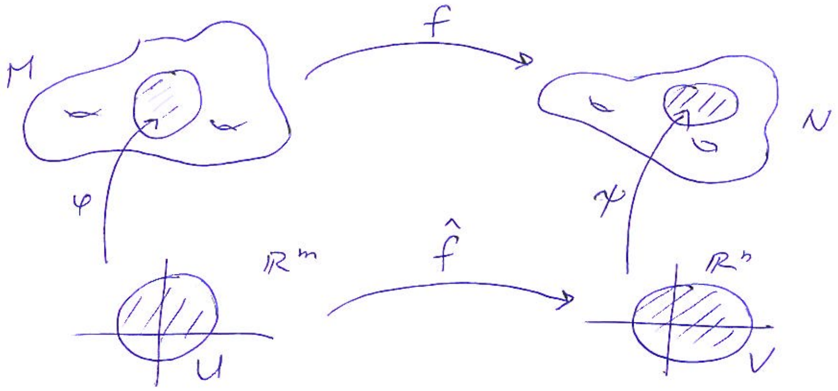
Differentiable maps (morphisms in the category of smooth man.)

differentiable, smooth $\equiv C^\infty$

Def 3: M, N - diff. man., $\dim M = m$, $\dim N = n$. A map $f: M \to N$ is differentiable (smooth, C^∞-) at $p \in M$, if $\exists$ parametrizations (U, φ_U) of M at p, $p \in \varphi(U)$, and (V, ψ_V) of N at $f(p)$, $f(\varphi_U(U)) \subseteq \psi_V(V)$, such that

$$\hat{f} := \psi_V^{-1} \circ f \circ \varphi_U : \underset{\mathbb{R}^m}{\overset{U}{\cap}} \to \mathbb{R}^n \quad \text{is smooth.}$$

f is differentiable on a subset of M if it is diff. at every point of this set.



Coordinate changes smooth $\Rightarrow$ Def 3 is independent on parametrization
$\hat{f} \equiv$ local repr. of $\hat{f}$ for (U, φ_U), $(V, \psi(V))$.

f is different. $\Rightarrow$ f is continuous;
f is diffeomorph. if bijective and f^{-1} is differentiable;
M, N diffeomorphic if $\exists$ a diffeomorphism $f: M \to N$;
f is local diffeomorphism on V: $f|_V : \underset{M}{\overset{V}{\cap}} \to \underset{N}{\overset{W}{\cap}}$ diffeom.

Tangent space

Def 4: Let $c: (-\epsilon, \epsilon) \to M$ be a diff. curve on a smooth man. M. The $C^\infty_p(M)$ denotes the set of all f-ions $M \to \mathbb{R}$ differentiable at $c(0) = p$. The tangent vector to c at $p \in M$ is the operator $\dot{c}(0): C^\infty_p(M) \to \mathbb{R}$

$$f \mapsto \dot{c}(0)(f) := \frac{d(f \circ c)}{dt}(0).$$

4

Choosing a parametrisation $(U, \varphi_U) : U \subseteq \mathbb{R}^n \to M$,
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad p\ \overset{\varphi}{}$

c is described as

$$\hat{c}(t) := (\varphi^{-1} \circ c)(t) = (x^1(t), \dots, x^n(t)) \subseteq U$$

Then

$$\dot{c}(0)(f) = \frac{d(f \circ c)}{dt}(0) = \frac{d}{dt}\Big|_{t=0}\Big(\underbrace{(f\circ\varphi)}_{\hat{f}}\circ\underbrace{(\varphi^{-1}\circ c)}_{\hat{c}}\Big) = \frac{d}{dt}\Big|_{t=0}\big(\hat{f}(x^1(t), \dots, x^n(t))\big)$$

$$= \sum_{i=1}^{n} \frac{\partial \hat{f}}{\partial x^i}(\hat{c}(0))\,\frac{dx^i}{dt}(0) = \left(\sum_{i=1}^{n} \dot{x}^i(0)\left(\frac{\partial}{\partial x^i}\right)_{\varphi^{-1}(0)}\right)(\hat{f}).$$

Here $\left(\frac{\partial}{\partial x^i}\right)_p$ denotes the operator associated to the vector tangent to c_i (i-th component of c in U) at p given in (U, φ_U) by

$$\hat{c}_i(t) = (x^1, \dots, x^{i-1}, x^i + t, x^{i+1}, \dots, x^n),$$
$$\varphi^{-1}(p) = (x^1, \dots, x^n).$$

~~Lemma 5: The vector space of~~

A tangent vector to M at p is a tangent vector to some diff. curve $c : (-\epsilon, \epsilon) \to M$, $0 \mapsto p$. The tangent space at p is the space T_pM of all tangent vectors at p.

<u>Lemma 5</u>: T_pM is an n-dim vector space.

$$(U, \varphi_U), \quad x_1, \dots, x_n \qquad T_pM = \left\langle \left(\frac{\partial}{\partial x_1}\right)_p, \left(\frac{\partial}{\partial x_2}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p \right\rangle$$

The tangent bundle

$$TM := \bigcup_{p\in M} T_pM = \{v \in T_pM \mid p \in M\}$$

admits canonical diff. structure (atlas) determined by M.
There is a natural projection $\pi : TM \to M$
$$(p, v) \mapsto p$$

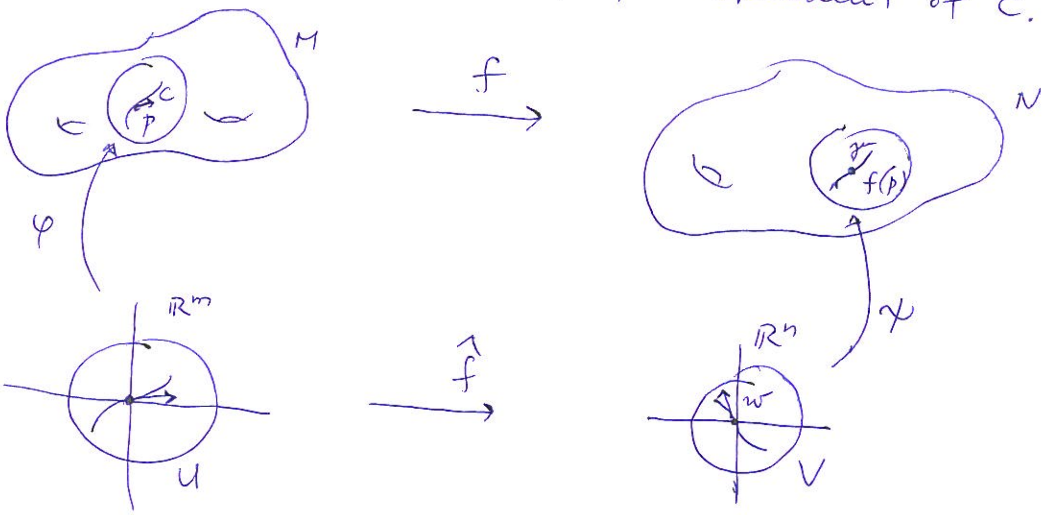
Differential $(df)_p$ of a differential map $f: M \to N$ at $p \in M$ is a linear transformation $(df)_p$ determined as follows:

<u>Def 6</u>: $f: M \to N$, $p \in M$. The differential (or, tangent map to f) at p is defined as

$$(df)_p : T_p M \to T_{f(p)} M$$
$$v \mapsto \frac{d(f \circ c)}{dt}(0)$$

$c: (-\epsilon, \epsilon) \to M$ is a curve satisfying $\begin{cases} c(0) = p \\ \dot{c}(0) = v. \end{cases}$

The map $(df)_p$ is a linear transform, independent of c.



$\gamma = f \circ c$

$\hat{c} = \varphi^{-1} \circ c \qquad c: (-\epsilon, \epsilon) \to M$

$\quad \overset{\shortparallel}{(x^1(t), \dots, x^m(t))}$

$\gamma = f \circ c \quad (-\epsilon, \epsilon) \to N$

$$\hat{\gamma} = (\psi^{-1} \circ \gamma) = (\psi^{-1} \circ f \circ \varphi) = (y^1 \circ x, \dots, y^n \circ x)$$

$\dot{\gamma}(0) \in T_{f(p)} N$

$\left.\begin{array}{l} v = (v^1, \dots, v^m) \text{ components of } v \text{ in } (U, \varphi_U) \\ w = (w^1, \dots, w^n) \quad \text{— " — } w \text{ in } (V, \psi_V) \end{array}\right\} \quad w = (df)_p(v)$

$$w^i = \sum_{j=1}^{m} \frac{\partial y^i}{\partial x^j} v^j$$

Here $\left(\frac{\partial y^i}{\partial x^j}\right)$ is an $(n \times m)$ Jacobian matrix of f in the local trivialization of f at $\varphi^{-1}(p)$.

df is called differential of f and defines a global map

$$(df) = f_* : TM \to TN.$$

There is chain rule for df:

$$\begin{tikzcd} \underset{p}{M} \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & \underset{f(p)}{N} \arrow[r, "g"] & \underset{g(f(p))}{P} \end{tikzcd}$$

is diff.

$$d(g\circ f)_p = (dg)_{f(p)} \circ (df)_p$$

Tangent bundle: $(U_\alpha, \varphi_\alpha)$ - diff. structure on M

$$\Phi_\alpha : U_\alpha \times \mathbb{R}^n \to TM|_{\varphi(U_\alpha)}$$

$$(x, v) \mapsto (d\varphi_\alpha)_x (v) \in T_{\varphi_\alpha(x)} M$$

The family $\{(U_\alpha \times \mathbb{R}^n, \Phi_\alpha)\}$ defines diff. structure on TM (TM is then smooth man. of dimension $2 \times \dim M$.)

Vector fields

A vector field on M is a ~~smooth~~ map $\quad X : M \to TM$

$$p \mapsto X(p) := X_p \in T_p M.$$

If X is differentiable, X is smooth vector field.

$\mathfrak{X}(M)$ ... the set of diff. vector fields.

Lemma: (U, φ_U), $W = \varphi_U(U) \subseteq M$, $x := \varphi^{-1} : W \to \mathbb{R}^n$ coordinate chart. Then, a map $X : W \to TW$ is a diff. vector field iff

$$X_p = X^1(p)\left(\frac{\partial}{\partial x^1}\right)_p + \dots + X^n(p)\left(\frac{\partial}{\partial x^n}\right)_p$$

for some diff. functions $X^i : M \to \mathbb{R}$, $i = 1, \dots, n$.

(+) Another characterization:

X is diff iff. $(X \cdot f) : M \to \mathbb{R}$ $\qquad f \in C^\infty(M)$

$$p \mapsto X_p \cdot f = X_p(f)$$

is differentiable function. $X \cdot f$ is directional derivative of f in the direction of f. Thus

$$X : C^\infty(M) \to C^\infty(M) \quad \text{is linear operator.}$$

Lemma:

Let $X, Y \in \mathfrak{X}(M)$ on M, then $\exists\ Z \in \mathfrak{X}(M)$ such that

$$Z \cdot f = (X \circ Y - Y \circ X) \cdot f \qquad \forall f \in C^\infty(M).$$

Pf: $x : W \subset M \to \mathbb{R}^n$ coordinate chart

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}, \quad Y = \sum_1^n Y^i \frac{\partial}{\partial x^i}.$$

Then

$$(X \circ Y - Y \circ X) \cdot f = X \cdot \left(\sum Y^i \frac{\partial \hat{f}}{\partial x^i} \right) - Y \cdot \left(\sum X^i \frac{\partial \hat{f}}{\partial x^i} \right)$$

$$= \sum \left((X \cdot Y^i) \frac{\partial \hat{f}}{\partial x^i} - (Y \cdot X^i) \frac{\partial \hat{f}}{\partial x^i} \right)$$

$$+ \underbrace{\sum_{i,j} \left(X^j Y^i \frac{\partial^2 \hat{f}}{\partial x^j \partial x^i} - Y^j X^i \frac{\partial^2 \hat{f}}{\partial x^j \partial x^i} \right)}_{= 0}$$

$$= \underbrace{\left(\sum_{i=1}^n (X \cdot Y^i - Y \cdot X^i) \frac{\partial}{\partial x^i} \right)}_{Z_p} \cdot f$$

and so $\exists\ Z \in \mathfrak{X}(M) : Z|_p = Z_p$ (because Z_p acts by derivation on $C^\infty(M)$ at each p)

This vector field is differentiable.

Z - Lie bracket of X, Y, $[X, Y]$

$X, Y \in \mathfrak{X}(M)$ commute if $[X, Y] = 0$.

Lemma: $X, Y, Z \in \mathfrak{X}(M)$

1/ Bilinearity: $\alpha, \beta \in \mathbb{R}$

$$[\alpha X + \beta Y, Z] = \alpha [X, Z] + \beta [Y, Z],$$

$$[X, \alpha Y + \beta Z] = \ldots$$

2/ Skew-symmetry:

$$[X, Y] = -[Y, X],$$

3/ Jacobi identity:

$$[[X, Y], Z] + [[Y, Z], Y] + [[Z, X], Y] = 0,$$

4/ Leibniz rule: $f, g \in C^\infty(M)$,

$$[fX, gY] = fg[X, Y] + f(X \cdot g)Y - g(Y \cdot f)X.$$

The space $\mathfrak{X}(M)$ of vector fields on M is an example of (infinite-dimensional) Lie algebra.

Def A vector space V equipped with an anti-symmetric bilinear map (Lie bracket)

$$[\ ,\ ] : V \times V \to V$$

satisfying the Jacobi identity is called a Lie algebra.
A linear map $F: V \to W$ between Lie algebras is a Lie algebra homomorphism if $F([v_1, v_2]) = [F(v_1), F(v_2)]$ $\forall v_1, v_2 \in V$.
If F is invertible, it is called Lie alg isomorphism.

$X \in \mathfrak{X}(M)$, $f: M \to N$ diffeomorphism,

$$(f_* X)_{f(p)} := (df)_p X_p$$

$$\begin{tikzcd} TM \arrow[r, dashed, "df"] & TN \\ M \arrow[u, "X"] \arrow[r, dashed, "f"'] \arrow[ur, phantom, "\circlearrowright"] & N \arrow[u, "(df)X"'] \end{tikzcd}$$

$X \in \mathfrak{X}(M)$ smooth vector field, an integral curve of X is a smooth curve $c: (-\epsilon, \epsilon) \to M$ such that $\dot{c}(t) = X_{c(t)}$. If $c(0) = p$, $c = c_p$ and call c_p an integral curve of X at p.

Local description: (U, φ_U), $\varphi_U : U \to M$

Integral curve is locally $\hat{c} := \varphi^{-1} \circ c \leftrightarrow c$, or

$$\dot{\hat{c}}(t) = \hat{X}(\hat{c}(t)) \quad \text{for } \hat{c} = \varphi^{-1} \circ c$$

$$\hat{X} = (d\varphi^{-1})(X \circ \varphi) = d\varphi^{-1} \circ X \circ \varphi \quad \text{local repr. of } X$$

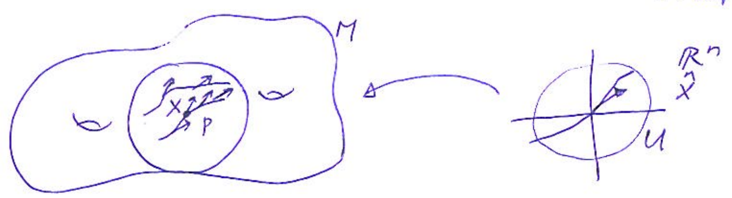
$\Updownarrow$

system of n ordinary diff. equat.:

$$\frac{d\hat{c}^i}{dt} = \hat{X}^i(\hat{c}(t)), \quad i = 1, \dots, n$$

The (local) existence and uniqueness follows from Picard–Lindelöf theorem of ordinary diff equations:

<u>Theorem</u>: M ... smooth manifold, $X \in \mathfrak{X}(M)$ ... smooth vector field on M. $p \in M$: $\exists$ an integral curve $c_p : I \to M$ of X at p, i.e., $\dot{c}_p(t) = X_{c_p(t)}$ for $t \in I = (-\epsilon, \epsilon)$ and $c_p(0) = p$. This curve is unique, i.e., any two such curves agree on the intersection of their domains.



Smooth dependence of ~~these~~ integral curves on $p \in M$ is the content of

<u>Theorem</u>: $X \in \mathfrak{X}(M)$. For $\forall p \in M$ $\exists$ $p \in W \subseteq M$, an interval $I = (-\epsilon, \epsilon)$ and a map $F : W \times I \to M$ such that

1/ For fixed $q \in W$, the curve $F(q, t)$, $t \in I$, is an integral curve of X at q: $F(q, 0) = q$, $\frac{\partial F(q,t)}{\partial t} = X_{F(q,t)}$;

2/ F is differentiable.

The map $F : W \times I \to M$ is called the <u>local flow</u> of X at $p \in M$.

Let us fix $t \in I$ and consider

$$\psi_t : W \to M$$
$$q \mapsto F(q, t) = c_q(t).$$

The maps $\psi_t : W \to M$ are local diffeomorphisms and satisfy

$$(\psi_t \circ \psi_s)(q) = \psi_{t+s}(q)$$

whenever $t, s, t+s \in I$, $\psi_s(q) \in W$.

(10) Let $X \in TM$, $\varphi = \varphi(X)$; a collection of diffeomorphisms $\{\varphi_t : M \to M\}_{t \in I}$, $I = (-\epsilon, \epsilon)$, satisfying $\varphi_t \circ \varphi_s = \varphi_{t+s}$ is called a local 1-parameter group of diffeomorphisms. When the interval of definition I of c_q is $\mathbb{R}$, this is 1-parameter group of diffeomorphisms of M. Such vector fields are called complete vector fields.

$X \in \mathfrak{X}(M)$, define Lie derivative of $f \in C^\infty(M, \mathbb{R})$:

$$(L_X f)(p) := \frac{d}{dt}\Big|_{t=0} (f \circ \varphi_t)(p),$$

when $\varphi_t = F(-, t)$, for F the local flow of X at p. We have

$$L_X f = X(f).$$

For $X, Y \in \mathfrak{X}(M)$, define the Lie derivative of Y in the direction of X as

$$L_X Y := \frac{d}{dt}\Big|_{t=0} \left((\varphi_{-t})_* Y \right),$$

where $(\varphi_t)_{t \in I}$ is the local flow of X. We have

1/ $L_X Y = [X, Y]$,

2/ $L_X [Y, Z] = [L_X Y, Z] + [Y, L_X Z]$, $\quad X, Y, Z \in \mathfrak{X}(M)$,

3/ $L_X \circ L_Y - L_Y \circ L_X = L_{[X,Y]}$.

Tensors and differential forms

V - n dim vectorspace (over $\mathbb{R}$, say). A k-tensor, $k \in \mathbb{N}$, is a multilinear map

$$\underbrace{V \times \dots \times V}_{k} \to \mathbb{R}.$$

This vector space is denoted $T^k(V^*)$.

1/ $T^1(V^*)$ = the dual space of V,

2/ $T^2(V^*)$ ... its elements are inner products on $\mathbb{R}^n$,

3/ $T^n(V^*)$ for $\dim_{\mathbb{R}} V = n$ ... its element is determinant.

$T \in T^k(V^*)$, $S \in T^m(V^*)$, tensor product is $(k+m)$-tensor $T \otimes S$:

$$(T \otimes S)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+m}) := T(v_1, \dots, v_k) S(v_{k+1}, \dots, v_{k+m})$$

for all $v_1, \dots, v_{k+m} \in V$.

$\{e_1^*, \dots, e_n^*\}$ – basis for $T^1(V^*) = V^*$, the set $\{e_{i_1} \otimes \dots \otimes e_{i_k} \mid 1 \le i_1, \dots, i_k \le n\}$ is a basis for $T^k(V^*)$ ($\Rightarrow \dim T^k(V^*) = n^k$)

If we start from V^* $\Rightarrow$ $T^k(V)$ [here $(V^*)^* \simeq V$] – contravariant tensors

Mixed tensors of type (k,m) – multilinear functions

$$\underbrace{V \times \dots \times V}_{k} \times \underbrace{V^* \times \dots \times V^*}_{m} \to \mathbb{R} \qquad T^{k,m}(V^*, V)$$

<u>Ex</u>: $T^{1,1}(V^*, V)$ … the space of linear maps $V \to V$

$$T^{1,1} \to \mathrm{End}(V)$$
$$T \mapsto (v \mapsto T(v, \circ))$$

Alternating tensors:

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -T(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$$
$$T \in \Lambda^k(V^*) \subseteq T^k(V^*)$$
$$T(v_1, \dots, v_k) = 0 \quad \text{if} \quad v_i = v_j,\ i \neq j \quad \text{for } T \in \Lambda^k(V^*)$$

<u>Ex</u>: The determinant is an alternating n-tensor on $\mathbb{R}^n$.

S_k … permutation group of $\{1, \dots, k\}$

$\sigma \in S_k$ set $\sigma(v_1, \dots, v_k) = (v_{\sigma(1)}, \dots, v_{\sigma(k)})$

For $T \in T^k(V^*)$, define an alternate k-tensor $\mathrm{Alt}(T) \in \Lambda^k(V^*)$ by

$$\mathrm{Alt}(T) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\,(T \circ \sigma)$$

, $\mathrm{sgn}(\sigma)$ is the sign of σ $= \pm 1$

This is projector: $\mathrm{Alt} \circ \mathrm{Alt} = \mathrm{Alt}$ ($\Leftarrow$ sgn is group homomorphism.)

Wedge product of alternating tensors:

$$\Lambda^k(V^*) \times \Lambda^m(V^*) \to \Lambda^k(V^*) \wedge \Lambda^m(V^*) \hookrightarrow \Lambda^{k+m}(V^*)$$
$$T, S \mapsto T \wedge S := \frac{(k+m)!}{k!\,m!} \mathrm{Alt}(T \otimes S)$$

<u>Lemma</u>: $T \in T^k(V^*)$, $S \in T^m(V^*)$. 1/ If $\mathrm{Alt}(T) = 0$ then

$$\mathrm{Alt}(T \otimes S) = \mathrm{Alt}(S \otimes T) = 0.$$

2/ $$\mathrm{Alt}(\mathrm{Alt}(T \otimes S) \otimes R) = \mathrm{Alt}(T \otimes S \otimes R) =$$
$$= \mathrm{Alt}(T \otimes \mathrm{Alt}(S \otimes R)).$$

proof: 1/ realize the sum over S_k and the quotient S_{k+m}/S_k.

2/ It follows from

$$\mathrm{Alt}(\mathrm{Alt}(S\otimes R) - S\otimes R) = 0$$

and apply it to the triple product $T\otimes S\otimes R$.

Lemma: For $T, S, R \in \Lambda^k(V^*), \Lambda^m(V^*), \Lambda^\ell(V^*)$, we have

$$(T\wedge S)\wedge R = T\wedge (S\wedge R).$$

Pf: We have $(T\wedge S)\wedge R = \dfrac{(k+m+\ell)!}{(k+m)!\,\ell!}\mathrm{Alt}((T\wedge S)\otimes R)$

$$= \frac{(k+m+\ell)!}{(k)!\,m!\,\ell!}\mathrm{Alt}(T\otimes S\otimes R).$$

Lemma: $\{e_1, \dots, e_n\}$ ... basis of V^*. Then the set

$$\{e_{i_1}\wedge\dots\wedge e_{i_k} \mid 1\le i_1<\dots<i_k\le n\}$$

is a basis for $\Lambda^k(V^*)$, and $\dim \Lambda^k(V^*) = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Moreover, $\wedge$ implies graded algebra structure on $\bigoplus \Lambda^*(V^*)$, with product

$$T\wedge S = (-)^{km} S\wedge T, \quad T\in\Lambda^k(V^*), \; S\in\Lambda^m(V^*).$$

A linear map $f: V\to W$ induces

$$f^*: T^k(W^*)\to T^k(V^*) \text{ by}$$

$$(f^*T)(v_1, \dots, v_k) = T(f(v_1), f(v_2), \dots, f(v_k)).$$

Lemma: V, W, Z vector spaces, $f: V\to W$, $h: W\to Z$ linear maps, $T\in T^k(W^*)$, $S\in T^m(W^*)$; Then

1/ $f^*(T\otimes S) = f^*T\otimes f^*S$,

2/ If T is alternating, then f^*T,

3/ $f^*(T\wedge S) = f^*T\wedge f^*S$,

4/ $(h\circ f)^* = f^*\circ h^*$.

(13) Let $T \in \Lambda^k(V^*)$, $v \in V$. We define the contraction of T by v, $i(v)T \in \Lambda^{k-1}(V^*)$, as

$$\big(i(v)T\big)(v_1, \dots, v_{k-1}) = T(v, v_1, \dots, v_{k-1}).$$

Show that
$$i(v_1)\big(i(v_2)T\big) = -\, i(v_2)\big(i(v_1)T\big),$$
if $T \in \Lambda^k(V^*)$, $S \in \Lambda^m(V^*)$, then
$$i(v)(T \wedge S) = \big(i(v)T\big) \wedge S + (-)^k\, T \wedge \big(i(v)S\big).$$

For $v_1^*, \dots, v_k^* \in V^*$, $w_1, \dots, w_k \in V$:
$$(v_1^* \wedge \dots \wedge v_k^*)[w_1 \wedge \dots \wedge w_k] = \det\big(v_i^*(w_j)\big) \in \mathbb{R}.$$

Tensor fields ....

vector field on M $\rightsquigarrow$ general tensor field on M

Def: We denote by T_p^*M the cotangent space of M at p, defined by $T_p^*M := \mathrm{Hom}(T_pM, \mathbb{R})$. A (k,m)- tensor field is a (smooth, differentiable) map, which to each $p \in M$ assigns a tensor $T \in T^{k,m}(T_p^*M, T_pM)$.

Ex: A vector field is a $(0,1)$ - tensor field (1- contravariant tensor field), assigning to each $p \in M$ tensor $X_p \in T_pM$.

Ex: $f: M \to \mathbb{R}$ diff. fcion. Define $(1,0)$ - tensor field (df):
$p \to (df)_p \in T_p^*M$, where $(df)_p : T_pM \to \underset{T_{f(p)}\mathbb{R}}{\overset{\parallel}{\mathbb{R}}}$. This is differential of f in the direction of T_pM (at $p \in M$), also called 1-form. For any $v \in T_pM$,
$(df)_p(v) = v(f)$.

In the coordinate chart $x: W \to \mathbb{R}^n$ we have $v = \sum v^i \left(\frac{\partial}{\partial x}\right)_p$, and so

$$(df)_p(v) = \sum_{i=1}^n v^i \frac{\partial \hat{f}}{\partial x^i}(x(p)),$$

where $\hat{f} = f \circ x^{-1}$. Taking the coordinate functions $x^i : W \to \mathbb{R}$, we obtain 1-forms dx^i on W. These satisfy

$$(dx^i)_p\left(\left(\frac{\partial}{\partial x^j}\right)_p\right) = \delta_{ij}$$

$\Rightarrow$ $\{dx^i\}_{i=1}^n$ is a basis of T_p^*M, dual to the coordinate basis $\left\{\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^n}\right)_p\right\}$ of T_pM.

Hence any $(1,0)$-tensor field on W can be written as $\omega = \sum \omega_i dx^i$, where $\omega_i : W \to \mathbb{R}$ is smooth fcion $\omega_i(p) = \omega_p\left(\left(\frac{\partial}{\partial x^i}\right)_p\right)$. In particular,

$$(df)_p = \sum_{i=1}^n \frac{\partial \hat{f}}{\partial x^i}(x(p))\,(dx^i)_p.$$

$T^*M = \bigcup_{p \in M} T_p^*M$ ... cotangent bundle (sheaf) of M

(15) The space of (k,m)-tensor fields is a vector space (since a lin. comb. of (k,m)-tensors is (k,m) tensor.) If W is a neighborhood of $p\in M$, $(dx^i)_p$ is a basis for T_p^*M, $\left(\frac{\partial}{\partial x^i}\right)_p$ is a basis for T_pM, hence

$$T_p = \sum a^{j_1\cdots j_m}_{i_1\cdots i_k}(p)(dx^{i_1})_p\otimes\cdots\otimes(dx^{i_k})_p\otimes\left(\frac{\partial}{\partial x^{j_1}}\right)_p\otimes\cdots\otimes\left(\frac{\partial}{\partial x^{j_m}}\right)_p$$

where $a^{j_1\cdots j_m}_{i_1\cdots i_k}: W\to\mathbb{R}$ are smooth functions, which at each $p\in W$ give compones of T_p relative to the bases of T_p^*M, T_pM.

An Important operation on covariant tensors is the pull-back by a smooth map:

Def: Let $f: M\to N$ be a diff. map between smooth manifolds M, N. Then $\forall$ differentiable k-covariant tensor field T on T defines a k-covariant tensor field (f^*T) on M:

$$(f^*T)_p(v_1,\dots,v_k) = T_{f(p)}((df)_p v_1,\dots,(df)_p v_k),$$

for $v_1,\dots,v_k\in T_pM$.

Remark: $(f^*T)_p$ is the image of $T_{f(p)}$ by $(df)_p^*: T^k(T^*_{f(p)}N)\to T^k(T_p^*M)$ induced by $(df)_p: T_pM\to T_{f(p)}N$. Therefore the properties

$$f^*(\alpha T+\beta S) = \alpha(f^*T)+\beta(f^*S)\qquad \alpha,\beta\in\mathbb{R}$$

$$f^*(T\otimes S) = (f^*T)\otimes(f^*S)$$ hold.

Ex: Lie derivative of a ~~ten~~ covariant tensor field:

$X\in\mathfrak{X}(M)$, Lie derivative of a k-covariant tensor field T along X as

$$L_XT := \frac{d}{dt}(\Psi_t^*T),$$

$\Psi_t = F(-,t)$ with F the local flow of T at p. Show that

$$L_X(T(Y_1,\dots,Y_k)) = (L_XT)(Y_1,\dots,Y_k) + T(L_XY_1,Y_2,\dots,Y_k)+\dots+T(Y_1,\dots,L_XY_k),$$ i.e.

$$X\cdot(T(Y_1,\dots,Y_k)) = (L_XT)(Y_1,\dots,Y_k) + T([X,Y_1],Y_2,\dots,Y_k)+\dots+T(Y_1,\dots,Y_{k-1},[X,Y_k])$$

$\forall\ Y_1,\dots,Y_k$.

<u>Def</u>: M - smooth manifold, a form of degree k on M is a field of alternating k-tensors, i.e. a map $\omega : M \to T^*M$

$$p \mapsto \omega_p \in \Lambda^k(T_p^*M)$$

The vector space of k-forms is denoted by $\Omega^k(M)$

In the coordinate chart $x : \underset{\substack{\cap \\ M}}{W} \to \mathbb{R}^n$, $p \in W$, a k-form can be written as

$$\omega = \sum_I \omega_I \, dx^I \quad , \quad I = (i_1, \dots, i_k) \text{ increasing sequence of integers in } \{1, \dots, n\}$$

$$dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

(ω is a smooth $(k,0)$-tensor, diff. form) $\quad \omega_I : W \to \mathbb{R}$ smooth fcions

$f : M \to N$, smooth map

<u>Lemma</u>: ω_1, ω_2 differentiable forms on N. Then

1/ $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$,

2/ $f^*(g\omega_1) = (g \circ f) f^*(\omega_1) = f^*(g) f^*(\omega_1) \quad \forall g \in C^\infty(N)$,

3/ $f^*(\omega_1 \wedge \omega_2) = (f^*\omega_1) \wedge (f^*\omega_2)$,

4/ $g^*(f^*\omega_1) = (f \circ g)^* \omega_1$, $\quad g \in C^\infty(L, M)$, L a diff. man.

$$\begin{tikzcd} L \arrow[r, "g"'] \arrow[rr, bend left, dotted, "f\circ g"] & M \arrow[r, "f"'] & N \end{tikzcd}$$

<u>Ex</u>: If $f : M \to N$ differentiable map, consider coordinate system $x : V \to \mathbb{R}^m$, $y : W \to \mathbb{R}^n$ resp. on M, N. We have $\hat{f}^i(x^1, \dots, x^m) = y^i$, $i = 1, \dots, n$ and $\hat{f} = y \circ f \circ x^{-1}$ the local repr. of f. If $\omega = \sum_I \omega_I \, dy^I$ is a k-form on W, then (by previous lemma)

$$(f^*\omega) = f^*\Big(\sum_I \omega_I \, dy^I\Big) = \sum_I (f^*\omega_I)(f^* dy^I) =$$

$$= \sum_I (\omega_I \circ f)(f^* dy^{i_1}) \wedge \dots \wedge (f^* dy^{i_k}).$$

Because for any $v \in T_pM$:

$$(f^*(dy^i))_p (v) = (dy^i)_{f(p)} ((df)_p v) = ((d(y^i \circ f))_p (v),$$

i.e. $f^*(dy^i) = d(y^i \circ f) = d f^*(y^i)$. Hence

$$(f^*\omega) = \sum_I (\omega_I \circ f)\, d(y^{i_1}\circ f)\wedge \cdots \wedge d(y^{i_k}\circ f)$$
$$= \sum_I (\omega_I \circ f)\, d(\hat{f}^{i_1}\circ x)\wedge \cdots \wedge d(\hat{f}^{i_k}\circ x).$$

Given any form ω on M and a parametrization $\varphi: U \to M$, can consider the pull-back of ω by φ and obtain a form on $U \subseteq \mathbb{R}^n$, called the local repr. of ω in that parametrization.

Ex. $x = \varphi^{-1}: \underset{\underset{M}{\cap}}{W} \to \mathbb{R}^n$ a coordinate system on M, consider the 1-form dx^i defined on W. The pull-back $\varphi^*(dx^i)$, $\varphi = x^{-1}$, is a 1-form on $U \subseteq \mathbb{R}^n$:

$$(\varphi^* dx^i)_x(v) = (\varphi^* dx^i)_x\left(\sum v^j \left(\frac{\partial}{\partial x^j}\right)_x\right) = (dx^i)_p\left(\sum_{j=1}^n v^j (d\varphi)_x\left(\frac{\partial}{\partial x^j}\right)_x\right)$$
$$= (dx^i)_p\left(\sum_{j=1}^n v^j\left(\frac{\partial}{\partial x^j}\right)_p\right) = v^i = (dx^i)_x(v).$$

$x \in U$, $p = \varphi(x)$, $v = \sum_{j=1}^n v^j\left(\frac{\partial}{\partial x^j}\right)_x \in T_x U$;

As we had $\left(\frac{\partial}{\partial x_i}\right)_p = (d\varphi)_x\left(\frac{\partial}{\partial x_i}\right)_x$, we now have $(dx^i)_x = \varphi^*(dx^i)_p$.

If $\omega = \sum_I \omega_I\, dx_I$ is a k-form on $U \subseteq \mathbb{R}^n$, we define a $(k+1)$-form called exterior derivative of ω: $d\omega = \sum_I d\omega_I \wedge dx_I$.

Lemma: $\alpha, \omega, \omega_1, \omega_2$ forms on $\mathbb{R}^n$. Then

1/ $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$,

2/ ω is k-form: $d(\omega\wedge\alpha) = d\omega\wedge\alpha + (-)^k \omega\wedge d\alpha$,

3/ $d(d\omega) = 0$,

4/ If $f: \mathbb{R}^m \to \mathbb{R}^n$ smooth func., $d(f^*\omega) = f^*(d\omega)$.

Riemannian manifolds

Introduce the concept of Riemm. metric, and then length, angle and volume.

On $\mathbb{R}^n$, all metric properties (distance, angles, volumes) are determined by canonical euclidean coordinates. On a general man. are no preferred coordinates, so have to introduce a Riemm. metric.

<u>Def</u>: A tensor $g_p \in T^2(T_p^*M)$ is said to be

1/ symmetric if $g_p(v,w) = g_p(w,v) \quad \forall\, v,w \in T_pM$,

2/ non-degenerate if $g_p(v,w) = 0 \quad \forall\, w \in T_pM \Rightarrow v = 0$,

3/ posit. def. if $g_p(v,v) > 0 \quad \forall\, x \in T_pM \setminus \{0\}$.

A covariant 2-tensor field is said to be symmetric, non-degenerate or pos. definite if g_p is — " — — " — $\forall p \in M$

If $x: V \to \mathbb{R}^n$ is a local chart, we have

$$g = \sum_{i,j=1}^{n} g_{ij}\, dx^i \otimes dx^j$$

in V, where

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \qquad \forall\, i,j = 1,\dots,n.$$

g is sym, non, pos. def. iff the matrix g_{ij} is of this form.

<u>Def</u>. A Riem. metric on M is a sym. pos. def. smooth covariant 2-tensor g.
A Riem. manifold is (M,g), where M is smooth man. and g is Riemm. metric.

We write $g_p(v,w) = \langle v, w\rangle_p$, $v,w \in T_pM$. For $M = \mathbb{R}^n$ is $g = \sum_{i,j=1}^{n} dx^i \otimes dx^i$ defines a Riem. metric.

<u>Prop</u>: Let (N,g) be a Riemm. man., $f: M \to N$ an immersion (a smooth map such that (df) is injective) Then $f^*(g)$ is a Riemm. metric on M (induced metric)

<u>Pf</u>: $p \in M$, $v, w \in T_pM$. Then

$$(f^*(g))_p(v,w) = g_{f(p)}((df)_p v, (df)_p w) \overset{\text{sym. of } g}{=} g_{f(p)}((df)_p w, (df)_p v) = (f^*g)_p(w,v)$$

(19) It is also clear that $(f^*g)_p(v,v) \geq 0 \quad \forall\, v \in T_pM$, and

$(f^*g)_p(v,v) = 0 \Rightarrow g_{f(p)}((df)_p v, (df)_p v) = 0 \Rightarrow (df_p)v = 0 \Rightarrow v = 0$

provided $(df)_p$ is injective. ∎

It is known that $\forall$ smooth Riemm. man. can be immersed in $\mathbb{R}^N$ for $N \gg 0$, and so any Riemm. metric on $\mathbb{R}^N$ induces a Riem. metric on M $\Rightarrow$ there are plenty of Riemm. metrics on M.

Ex: Computing f^*g, where g is Riemm. metric on $\mathbb{R}^{n+1}$ $\left(g = \sum_{i=1}^{n+1} dx^i \otimes dx^i\right)$ and $f: U \to \mathbb{R}^{n+1}$ is an immersion.

$U = \{x \in S^n \mid x^{n+1} > 0\}$

$$(f^*g)_{ij} = g_{ij} = \left\langle \frac{\partial \varphi}{\partial x^i}, \frac{\partial \varphi}{\partial x^j} \right\rangle_{\mathbb{R}^{n+1}} = \delta_{ij} + \frac{x^i x^j}{1-(x^1)^2-\ldots-(x^n)^2}$$

for $\varphi(x^1,\ldots,x^n) = \left(x^1,\ldots,x^n, \sqrt{1-(x^1)^2-\ldots-(x^n)^2}\right)$

Def: (M,g), (N,h) ... Riemm man. A diffeom. $f: M \to N$ is said to be an isometry if $f^*(h) = g$. Similarly for local diffeomorph + local isometry.

A Riemm. metric allows to compute the length $\|v\| = \langle v,v \rangle^{1/2}$, $v \in T_pM$, as well as an angle of two vectors. So

Def: If $(M, \langle\ ,\ \rangle)$ is a Riemm. man. and $c: [a,b] \to M$ a diff. curve. The length of c is
$$\ell(c) = \int_a^b \|\dot{c}(t)\|\, dt.$$

$\left.\begin{array}{l} a \to c(a) \\ b \to c(b) \end{array}\right\}$ pts on M

It is independent on parametrization.

Affine connections

For $X, Y \in \mathfrak{X}(\mathbb{R}^n)$ and Euclidean coordinates, one can define the directional derivative of (in fact, any tensor) Y along X. The non-existence of canonical coordinates and the local concept of coordinate chart leads to

Def: M... diff. manifold. An affine connection on M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

1/ $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,

2/ $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$,

3/ $\nabla_X (fY) = X(f)Y + f\nabla_X Y$,

for all $X, Y, Z \in \mathfrak{X}(M)$, $f, g \in \mathcal{F}(M)$.

$\nabla_X Y$ is a covariant derivative of Y along X.

Proposition: ∇... affine conn. on M, $X, Y \in \mathfrak{X}(M)$, $p \in M$. Then $(\nabla_X Y)_p \in T_p M$ depends only on X_p and the value of Y along a curve tangent to X at p. If $x : W \to \mathbb{R}^n$ are local coordinates on $W \subseteq M$, $p \in M$,

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}, \quad Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial x^i}, \quad \dim M = n$$

we have

$$\nabla_X Y = \sum_{i=1}^n \left(X \cdot Y^i + \sum_{j,k=1}^n \Gamma^i_{jk} X^j Y^k \right) \frac{\partial}{\partial x^i},$$

where n^3 - diff. functions $\Gamma^i_{jk} : W \to \mathbb{R}$, called Christoffel symbols, are defined by

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \sum_{i=1}^n \Gamma^i_{jk} \frac{\partial}{\partial x^i}.$$

Pf: The affine connection is local, i.e. if $X, Y \in \mathfrak{X}(M)$ coincide with $\tilde{X}, \tilde{Y} \in \mathfrak{X}(M)$ on some open neighborhood $W \subseteq M$, then $\nabla_X Y = \nabla_{\tilde{X}} \tilde{Y}$ on W (use the cut-off = bump fcion and 1/, 2/, 3/ above.)
$\Rightarrow$ $\nabla_X Y$ can be computed for vector fields supported on W only.
Let $x : W \to \mathbb{R}^n$ local coordinates and define Christ. symbols by $\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \sum_{i=1}^n \Gamma^i_{jk} \frac{\partial}{\partial x^i}$; then 1/, 2/, 3/ applied to

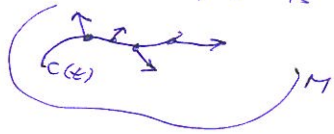
$$\nabla_X Y = \nabla_{\left(\sum_{i=1}^n X^i \frac{\partial}{\partial x^i}\right)} \left(\sum_{j=1}^n Y^j \frac{\partial}{\partial x^j} \right)$$

implies the required local formula and also ~~the~~ its dependence $\overset{(\nabla_X Y)_p}{\shortparallel}$

 on $X^i(p), Y^i(p)$ and $(X\cdot Y^i)(p) = X(Y^i)(p)$. Here $X^i(p), Y^i(p)$ depend on X_p, Y_p, and $(X\cdot Y^i)(p) = \frac{d}{dt}\Big|_{t=0} Y^i(c(t))$, depends on the values of Y^i (or Y) along c, $c(0)=p$, $c'(0)=X_p$. ∎

The choices of Christ. symbols on diff. charts are not independent, the covariant derivative must agree on the overlap.

A vector field defined along a diff. curve $c: I \to M$ is a diff. map $V: I \to TM$ such that $V(t) \in T_{c(t)}M$ $\forall t \in I$. An example is the tangent vector field $\dot{c}(t)$ to $c(t)$.



If V is a vector field defined along $c: I \to M$, $\dot{c} \neq 0$, its covariant derivative along c is the vector field (defined along c)

$$\frac{DV}{dt}(t) := \nabla_{\dot{c}(t)} V = \left(\nabla_X Y\right)_{c(t)}$$

$\forall X, Y \in \mathfrak{X}(M)$, $X_{c(t)} = \dot{c}(t)$ and $Y_{c(s)} = V(s)$ with $s \in (t-\epsilon, t+\epsilon)$ for some $\epsilon > 0$.
(such extensions always exists for $\dot{c}(t) \neq 0$)

In local coordinates $x: W \to \mathbb{R}^n$ with $x^i(t) = x^i(c(t))$ and

$$V(t) = \sum_{i=1}^n V^i(t) \left(\frac{\partial}{\partial x^i}\right)_{c(t)},$$

$$\left(\frac{D}{dt}V\right)(t) := \sum_{i=1}^n \left(\dot{V}^i(t) + \sum_{j,k=1}^n \Gamma^i_{jk}(c(t))\, \dot{x}^j(t)\, V^k(t)\right)\left(\frac{\partial}{\partial x^i}\right)_{c(t)}.$$

<u>Def:</u> A vector field V along $c: I \to M$ is parallel along c if

$$\left(\frac{D}{dt}V\right)(t) = 0, \qquad \forall t \in I.$$

The curve c is called geodesic of the connection ∇ if $\dot{c}$ is parallel along c, i.e.

$$\left(\frac{D}{dt}\dot{c}\right)(t) = 0, \quad \forall t \in I.$$

In local coordinates $x: W \to \mathbb{R}^n$, parallel condition is

$$\dot{V}^i + \sum \Gamma^i_{jk}\, \dot{x}^j\, V^k = 0 \qquad \forall i = 1, \dots, n.$$

(22) This is 1st order system of ODE's (for V^i, components of V). By Picard-Lindelöf theorem, for $c: I \to M$, $p \in c(I)$, $v \in T_pM$ $\exists!$ $V: I \to TM$ parallel along c and $V(0) = v$ – it is called parallel transport of v along c.

The geodesic equation is

$$\ddot{x}^i + \sum_{j,k=1}^{n} \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \quad i = 1, \dots, n.$$

2nd order (non-linear) ODE's for the coordinates $c(t)$. Then Picard-Lindelöf theorem implies, given $p \in M$ and $v \in T_pM$, there exists a unique geodesic $c: I \to M$ defined on a maximal open interval I such that $0 \in I$ and $c(0) = p$, $\dot{c}(0) = v$.

In local coordinates we have $(x: W \to \mathbb{R}^n)$

$$\nabla_X Y - \nabla_Y X = \sum_{i=1}^{n} \Big(X \cdot Y^i - Y \cdot X^i + \sum_{j,k=1}^{n} \Gamma^i_{kj} (X^j Y^k - Y^j X^k) \Big) \frac{\partial}{\partial x^i}$$

$$= [X, Y] + \sum_{i,j,k=1}^{n} (\Gamma^i_{jk} - \Gamma^i_{kj}) X^j Y^k \frac{\partial}{\partial x^i}.$$

Def: The torsion of an affine connection ∇ on M is $T: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ an operator given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \qquad \forall X, Y \in \mathfrak{X}(M)$$

The affine connection ∇ is symmetric if $T = 0$.

$T(X, Y)_p$ depends linearly on X_p, Y_p only, so defines $(2,1)$-tensor field on M; in local coordinates

$$T = \sum_{i,j,k=1}^{n} (\Gamma^i_{jk} - \Gamma^i_{kj}) dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}$$

(Note $\forall$ $(2,1)$ tensor $T \in T^{(2,1)}(V^*, V)$ is naturally identified with a bilinear map $\Phi_T: V^* \times V^* \to V \simeq (V^*)^*$ through $\Phi_T(v, w)(\alpha) := T(v, w, \alpha)$ $\forall v, w \in V$, $\alpha \in V^*$.)

In local coordinates, symmetric condition for ∇ is

$$\Gamma^i_{jk} = \Gamma^i_{kj} \quad \forall i, j, k = 1, \dots, n.$$

Exercise: ∇ - affine connection on M, $\omega \in \Omega^1(M)$, $X \in \mathfrak{X}(M)$ we define the covariant derivative of ω along X, $\nabla_X \omega \in \Omega^1(M)$, by

$$(\nabla_X \omega)(Y) = X\cdot(\omega(Y)) - \omega(\nabla_X Y) \qquad \forall\, Y \in \mathfrak{X}(M).$$

1/ Show that the formula for $\nabla_X \omega$ is a 1-form, i.e. $\big((\nabla_X \omega)(Y)\big)(p)$ is a linear function of Y_p.

2/
$$\nabla_{(fX+gY)}\omega = f\nabla_X \omega + g\nabla_Y \omega,$$
$$\nabla_X(\omega+\eta) = \nabla_X \omega + \nabla_X \eta,$$
$$\nabla_X(f\omega) = (X\cdot f)\,\omega + f\nabla_X \omega,$$
$\forall\, X, Y \in \mathfrak{X}(M)$, $f \in C^\infty(M)$, $\omega, \eta \in \Omega^1(M)$.

3/ Let $x: W \to \mathbb{R}^n$ local coordinates on an open set $W \subseteq M$, $\omega = \sum_{i=1}^n \omega_i\, dx^i$ a 1-form. Then

$$\nabla_X \omega = \sum_{i=1}^n \Big(X\cdot \omega_i - \sum_{j,k=1}^n \Gamma_{ji}^k X^j \omega_k \Big) dx^i, \qquad X = \sum_{j=1}^n X^j \frac{\partial}{\partial x^j}$$

a vector field.

Levi-Civita connection

In the case of Riemm. manifolds, there is a particular choice of connection, called Levi-Civita connection.

<u>Def:</u> A connection ∇ of a Riemannian ~~connection~~ manifold $(M, \langle -,- \rangle)$ is said to be compatible with the metric if

$$\forall X, Y, Z \in \mathfrak{X}(M). \quad X \cdot \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

If ∇ is compatible with the metric, the inner product of two vector fields V_1, V_2, parallel along a curve, is constant along the curve:

$$\frac{d}{dt} \langle V_1(t), V_2(t) \rangle = \langle \nabla_{\dot{c}(t)} V_1(t), V_2(t) \rangle + \langle V_1(t), \nabla_{\dot{c}(t)} V_2(t) \rangle = 0$$

Therefore, if $c: I \to M$ is a geodesic, then $\|\dot{c}(t)\| = k$ is a constant. The length s of the geodesic segment is

$$s = \int_{t_1}^{t_2} \|\dot{c}(t)\| \, dt = \int_{t_1}^{t_2} k \, du = k(t_2 - t_1),$$

so t is an affine function of its arclength s ($\Rightarrow$ t is affine parameter.)

<u>Th. (Levi-Civita):</u> If $(M, \langle , \rangle)$ is a Riemm. manifold, then there exists a unique connection ∇ on M which is symmetric and compatible with $\langle -,- \rangle$. In local coordinates $(x^1, \dots, x^m)$, $m = \dim M$, the Christoffel symbols are

$$\Gamma^i_{jk} = \frac{1}{2} \sum_{l=1}^{m} g^{il} \left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right),$$

where $g^{ij} = (g_{ij})^{-1}$.

<u>Pf:</u> $X, Y, Z \in \mathfrak{X}(M)$. If $\exists$ Levi-Civita connection, then

$$\left.\begin{aligned} X \cdot \langle Y, Z \rangle &= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \\ Y \cdot \langle X, Z \rangle &= \langle \nabla_Y X, Z \rangle + \langle X, \nabla_Y Z \rangle, \\ - Z \cdot \langle X, Y \rangle &= - \langle \nabla_Z X, Y \rangle - \langle X, \nabla_Z Y \rangle. \end{aligned}\right\} \Leftarrow \nabla \text{ is compatible with } g$$

Their sum is

$$2 \langle \nabla_X Y, Z \rangle = X \cdot \langle Y, Z \rangle + Y \cdot \langle X, Z \rangle - Z \cdot \langle X, Y \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle + \langle [X, Y], Z \rangle.$$

(continue $\langle Y, \nabla_X Z \rangle - \langle \nabla_Z X, Y \rangle = \langle Y, \nabla_X Z - \nabla_Z X \rangle = \langle Y, [X, Z] \rangle$, etc.)

(25) Since ∇ is symmetric,

$$
\left.\begin{aligned}
-\langle [X,Z], Y\rangle &= -\langle \nabla_X Z, Y\rangle + \langle \nabla_Z X, Y\rangle, \\
-\langle [Y,Z], X\rangle &= -\langle \nabla_Y Z, X\rangle + \langle \nabla_Z Y, X\rangle, \\
\langle [X,Y], Z\rangle &= \langle \nabla_X Y, Z\rangle - \langle \nabla_Y X, Z\rangle.
\end{aligned}\right\} \begin{array}{l} \nabla \text{ is symmetric, i} \\ T = \text{torsion is trivi} \end{array}
$$

The sum of previous expressions gives (Koszul formula)

$$
\begin{aligned}
2\langle \nabla_X Y, Z\rangle = {} & X\cdot\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\
& - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle + \langle [X,Y], Z\rangle.
\end{aligned}
$$

Since $\langle -, -\rangle$ is non-degenerate, Z is arbitrary $\Rightarrow$ $\nabla_X Y$ is determined (it is unique provided ∇ exists.)

The existence of Levi-Civita connection ∇: define it by Koszul formula. One easily proves it is a connection, and because

$$2\langle \nabla_X Y - \nabla_Y X, Z\rangle = 2\langle \nabla_X Y, Z\rangle - 2\langle \nabla_Y X, Z\rangle = 2\langle [X,Y], Z\rangle$$

hence ∇ is symmetric. Again, the same Koszul formula gives

$$2\langle \nabla_X Y, Z\rangle + 2\langle Y, \nabla_X Z\rangle = 2X\cdot\langle Y, Z\rangle$$

$\Rightarrow$ the connection is compatible with the metric.

In local coordinates $x = (x^1, \dots, x^m) : W \to \mathbb{R}$, we have

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0, \quad \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle = g_{ij},$$

so the Koszul formula yields

$$2\left\langle \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right\rangle = \frac{\partial}{\partial x^j} g_{kl} + \frac{\partial}{\partial x^k} g_{jl} - \frac{\partial}{\partial x^l} g_{jk}$$

$$\Leftrightarrow \left\langle \sum_{i=1}^m \Gamma^i_{jk} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^l}\right\rangle = \frac{1}{2}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

$$\Leftrightarrow \sum_{i=1}^m \Gamma^i_{jk} g_{il} = \frac{1}{2}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

and this is equivalent to the claim made above. ▨

(26) Example: Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Show that g is parallel along any curve, i.e.,

$$\nabla_X g = 0 \qquad \forall\, X \in \mathfrak{X}(M).$$

Use the defining property

$$(\nabla_X g)(Y, Z) = X \cdot g(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z).$$

Example: Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇, $\varphi_t : M \to M$ be a 1-parameter group of isometries, $\langle \varphi_* X, \varphi_* Y \rangle = \langle X, Y \rangle$. The vector field $X \in \mathfrak{X}(M)$, defined by

$$X_p := \frac{d}{dt}\Big|_{t=0} \varphi_t(p)$$

is called the Killing vector field associated to φ_t.

Show that

1/ $L_X g = 0$,

2/ $\langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle = 0$ for all $Y, Z \in \mathfrak{X}(M)$

Curvature

We introduce the curvature operator of a general affine connection, then we specialize to Riemannian manifolds and Ricci + scalar curvature.

It can be easily shown that no open subset of 2-sphere S^2 with standard round metric is isometric to an open subset of $\mathbb{R}^2$. The geometric object which distinguishes these two Riemannian manifolds is so called curvature operator.

(Notice isometric does not mean homeomorphic, conformal, etc.)

Def: The curvature of an affine connection ∇ is a map

$$R : \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$$

$$R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Hence R measures the deviation of ∇ preserving $\underset{\downarrow\nabla}{X\circ Y - Y\circ X = [X,Y]}$,

R is $(3,1)$-tensor, called Riemann tensor, meaning

1/ $R(fX_1+gX_2, Y)Z = fR(X_1,Y)Z + gR(X_2,Y)Z$,

2/ $R(X, fY_1+gY_2)Z = fR(X,Y_1)Z + gR(X,Y_2)Z$,

3/ $R(X,Y)(fZ_1+gZ_2) = fR(X,Y)Z_1 + gR(X,Y)Z_2$,

$\forall\, X,Y,Z\in\mathfrak{X}(M)$, $\forall\, f,g\in C^\infty(M)$.
X_1, Y_1, Z_1
X_2, Y_2, Z_2

For example, we have $\left([fX,Y] = f[X,Y] - Y(f)X\right)$

$$\nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX} - \nabla_{[fX,Y]}Z = f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z)$$

$$-f\nabla_{[X,Y]}Z + Y(f)\nabla_X Z = f\nabla_X\nabla_Y Z - Y(f)\nabla_X Z - f\nabla_Y\nabla_X Z$$

$$-f\nabla_{[X,Y]}Z + Y(f)\nabla_X Z = f(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z$$

and similarly in the other cases.

Choosing the coordinate system $x : \underset{W}{\overset{M}{\cup}} \to \mathbb{R}^m$, the curvature tensor can be written as

$$R = \sum_{i,j,k,l=1}^{m} R_{ijk}{}^{l}\, dx^i\otimes dx^j\otimes dx^k\otimes,$$

where each function (coefficient) $R_{ijk}{}^{l}$ is the l-th coordinate of the vector field $R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}$: $R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^j} = \sum_{\ell=1}^{n} R_{ijk}{}^{\ell}\frac{\partial}{\partial x^\ell}$.

Because $\left[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right]=0$ for $\forall i,j,\ i\neq j$,

$$R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = \nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k} =$$

$$= \nabla_{\frac{\partial}{\partial x^i}}\left(\sum_{\ell=1}^{m}\Gamma_{jk}^{\ell}\frac{\partial}{\partial x^\ell}\right) - \nabla_{\frac{\partial}{\partial x^j}}\left(\sum_{\ell=1}^{m}\Gamma_{ik}^{\ell}\frac{\partial}{\partial x^\ell}\right)$$

$$= \sum_{\ell=1}^{m}\left(\frac{\partial}{\partial x^i}\Gamma_{jk}^{\ell} - \frac{\partial}{\partial x^j}\Gamma_{ik}^{\ell}\right)\frac{\partial}{\partial x^\ell} + \sum_{\ell,s=1}^{m}\left(\Gamma_{jk}^{s}\Gamma_{is}^{\ell} - \Gamma_{ik}^{s}\Gamma_{js}^{\ell}\right)\frac{\partial}{\partial x^\ell}$$

$$= \sum_{\ell=1}^{m}\left(\frac{\partial\Gamma_{jk}^{\ell}}{\partial x^i} - \frac{\partial\Gamma_{ik}^{\ell}}{\partial x^j} + \sum_{s=1}^{m}\Gamma_{jk}^{s}\Gamma_{is}^{\ell} - \sum_{s=1}^{m}\Gamma_{jk}^{s}\Gamma_{js}^{\ell}\right)\frac{\partial}{\partial x^\ell}$$

$$\Rightarrow\quad R_{ijk}{}^{\ell} = \frac{\partial\Gamma_{jk}^{\ell}}{\partial x^i} - \frac{\partial\Gamma_{ik}^{\ell}}{\partial x^j} + \sum_{s=1}^{m}\Gamma_{jk}^{s}\Gamma_{is}^{\ell} - \sum_{s=1}^{m}\Gamma_{ik}^{s}\Gamma_{js}^{\ell}$$

Ex: $M=\mathbb{R}^m$, $x=(x^1,\dots,x^m)$, then $\Gamma_{jk}^{i}=0$ $\forall i,j,k=1,\dots,m$
and so $R_{ijk}{}^{\ell}=0$ $\forall i,j,k,\ell=0,\dots,m$.

When the connection is symmetric (as in the case of Levi-Civita connection) R satisfies the (so called) Bianchi identities):

Prop.: (Bianchi identity) If M is a smooth manifold with symmetric connection, then the associated curvature satisfies

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

$\forall X,Y,Z\in\mathfrak{X}(M)$.

Pf: A direct consequence of the Jacobi identity for vector fields:

$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y =$

$\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X$

$+\nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y =$

$\nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z) + \nabla_Z(\nabla_X Y - \nabla_Y X)$

$-\nabla_{[X,Y]}Z - \nabla_{[Y,Z]}X - \nabla_{[Z,X]}Y \overset{\nabla \text{ is symmetric}}{=}$

$$\nabla_X[Y,Z] + \nabla_Y[Z,X] + \nabla_Z[X,Y] - \nabla_{[Y,Z]}X - \nabla_{[Z,X]}Y - \nabla_{[X,Y]}Z \overset{\nabla \text{ is symmetr.}}{=}$$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \overset{\text{Jacobi ident.}}{=} 0$$

$\boxtimes$

Let (M,g) be a Riemman. man., ∇ - Levi-Civita connection. We define a new covariant 4-tensor out of Riemm. curvature operator $(3,1)$, known as a curvature tensor:

$$R(X,Y,Z,W) := g(R(X,Y)Z,W) \quad \forall X,Y,Z,W \in \mathcal{X}(M)$$

(Due to non-degeneracy has the same info as curvature operator.)

In the coordinate system $x: \overset{M}{\underset{}{\overset{\cup}{W}}} \to \mathbb{R}^m$,

$$R(X,Y,Z,W) = \Big(\sum_{i,j,k,l=1}^{m} R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l\Big)(X,Y,Z,W),$$

where

$$R_{ijkl} = g\Big(R\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\Big) = g\Big(\sum_{s=1}^{m} R_{ijk}{}^{s}\frac{\partial}{\partial x^s},\frac{\partial}{\partial x^l}\Big) =$$

$$= \sum_{s=1}^{m} R_{ijk}{}^{s} g_{sl}.$$

This tensor satisfies several symmetry properties.

<u>Prop:</u> $X,Y,Z,W \in \mathcal{X}(M)$ on smooth manifold M, ∇ - Levi-Civita connection, then

1/ $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0,$

2/ $R(X,Y,Z,W) + R(Y,X,Z,W) = 0,$

3/ $R(X,Y,Z,W) = -R(X,Y,W,Z),$

4/ $R(X,Y,Z,W) = R(Z,W,X,Y).$

<u>Pf:</u> 1/ $\Leftarrow$ Bianchi identity,

2/ $\Leftarrow$ elementary

3/ This property is equivalent to showing

$$R(X,Y,Z,Z) = 0;$$

Indeed, if 3/ is true $\Rightarrow R(X,Y,Z,Z) = 0$;

Conversely, polarization in the last two components gives

(30) $R(X,Y,Z+W,Z+W)=0 \Leftrightarrow R(X,Y,Z,W)+R(X,Y,W,Z)=0$

$\forall X,Y,Z,W \in \mathfrak{X}(M)$

Now the metric compatibility of the Levi-Civita connection gives

$$X\cdot\langle\nabla_Y Z, Z\rangle = \langle\nabla_X\nabla_Y Z, Z\rangle + \langle\nabla_Y Z, \nabla_X Z\rangle,$$

$$[X,Y]\langle Z,Z\rangle = \langle\nabla_{[X,Y]}Z, Z\rangle + \langle Z, \nabla_{[X,Y]}Z\rangle$$
$$= 2\langle\nabla_{[X,Y]}Z, Z\rangle.$$

Hence,

$$R(X,Y,Z,Z) = \langle\nabla_X\nabla_Y Z, Z\rangle - \langle\nabla_Y\nabla_X Z, Z\rangle - \langle\nabla_{[X,Y]}Z, Z\rangle$$

$Y\cdot\langle Z,Z\rangle = \langle\nabla_Y Z, Z\rangle + \langle Z, \nabla_Y Z\rangle$
$= 2\langle\nabla_Y Z, Z\rangle$

$$= X\cdot\langle\nabla_Y Z, Z\rangle - \langle\nabla_Y Z, \nabla_X Z\rangle - Y\cdot\langle\nabla_X Z, Z\rangle$$
$$+ \langle\nabla_X Z, \nabla_Y Z\rangle - \tfrac{1}{2}[X,Y]\langle Z,Z\rangle$$
$$= \tfrac{1}{2}X\cdot(Y\cdot\langle Z,Z\rangle) - \tfrac{1}{2}Y\cdot(X\cdot\langle Z,Z\rangle) - \tfrac{1}{2}[X,Y]\langle Z,Z\rangle$$
$$= \tfrac{1}{2}[X,Y]\langle Z,Z\rangle - \tfrac{1}{2}[X,Y]\langle Z,Z\rangle = 0,$$

and 3/ is proved.

To prove 4/, 1/ $\Rightarrow$

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0,$$
$$R(Y,Z,W,X) + R(Z,W,Y,X) + R(W,Y,Z,X) = 0,$$
$$R(Z,W,X,Y) + R(W,X,Z,Y) + R(X,Z,W,Y) = 0,$$
$$R(W,X,Y,Z) + R(X,Y,W,Z) + R(Y,W,X,Z) = 0,$$

and adding these + 3/ $\Rightarrow$

$$R(Z,X,Y,W) + R(W,Y,Z,X) + R(X,Z,W,Y) + R(Y,W,X,Z) = 0.$$

By 2/ and 3/ $\Rightarrow$

$$2\big(R(Z,X,Y,W) - R(Y,W,Z,X)\big) = 0$$

and the result follows. ∎

An equivalent way to encode curvature on Riemm. manifold follows from

Def: $\Pi \subseteq T_pM$ be a 2-dim subspace, X_p, Y_p a basis of Π. Then the sectional curvature of Π is defined by

$$K(\Pi) := - \frac{R(X_p, Y_p, X_p, Y_p)}{\|X_p\|^2 \|Y_p\|^2 - \langle X_p, Y_p\rangle^2}$$

The result of the change of basis of Π results in the change by square of determinant of the change of basis matrix of both nominator and denominator.

Prop: The Riemm. curvature tensor at p is uniquely determined by the value of the sectional curvatures on subspaces $\Pi \subseteq T_pM \times T_pM$.

Pf: Let R_1, R_2 be two covariant 4-tensors satisfying the (symmetry) conditions in the previous Proposition. Then $T := R_1 - R_2$ is a tensor with the same properties as R_1, R_2.
We prove that if $R_1(X_p, Y_p, X_p, Y_p) = R_2(X_p, Y_p, X_p, Y_p)$ $\forall X_p, Y_p \in T_pM$, i.e., $T(X_p, Y_p, X_p, Y_p) = 0$ $\forall X_p, Y_p \in T_pM$, then $R_1 = R_2$ (that is, $T = 0$). Indeed, for all $X_p, Y_p, Z_p \in T_pM$ we have

$$0 = T(X_p + Z_p, Y_p, X_p + Z_p, Y_p) = T(X_p, Y_p, Z_p, Y_p) + T(Z_p, Y_p, X_p, Y_p)$$
$$= 2\,T(X_p, Y_p, Z_p, Y_p)$$

$\Rightarrow$

$$0 = T(X_p, Y_p + W_p, Z_p, Y_p + W_p) = T(X_p, Y_p, Z_p, W_p) + T(X_p, W_p, Z_p, Y_p)$$
$$= T(Z_p, W_p, X_p, Y_p) - T(W_p, X_p, Z_p, Y_p)$$

$\Rightarrow T(Z_p, W_p, X_p, Y_p) = T(W_p, X_p, Z_p, Y_p)$

$\Rightarrow$ T is invariant by cyclic permutations of first three elements

$\Rightarrow$ Bianchi identity $3\,T(X_p, Y_p, Z_p, W_p) = 0$. The proof is complete. $\square$

For M, $\dim M = 2$, the sectional curvature $K(p) = K_p$ is called Gauss curvature. For instance, its integral over a disk $D \subseteq M$ measures angle of difference (rotation) of a vector when $\parallel$ transported around the boundary of D.

Prop: Let K_p be constant on $\Pi \subseteq T_pM \times T_pM$, and $x: W \to \mathbb{R}^n$ be a coordinate system around $p \in M$. Then

$$R_{ijkl}(p) = -K_p(g_{ik}g_{jl} - g_{il}g_{jk}).$$

Pf: The 4-tensor on T_pM,

$$A := \sum_{i,j,k,l=1}^{m} -K_p(g_{ik}g_{jl} - g_{il}g_{jk})\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

has the symmetries of key Proposition. Moreover,

$$A(X_p, Y_p, X_p, Y_p) = \sum_{i,j,k,l=1}^{m} -K_p(g_{ik}g_{jl} - g_{il}g_{jk}) X_p^i Y_p^j X_p^k Y_p^l$$

$$= -K_p(\langle X_p, X_p\rangle\langle Y_p, Y_p\rangle - \langle X_p, Y_p\rangle^2)$$

$$= R(X_p, Y_p, X_p, Y_p), \quad \forall X_p, Y_p \in T_pM,$$

and the previous proposition implies $A = R$. ☒

Def: A Riemann. manifold is called a manifold of constant curvature if the sectional curvature K_p is constant on $\Pi \subseteq T_pM \times T_pM$ and K_p is the same at all $p \in M$.

Def: The Ricci curvature tensor is the covariant $(2,0)$-tensor, in a local chart $x: W \to \mathbb{R}^m$ defined by

$$Ric(X, Y) := \sum_{k=1}^{m} dx^k\left(R\left(\frac{\partial}{\partial x^k}, X\right)Y\right).$$

This definition is independent of the choice of coordinates, because it is trace of the linear map

$$T_pM \longrightarrow T_pM,$$
$$Z_p \mapsto Ric_p(Z_p, X_p)Y_p,$$

hence independent of a basis.

It is symmetric, because for $\{E_1, \dots, E_m\}$ an ON-basis of T_pM,

$$Ric_p(X_p, Y_p) = \sum_{k=1}^{m} g(R(E_k, X_p)Y_p, E_k) = \sum_{k=1}^{m} R(E_k, X_p, Y_p, E_k)$$

$$= \sum_{k=1}^{m} R(Y_p, E_k, E_k, X_p) = \sum_{k=1}^{m} R(E_k, Y_p, X_p, E_k)$$

$$= Ric_p(Y_p, X_p).$$

Locally, we can write

$$Ric = \sum_{i,j=1}^{m} R_{ij}\, dx^i \otimes dx^j ,$$

$$R_{ij} := Ric\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_{k=1}^{m} dx^k\left(R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i}\right)\frac{\partial}{\partial x^j}\right)$$

$$= \sum_{k=1}^{m} R_{kij}{}^{k}, \qquad R_{ij} := \sum_{k=1}^{m} R_{kij}{}^{k}.$$

Ric is an example of $\quad (3,1)\text{-tensor} \xrightarrow{\text{contraction}} (2,0)\text{-tensor}$

Similarly, we define contraction of Ricci tensor and produce scalar curvature by $\quad (2,0) \xrightarrow{g} (1,1) \xrightarrow{\text{contraction}} (0,0)$:

define $(1,1)$ tensor T by $\quad T(X,\omega) := Ric(X,Y),$

$$\omega(Z) = g(Y,Z) \quad \forall\, Z \in \mathfrak{X}(M),$$

and then the scalar curvature $S: M \to \mathbb{R}$ in $x: W \to \mathbb{R}^m$ is

$$S(p) = \sum_{k=1}^{m} T\left(\frac{\partial}{\partial x^k}, dx^k\right) = \sum_{k=1}^{m} Ric\left(\frac{\partial}{\partial x^k}, Y_k\right),$$

where $\forall$ vector field Z on W,

$$Z^k = dx^k(Z) = g\left(Z, \frac{\partial}{\partial x^k}\right)$$

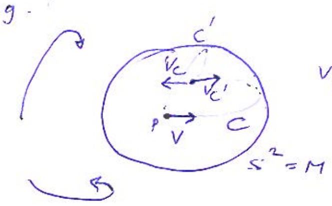
The scalar curvature is

$$S(p) = \sum_{k=1}^{m} Ric\left(\frac{\partial}{\partial x^k}, \sum_{i=1}^{m} g^{ik}\frac{\partial}{\partial x^i}\right) = \sum_{i,k=1}^{m} R_{ki}\, g^{ik} = \sum_{i,k=1}^{m} g^{ik} R_{ik}.$$

(34) Geometrical meaning of Riemann curvature and torsion tensors

(M,g) .. Riemmann, $\parallel$ transport of a vector V at p to q, $p,q \in M$, $V \in T_pM$, along two different curves $C, C' \subseteq M$ gives different vector at $q \in M$.

E.g.:



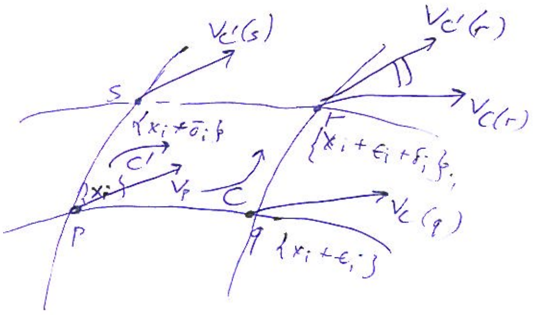
$V_C \neq V_{C'}$ in the case of two geodesic and V a vector tangent to C at p

This differs from $\mathbb{R}^n$ with trivial connection, where the notion of $\parallel$ transport does not depend on a curves between any 2 points.

$\Rightarrow$ intrinsic characterization of curvature.

Take $p,q,r,s \in M$,

$$\left.\begin{aligned} p &= \{x_i\}_i \\ q &= \{x_i+\epsilon_i\}_i \\ s &= \{x_i+\delta_i\}_i \\ r &= \{x_i+\epsilon_i+\delta_i\}_i \end{aligned}\right\} \text{infin. closed } (\epsilon_i \to 0,\ \delta_i \to 0)$$



$\parallel$ transport $V_0 \in T_pM$ along $C = (pqr) \Rightarrow V_C(r) \in T_rM$

is given by the composition along C:

$p \to q$: $V^i_C(q) = V^i_0 - V^j_0 \Gamma^i_{kj} \epsilon^k$

$q \to r$:
$$\begin{aligned} V^i_C(r) &= V^i_C(q) - V^k_C(q)\Gamma^i_{jk}(q)\delta^j \\ &= V_0^i - V_0^j \Gamma^i_{ej}(p)\epsilon^e - \left[V^k_0 - V^r_0 \Gamma^k_{sr}(p)\epsilon^s\right]\cdot \\ &\qquad \left[\Gamma^i_{jk}(p) + \partial_e \Gamma^i_{jk}(p)\epsilon^e\right]\delta^j \\ &= V^i_0 - V^k_0 \Gamma^i_{jk}(p)\epsilon^j - V^k_0\Gamma^i_{jk}(p)\delta^j \\ &\quad - V^j_0\left[\partial_e \Gamma^i_{rj}(p) - \Gamma^s_{ej}(p)\Gamma^i_{rs}(p)\right]\epsilon^e\delta^r \\ &\quad + O(\epsilon^2, \delta^2) \end{aligned}$$

Similarly, $\parallel$ transport of $V_0 \in T_pM$ along $C' = (psr)$ gives another vector $V_{C'}(r) \in T_rM$:

$$\begin{aligned} V^i_{C'}(r) &= V^i_0 - V^k_0 \Gamma^i_{jk}(p)\delta^j - V^k_0 \Gamma^i_{jk}(p)\epsilon^j \\ &\quad - V^k_0\left[\partial_e \Gamma^i_{rk}(p) - \Gamma^s_{ek}(p)\Gamma^i_{rs}(p)\right]\epsilon^r\delta^e \end{aligned}$$

(35) and the difference

$$V^i_c(r) - V^i_{c'}(r) = V^k_0 [\partial_\ell \Gamma^i_{jk}(p) - \partial_j \Gamma^i_{\ell k}(p)$$
$$- \Gamma^s_{\ell k}(p)\Gamma^i_{js}(p) + \Gamma^s_{jk}(p)\Gamma^i_{\ell s}(p)] \epsilon^\ell \delta^j$$

$$= V^k_0 R^{*i}_{k\ell j}(p) \epsilon^\ell \delta^j$$

characterization of torsion:

$p \in M$, $p = \{x_i\}_i$ in the chart (U, x), $X_p, Y_p \in T_p M$ $\quad Y_p = \sum_j \delta^j \left(\frac{\partial}{\partial x^j}\right)_p$, $X_p = \sum_j \epsilon^j \left(\frac{\partial}{\partial x^j}\right)_p$

$\epsilon^i \left(\frac{\partial}{\partial x_i}\right)_p$ ... integral curve for X_p defines the point $q = \{x_i + \epsilon_i\}_i$

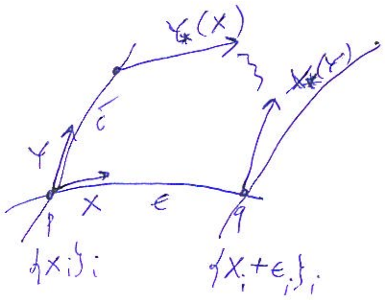
$\delta^i \left(\frac{\partial}{\partial x_i}\right)_p$ ——— " ——— $s = \{x_i + \delta_i\}_i$

and now 1/ ∥ transport X along ps get a vector $\left(\epsilon^i + \epsilon^j \Gamma^i_{\ell j} \delta^\ell\right)\frac{\partial}{\partial x^i}$

2/ ∥ —"— Y —"— pq —"— $\left(\delta^i - \delta^j \Gamma^i_{\ell j} \epsilon^\ell\right)\frac{\partial}{\partial x^i}$

and the difference

$$Y + Y_*(X) - (X + X_*(Y))$$
$$= \left(\Gamma^i_{jk} - \Gamma^i_{kj}\right)\epsilon^k \delta^j + O(\epsilon^2, \delta^2)$$
$$= T^i_{kj} \epsilon^k \delta^j + O(\epsilon^2, \delta^2)$$



T = torsion = measure of non-closedness of the parallelogram build out of small displacement and the ∥ transport.

$\boxed{\text{Minimizing properties of geodesics \& normal neighborhood \& exp-map}}$:

M... smooth man., ∇ ... affine connection,

$\forall p \in M$, $v \in T_p M$ $\exists!\ c_v : I \to M$ a unique geodesic defined on maximal $I \subseteq \mathbb{R}$: $c_v(0) = p$, $\dot{c}_v(0) = v$.

Consider the curve $\gamma_a : J \to M$

$t \mapsto \gamma(t) := c_v(at)$, $a \in \mathbb{R}$, J is the preimage of I by map $t \to at$

We have $\dot{\gamma}(t) = a\, \dot{c}_v(at)$ $\Rightarrow$

$\nabla_{\dot\gamma} \dot\gamma = \nabla_{a\dot{c}(t)}(a\dot{c}_v) = a^2 \nabla_{\dot{c}_v}\dot{c}_v = 0$ $\Rightarrow$ γ is geodesic as well.

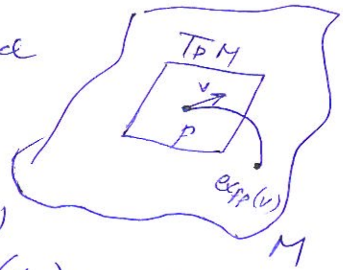
 Since $\gamma(0) = c_v(0) = p$, $\dot\gamma(0) = a\dot c_v(0) = av$ $\Rightarrow$ γ is the unique geodesic with $av \in T_pM$ boundary cond. ($\gamma = c_{av}$). Therefore, $c_{av}(t) = c_v(at)$, $\forall t \in I$, so $c_{av}(-) = c_v(a-)$ referred to as the homogeneity of geodesics. I can be arbitrarily large by making $\underline{a}$ arbitrarily small.

If $1 \in I$, define $\exp_p(v) = c_v(1)$; by homogeneity of geodesics, can define $\exp_p(v)$ for an open neigh. $U \subseteq T_pM$. The map $\exp_p : U \subseteq T_pM \to M$ is the exponential map at p.

<u>Lemma</u>: There exists an open set $U \subseteq T_pM$, $0 \in U$, such that $\exp_p : U \to M$ is a diffeomorphism onto an open subset $V \subseteq M$ containing $p \in M$ (a normal neighborhood.)

<u>Pf</u>:

The exp. map is diff. $\Leftarrow$ smooth dependence of an ODE on initial data.



If $v \in T_pM$ such that $\exp_p(v)$ is defined, we have (by homogeneity) $\exp_p(tv) = c_{tv}(1) = c_v(t)$, so

$$(d\exp_p)_0\, v = \frac{d}{dt}\exp_p(tv)\Big|_{t=0} = \frac{d}{dt}\Big|_{t=0} c_v(t) = v,$$

$$(d\exp_p)_0 : T_0(T_pM) \simeq T_pM \to T_pM,$$

so $(d\exp_p)_0$ is the identity map. By inverse function theorem, $\exp_p$ is diff. of some open neighb. U of $0 \in T_pM$ onto $V \subseteq M$ containing $p = \exp_p(0)$. $\boxtimes$

<u>Ex</u>: LC-connection in S^2, standard metric, $p \in S^2$. Then $\exp_p(v)$ is well-defined $\forall v \in T_pS^2$, but is not diff. (it is not injective.) Its restriction to $B_\pi(0) \subseteq T_pS^2$ is a diff. onto $S^2 \setminus \{-p\}$.

(37) (M,g) ... Riemm man, ∇ .- LC connection; $\langle\, ,\,\rangle = g_p$ makes T_pM m-dim. unitary space. Let E be ~~...~~ the vector ~~...~~ on $T_pM\setminus\{0\}$, defined by $E_v = \frac{v}{\|v\|}$ $(g_p(E_v,E_v)=1)$ $\forall v\in T_pM\setminus\{0\}$.

Define $\underset{\overset{\cap}{TM}}{X} := (\exp_p)_* E$ on $V\setminus\{p\}$, $V\subseteq M$ a normal neigh

We have

$$X_{\exp_p(v)} = (d\exp_p)_v E_v = \frac{d}{dt}\Big|_{t=0} \exp_p\left(v + t\frac{v}{\|v\|}\right)$$

$$= \frac{d}{dt}\Big|_{t=0} c_v\left(1+\frac{t}{\|v\|}\right) = \frac{1}{\|v\|}\dot{c}_v(1).$$

Since $\|\dot{c}_v(1)\| = \|\dot{c}_v(0)\| = \|v\|$, $X_{\exp_p(v)}$ is the unit tangent vector to the geodesic c_v, in particular satisfies $\nabla_X X = 0$.

For $\epsilon>0$ such that $\overline{B_\epsilon(0)} \subseteq U := \exp_p^{-1}(V)$, we define the normal ball with center $p\in M$ and radius $\epsilon>0$ as an open set $B_\epsilon(p) := \exp_p(B_\epsilon(0))$, and the normal sphere of radius ϵ at p as the compact submanifold $S_\epsilon(p) := \exp_p(\partial\overline{B_\epsilon(0)})$. One can easily prove that X is (so the geodesics through $p\in M$ are) orthogonal to normal spheres (with respect to g.)

Cartan structure equations

Properties of the L-C connection & Riemann curvature tensor might be reformulat in terms of differential forms. (M, g) Riemann.

$V \subseteq M$ open subset, field frames $\{X_1, \dots, X_n\}$: $\forall i$ $X_i \in TM|_V$

$\forall p \in V$: $\{X_1|_p, \dots, X_n|_p\}$ is a basis of T_pM

$\{X_1, \dots, X_n\}$ might be coordinate vector fields, for example.

Fields of dual coframes, 1-forms $\{\omega^1, \dots, \omega^n\}$ on V : $\omega^i(X_j) = \delta^i_j$.

This implies $\{\omega^1|_p, \dots, \omega^n|_p\}$ is a basis of T_p^*M.

Assume $\nabla_{X_i} X_j = \sum_{k=1}^m f^k_{ij} X_k$, where f^k_{ij} is the coefficient of $\nabla_{X_i} X_j$ in the basis X_k.

Because $\{X_i\}_i$ are not the coordinate vector fields in general, f^k_{ij} is not necessarily symmetric.

Define 1-forms ω^k_j on V by $\omega^k_j := \sum_{i=1}^m f^k_{ij} \omega^i$; then $f^k_{ij} = \omega^k_j(X_i)$.

The values of L-C connection on ~~X_i~~, any vector field is:

$X = \sum_{i=1}^m a^i X_i$, $Y = \sum_{i=1}^m b^i X_i$ for some $a^i, b^i \in C^\infty(V)$,

- $$\nabla_X X_j = \nabla_{\sum a^i X_i} X_j = \sum_{i=1}^m a^i \nabla_{X_i} X_j = \sum_{i,k} a^i \Gamma^k_{ij} X_k = \sum_{i,k} a^i \omega^k_j(X_i) X_k = \sum_{k=1}^m \omega^k_j(X) X_k,$$
- $$\nabla_X Y = \nabla_X \left(\sum b^i X_i\right) = \sum_i \left(X(b^i) X_i + b^i \nabla_X X_i \right) = \sum_{j=1}^m \left(X(b^j) + \sum b^i \omega^j_i(X) \right) X_j .$$

Notice that $\omega^i_j(X) = \omega^i(\nabla_X X_j)$, and ω^i_j are called connection forms.

For the L-C connection, ω^i_j can not be arbitrary:

Theorem (Cartan) (M, g), $V \subseteq M$, $\{X_1, \dots, X_n\}$ frame fields, $\{\omega^1, \dots, \omega^n\}$ co-frame fields. Then the connection forms of the L-C connection are the unique solution of the equations

1/ $d\omega^i = \sum_{j=1}^m \omega^j \wedge \omega^i_j$

2/ $dg_{ij} = \sum_{k=1}^m (g_{kj} \omega^k_i + g_{ki} \omega^k_j)$, $g_{ij} := g(X_i, X_j)$.

(39)

Pf: We use the fundamental formula for the differential of a form: $\omega \in \Omega^1(M)$, $X, Y \in TM$.

$$(d\omega)(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

We have

$$\nabla_Y X = \nabla_Y\left(\sum_j \omega^j(X) X_j\right) = \sum_j \left(Y(\omega^j(X)) X_j + \omega^j(X) \nabla_Y X_j\right),$$

and so

$$\omega^i(\nabla_Y X) = Y(\omega^i(X)) + \sum_j \omega^j(X)\, \omega^i(\nabla_Y X_j).$$

Then

$$\left(\sum_j \omega^j \wedge \omega^i_j\right)(X,Y) = \sum_j \left(\omega^j(X)\omega^i_j(Y) - \omega^j(Y)\omega^i_j(X)\right)$$

$$= \sum_j \left(\omega^j(X)\,\omega^i(\nabla_Y X_j) - \omega^j(Y)\,\omega^i(\nabla_X X_j)\right)$$

$$= \omega^i(\nabla_Y X) - Y(\omega^i(X)) - \omega^i(\nabla_X Y) + X(\omega^i(Y))$$

and so

$$\left(d\omega^i - \sum_j \omega^j \wedge \omega^i_j\right)(X,Y) =$$

$$= X(\omega^i(Y)) - Y(\omega^i(X)) - \omega^i([X,Y]) - \left(\sum_j \omega^j \wedge \omega^i_j\right)(X,Y)$$

$$= \omega^i\big(\underbrace{\nabla_X Y - \nabla_Y X - [X,Y]}_{T(X,Y)}\big) = 0.$$

The first equation is equivalent to the symmetry of ∇.

To prove 2/, observe $(dg_{ij})(Y) = Y(\langle X_i, X_j\rangle)$. On the other hand,

$$\left(\sum_{k=1}^m \left(g_{kj}\omega_i^k + g_{ki}\,\omega^k_j\right)\right)(Y) =$$

$$= \sum_k \left(g_{kj}\,\omega^k_i(Y) + g_{ki}\,\omega^k_j(Y)\right)$$

$$= \left\langle \sum_k \omega^k_i(Y) X_k, X_j\right\rangle + \left\langle \sum_k \omega^k_j(Y) X_k, X_i\right\rangle$$

$$= \langle \nabla_Y X_i, X_j\rangle + \langle \nabla_Y X_j, X_i\rangle$$

equivalent to $Y(\langle X_i, X_j\rangle) = \langle \nabla_Y X_i, X_j\rangle + \langle X_i, \nabla_Y X_j\rangle \ \forall Y \in T_x$

This is compatibility of ∇ with g.

(40) The uniqueness of ω^i_j is equivalent to uniqueness of ∇, the L-C connection. ▨

Apart from connection forms, we define curvature forms: $V \subseteq M$, $\{X_1, \dots, X_m\}$, $\{\omega^1, \dots, \omega^m\}$, 2-forms Ω^l_k $(k, l = 1, \dots, m)$ are defined by

$$\Omega^l_k(X, Y) = \omega^l(R(X, Y)X_k), \quad \forall X, Y \in TM|_V.$$

This means $R(X, Y)X_k = \sum_{l=1}^m \Omega^l_k(X, Y)X_l$. Using the basis of 2-forms $\{\omega^i \wedge \omega^j\}_{i<j}$,

$$\Omega^l_k = \sum_{i<j} \Omega^l_k(X_i, X_j)\,\omega^i \wedge \omega^j = \sum_{i<j} \omega^l(R(X_i, X_j)X_k)\,\omega^i \wedge \omega^j$$
$$= \sum_{i<j} R_{ijk}{}^l\,\omega^i \wedge \omega^j = \frac{1}{2}\sum_{i,j=1}^m R_{ijk}{}^l\,\omega^i \wedge \omega^j$$

where

$$R(X_i, X_j)X_k = \sum_{j=1}^m R_{ijk}{}^l X_l \quad (\text{coeff. of } R \text{ relative to } \{X_1, \dots, X_m\})$$

The curvature forms satisfy

<u>Theorem</u> (Cartan) $\quad \Omega^j_i = d\omega^j_i - \sum_{k=1}^m \omega^k_i \wedge \omega^j_k \quad \forall i, j = 1, \dots, m.$

<u>Pf</u>: We will show

$$R(X, Y)X_i = \sum_{j=1}^m \Omega^j_i(X, Y)X_j = \left(\sum_{j=1}^m \left(d\omega^j_i - \sum_{k=1}^m \omega^k_i \wedge \omega^j_k\right)(X, Y)\right)X_j$$

$\forall X, Y \in TM.$

$$R(X, Y)X_i = \nabla_X \nabla_Y X_i - \nabla_Y \nabla_X X_i - \nabla_{[X,Y]}X_i$$
$$= \nabla_X\left(\sum_k \omega^k_i(Y)X_k\right) - \nabla_Y\left(\sum_k \omega^k_i(X)X_k\right) - \sum_k \omega^k_i([X, Y])X_k$$
$$= \sum_k \left(X(\omega^k_i(Y)) - Y(\omega^k_i(X)) - \omega^k_i([X, Y])\right)X_k$$
$$+ \sum_k \omega^k_i(Y)\nabla_X X_k - \sum_k \omega^k_i(X)\nabla_Y X_k$$
$$= \sum_k d\omega^k_i(X, Y)X_k + \sum_{k,j}\left(\omega^k_i(Y)\omega^j_k(X)X_j - \omega^k_i(X)\omega^j_k(Y)X_j\right)$$
$$= \sum_j \left(d\omega^j_i(X, Y) - \sum_k (\omega^k_i \wedge \omega^j_k)(X, Y)\right)X_j.$$

▨

(41) The equations

$$d\omega^i = \sum_j \omega^j \wedge \omega^i_j,$$

$$dg_{ij} = \sum_k \left(g_{kj}\, \omega^k_i + g_{ki}\, \omega^k_j \right),$$

$$d\omega^j_i = \Omega^j_i + \sum_k \omega^k_i \wedge \omega^j_k,$$

are called Cartan structure equations.

for $\omega^i(X_j) = \delta^i_j$, $\omega^k_j = \sum f^k_{ij}\, \omega^i$, $\Omega^j_i = \sum_{k<l} R_{kli}{}^j\, \omega^k \wedge \omega^l$.

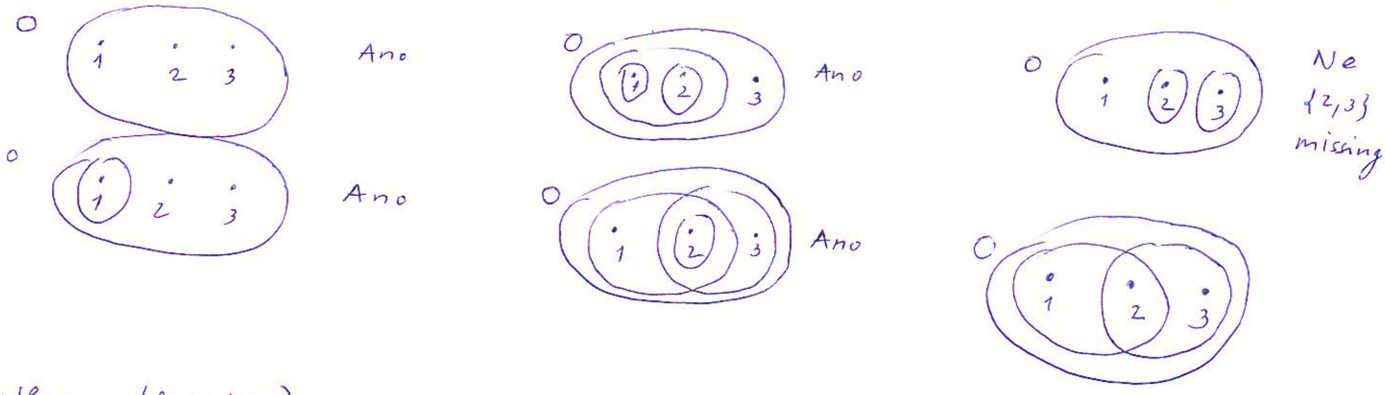
$f^k_{ij} = \Gamma^k_{ij}$

Exercises

(1)

Exercises

Př: Jsou následující kolekce podmnožin množiny $\{1,2,3\}$ topologiemi?



Př: (atlas on the sphere)
$S^m \hookrightarrow \mathbb{R}^{m+1}$ unit sphere,
$S^m = \{p \in \mathbb{R}^{m+1} \mid \sum_{i=1}^{m+1} p_i^2 = 1\}$ equiped with subset topology
(i.e., $U \subseteq S^m$ is open if $\exists\ V \subseteq \mathbb{R}^{m+1}$ open such that $U = S^m \cap V$.)

Let $N = (1,0) \in \mathbb{R} \times \mathbb{R}^m$
$S = (-1,0) \in \mathbb{R} \times \mathbb{R}^m$ $\in S^m \subseteq \mathbb{R}^{m+1}$

$U_N := S^m \setminus \{N\}$
$U_S := S^m \setminus \{S\}$

$\varphi_N : U_N \to \mathbb{R}^m \quad (p_1, \dots, p_{m+1}) \mapsto \frac{1}{1-p_1}(p_2, \dots, p_{m+1})$

$\varphi_S : U_S \to \mathbb{R}^m \quad (p_1, \dots, p_{m+1}) \mapsto \frac{1}{1+p_1}(p_2, \dots, p_{m+1})$

Then the transition maps

$$\varphi_S \circ \varphi_N^{-1},\ \varphi_N \circ \varphi_S^{-1} : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$$
$$x \mapsto \frac{x}{|x|^2}$$

$|\ |$ - standard norm on $\mathbb{R}^m$

is C^ω (analytic)

$\mathcal{A} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$ is C^ω ($\Rightarrow C^\infty$) atlas on S^m.

Př: (atlas on the projective space)
$\mathbb{R}^{m+1} \setminus \{0\}$ define $\equiv$ equiv. rel.

$p \equiv q$ iff $\exists\ \lambda \in \mathbb{R}^* : p = \lambda q$

$\mathbb{R}P^m$ be the quotient space $(\mathbb{R}^{m+1} \setminus \{0\})/_\equiv$

$\pi : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{R}P^m$

be the natural projection $p \in \mathbb{R}^{m+1} \setminus \{0\} \mapsto [p] \in \mathbb{R}P^m$
(line, $[p] = \{\lambda p \in \mathbb{R}^{m+1} \mid \lambda \in \mathbb{R}^*\}$)

(2)

Exercises

We equip $\mathbb{R}P^m$ with the quotient topology ($U \subseteq \mathbb{R}P^m$ is open iff $\pi^{-1}(U) \subseteq \mathbb{R}^{m+1}$ is open)

For $k \in \{1, \dots, m+1\}$, define open subset $\underset{\mathbb{R}P^m}{\overset{U_k}{\cap}} = \{[p] \in \mathbb{R}P^m \mid p_k \neq 0\}$

and charts
$$\varphi_k : U_k \to \mathbb{R}^m$$
$$[p] \mapsto \left(\frac{p_1}{p_k}, \dots, \frac{p_{k-1}}{p_k}, 1, \frac{p_{k+1}}{p_k}, \dots, \frac{p_{m+1}}{p_k}\right).$$

If $[p]=[q]$ then $p = \lambda q$ for some $\lambda \in \mathbb{R}^*$, so $\frac{p_\ell}{p_k} = \frac{q_\ell}{q_k}$ for $\forall \ell = 1, \dots, m+1$.
($\Rightarrow$ φ_k is well-defined.) The transition maps
$$\varphi_k \circ \varphi_\ell^{-1}\big|_{\varphi_\ell(U_\ell \cap U_k)} : \varphi_\ell(U_\ell \cap U_k) \to \mathbb{R}^m$$

are given by
$$\left(\frac{p_1}{p_\ell}, \dots, \frac{p_{\ell-1}}{p_\ell}, 1, \frac{p_{\ell+1}}{p_\ell}, \dots, \frac{p_{m+1}}{p_\ell}\right) \mapsto \left(\frac{p_1}{p_k}, \dots, \frac{p_{k-1}}{p_k}, 1, \frac{p_{k+1}}{p_k}, \dots, \frac{p_{m+1}}{p_k}\right)$$

The collection $\mathcal{A} = \{(U_k, \varphi_k) \mid k = 1, \dots, m+1\}$ is C^ω-atlas on $\mathbb{R}P^m$

Examples

Ex: Let $X, Y \in \mathfrak{X}(\mathbb{R}^3)$ smooth vector fields:

$$X(x_1,x_2,x_3) = (2x_3-x_2)\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - 2x_1\frac{\partial}{\partial x_3},$$

$$Y(x_1,x_2,x_3) = x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}.$$

1/ Calculate the Lie bracket $[X,Y]$:

For $X = \sum f_i \frac{\partial}{\partial x_i}$, $Y = \sum g_i \frac{\partial}{\partial x_i}$

$$[X,Y] = \sum_i \left(\sum_j f_j \frac{\partial g_i}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j} \right)\frac{\partial}{\partial x_i}$$

Then

$$[X,Y] = \left(-x_3 \frac{\partial (2x_3-x_2)}{\partial x_2} + x_2 \frac{\partial (2x_3 - x_2)}{\partial x_3} \right)\frac{\partial}{\partial x_1} \qquad \langle\, , \rangle = \langle\, , \rangle_{\mathbb{R}^3}$$

$$+ \left(-2x_1 \frac{\partial x_3}{\partial x_3}\right)\frac{\partial}{\partial x_2} + \left(-x_1 \frac{\partial x_2}{\partial x_2}\right)\frac{\partial}{\partial x_3}$$

$$= (x_3+2x_2)\frac{\partial}{\partial x_1} - 2x_1\frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3}$$

2/ $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ standard unit sphere. Show that $X|_{S^2}$, $Y|_{S^2}$ are vector fields on S^2 (i.e., $\forall x \in S^2$, $X(x), Y(x) \in T_x S^2$):

$e_i \leftrightarrow \frac{\partial}{\partial x_i}$ the basis of $\mathbb{R}^3$

$X = (2x_3 - x_2, x_1, -2x_1)^T$, $Y(x_1,x_2,x_3) = (0, x_3, -x_2)^T$

The tangent vectors in $T_x S^2$ are characterized by

$\langle x, z \rangle = 0$, $z \in T_x S^2$, $x = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}$ ($= \frac{\partial}{\partial r}$ in spherical coordinates)

$\langle (x_1,x_2,x_3), X(x_1,x_2,x_3) \rangle = x_1(2x_3-x_2) + x_2x_1 - 2x_3x_1 = 0$,

$\langle (x_1,x_2,x_3), Y(x_1,x_2,x_3) \rangle = x_2x_3 - x_3x_2 = 0$.

3/ Check that the restriction of $[X,Y]$ to S^2 is a vector field on S^2:

Have to check $\langle [X,Y], x \rangle = 0$. Because $[X,Y] = (x_3+2x_2, -2x_1, -x_1)^T$, we have

$$\langle [X,Y], x \rangle = x_1(x_3+2x_2) - 2x_2x_1 - x_3x_1 = 0.$$

Ex: Show the properties of the Lie bracket:

1/ $[X,Y] = -[Y,X]$,

2/ $[aX + bY, Z] = a[X,Z] + b[Y,Z]$, $a, b \in \mathbb{R}$

3/ Jacobi's identity for Lie bracket:

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

(4) 1/ $[X,Y]f = XYf - YXf = -(YXf - XYf) = -[Y,X]f$, $\forall f \in C^\infty(M,\mathbb{R})$ and the claim follows.

2/ $Z(ag) = a\,Z(g)$, $a \in \mathbb{R}$, since
$$\frac{\partial}{\partial x_i}(ag) = a\frac{\partial g}{\partial x_i},$$
and so
$$[aX+bY, Z]f = (aX+bY)Zf - Z(aX+bY)f$$
$$= aXZf + bYZf - aZXf - bZYf$$
$$= a(XZf - ZXf) + b(YZf - ZYf)$$
$$= a[X,Z]f + b[Y,Z]f$$

3/ $[[X,Y],Z]f = [X,Y]Zf - Z[X,Y]f = XYZf - YXZf - ZXYf + ZYXf$.

$[X,[Y,Z]]f + [Y,[Z,X]]f =$
$$= XYZf - XZYf - YZXf + ZYXf + YZXf - YXZf - ZXYf + XZYf$$
$$= XYZf + ZYXf - YXZf - ZXYf.$$

Ex: Prove the tangent space of the Lie group $SO(n) \subseteq Mat(n\times n,\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ at $e \in SO(n)$ is given by
$$T_e SO(n) = \{A \in M(n,\mathbb{R}) \mid A^T = -A\},$$
i.e. the space of skew-symmetric matrices.

Note: $Q: \mathbb{R}^n \to \mathbb{R}$ quadratic form, B_Q its bilinear form.
$A \in SO(Q,n) \Leftrightarrow AQ = Q \Leftrightarrow AQ(x,y) = Q(x,y) \Leftrightarrow Q(Ax,Ay) = Q(x,y)$ $\forall x,y \in \mathbb{R}^n$.
$AA^T = Id_n \Leftrightarrow$

Let $A: (-\epsilon,\epsilon) \to SO(n)$ a diff. curve on $SO(n)$, $A(0) = \underset{Id_n}{\overset{\shortparallel}{e}} \in SO(n)$. This means
$A(t)A(t)^T = e = Id_n$. Differentiating $\frac{d}{dt}\big|_{t=0}$ gives
$$A'(0)(A(0))^T + A(0)(A'(0))^T = A'(0)\underset{Id_n^T}{\overset{\shortparallel}{e^T}} + \underset{Id_n}{\overset{\shortparallel}{e}}A'(0)^T$$
$A'(0) \in T_e SO(n)$ $\qquad = A'(0) + A'(0)^T = 0$

$\Rightarrow \quad T_e\, SO(n) \subseteq \{ B \in Mat(n\times n, \mathbb{R}) \mid B + B^T = 0 \}$,

$\dim(\quad) = \frac{n(n-1)}{2} = \dim SO(n)$)

Ex: $GL(n,\mathbb{R})$ ... the group of $n\times n$ invertible matrices,

$A \in GL(n,\mathbb{R})$

$$f : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}$$
$$A \longmapsto \det A$$
is differentiable function.

Let $B \in T_A GL(n,\mathbb{R}) \simeq Mat(n\times n, \mathbb{R})$.

Compute $(df)_A(B) \in \mathbb{R}$ $(= T_{f(A)}\mathbb{R} \simeq \mathbb{R})$:

$$(df)_A(B) = \lim_{t\to 0} \frac{\det(A+tB) - \det(A)}{t}$$

$$= \lim_{t\to 0} \frac{\det A\,(\det(I + tA^{-1}B)) - \det A}{t}$$

$$= (\det A) \lim_{t\to 0} \frac{\det(I + tA^{-1}B)}{t}$$

$$= (\det A) \lim_{t\to 0} \frac{1 + t\,Tr(A^{-1}B) + o(t) - 1}{t}$$

$$= (\det A)\, Tr(A^{-1}B).$$

(6)

Exercises

<u>Ex</u>: M, diff. manifold, $U \subseteq M$ open, $\varphi^{-1} \equiv (x_1, \dots, x_n) : U \to V_1 \subseteq \mathbb{R}^n$
$\psi^{-1} \equiv (y_1, \dots, y_n) : U \to V_2 \subseteq \mathbb{R}^n$ } coordinate charts

Show that for $p \in U$:

$$\frac{\partial}{\partial x_i}\Big|_p = \sum_{j=1}^{n} \frac{\partial (y_j \circ \varphi)}{\partial x_i}(\varphi^{-1}(p)) \cdot \frac{\partial}{\partial y_j}\Big|_p$$

(where $y_j \circ \varphi : V_1 \to \mathbb{R}$ and $\frac{\partial (y_j \circ \varphi)}{\partial x_i}$ is the classical partial derivative in the coordinate ~~...~~ direction x_i of $\mathbb{R}^n$.)

Hint: For $f \in C^\infty(M, \mathbb{R})$, expand $f \circ \varphi = f \circ \psi \circ \psi^{-1} \circ \varphi$ and use the chain rule.

For any $C^\infty(M, \mathbb{R})$ we have

$$\frac{\partial}{\partial x_i}\Big|_p (f) = \frac{\partial (f \circ \varphi)}{\partial x_i}(\varphi^{-1}(p)) = \frac{\partial}{\partial x_i}(f \circ \psi \circ \psi^{-1} \circ \varphi)(\varphi^{-1}(p))$$

This is partial derivative ~~...~~ in x_i of the composition of

$\psi^{-1} \circ \varphi : V_1 \subseteq \mathbb{R}^n \to V_2 \subseteq \mathbb{R}^n$

$f \circ \psi : V_2 \subseteq \mathbb{R}^n \to \mathbb{R}$

The chain rule:

$$\frac{\partial}{\partial x_i}(f \circ \psi \circ \psi^{-1} \circ \varphi)(\varphi^{-1}(p)) = \sum_{j=1}^{n} \frac{\partial (f \circ \psi)}{\partial y_j}(\psi^{-1}(p)) \cdot \frac{\partial (y_j \circ \varphi)}{\partial x_i}(\varphi^{-1}(p))$$

$\frac{\partial}{\partial y_j}$ ... j-th coordinate partial derivative for $V_2 \subseteq \mathbb{R}^n$

y_j in $y_j \circ \varphi$ denotes j-th component function of ψ^{-1}.

Finally,

$$\frac{\partial}{\partial x_i}\Big|_p (f) = \sum_{j=1}^{n} \frac{\partial (y_j \circ \varphi)}{\partial x_i}(\varphi^{-1}(p)) \cdot \frac{\partial}{\partial y_j}\Big|_p (f)$$

(+)

Exercises

Ex: $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$, $\mathbb{RP}^2 = \{\ell = \text{line in } \mathbb{R}^3\}$

Euclidean norm on $\mathbb{R}^3$

$\pi : S^2 \to \mathbb{RP}^2$

$p = (x_1, x_2, x_3) \longmapsto [(x_1, x_2, x_3)]$ $\qquad [x] \sim [y] \iff x = \pm y$

canonical projection

Let $c : (-\epsilon, \epsilon) \to S^2$

$t \longmapsto c(t) = (\cos t \cos(2t), \cos t \sin(2t), \sin t)$,

and

$f : \mathbb{RP}^2 \to \mathbb{R}$

$\mathbb{R}\langle(x_1, x_2, x_3)\rangle \longmapsto \dfrac{(x_1 + x_2 + x_3)^2}{x_1^2 + x_2^2 + x_3^2}$.

1/ Let $\gamma := \pi \circ c$

$(-\epsilon, \epsilon) \longrightarrow \mathbb{RP}^2$. Calculate $\gamma'(0)(f)$ (i.e. the vector field $\gamma'(0)$ acting on $\mathbb{RP}^2$)

2/ Let (φ_U, U) be the following chart of $\mathbb{RP}^2$:

$U = \{\mathbb{R}\langle(x_1, x_2, x_3)\rangle \mid x_1 \neq 0\} \subseteq \mathbb{RP}^2$, and

$\varphi : U \to \mathbb{R}^2$

$\mathbb{R}\langle(x_1, x_2, x_3)\rangle \longmapsto \left(\dfrac{x_2}{x_1}, \dfrac{x_3}{x_1}\right)$. Set $\varphi = (z_1, z_2)$. Express $\gamma'(t)$ in the form $\alpha_1(t) \dfrac{\partial}{\partial z_1}\Big|_{\gamma(t)} + \alpha_2(t) \dfrac{\partial}{\partial z_2}\Big|_{\gamma(t)}$

Sol 1: We have $\gamma(t) = \mathbb{R}\cdot\langle \cos t \cos 2t, \cos t \sin 2t, \sin t\rangle$.

1/ Since $(\cos t \cos 2t)^2 + (\cos t \sin 2t)^2 + (\sin t)^2 = 1$,

we obtain

$$\gamma'(0)(f) = \frac{d}{dt}\Big|_{t=0} (\cos t \cos 2t + \cos t \sin 2t + \sin t)^2$$

$= 2 \cdot 3 = 6.$

2/ Let $(\varphi \circ \gamma)(t) = (z_1(t), z_2(t))$, then

$$z_1(t) = \tan 2t, \quad z_2(t) = \frac{\tan t}{\cos 2t}.$$

(8) This implies

$$\gamma'(t) = z_1'(t)\frac{\partial}{\partial z_1}\Big|_{\gamma(t)} + z_2'(t)\frac{\partial}{\partial z_2}\Big|_{\gamma(t)} =$$

$$= 2\left(1+\tan^2(2t)\right)\frac{\partial}{\partial z_1}\Big|_{\gamma(t)} + \frac{(1+\tan^2 t)\cos 2t + 2\tan t \sin 2t}{\cos^2(2t)}\frac{\partial}{\partial z_2}\Big|_{\gamma(t)}$$

Ex: M, $\dim M = m$; N, $\dim N = n$ } diff manifolds

Show that the cartesian product

$$M \times N := \{ (x,y) \mid x \in M, y \in N \}$$

is diff. man., $\dim(M\times N) = m+n$.

$(U_\alpha, \varphi_\alpha)_{\alpha\in A}$, $(\tilde{U}_\beta, \tilde{\varphi}_\beta)_{\beta \in B}$ ... atlases of M resp. N

Then atlas on $M\times N$: $\{ (U_\alpha \times \tilde{U}_\beta), \psi_{\alpha,\beta} \}_{(\alpha,\beta)\in A\times B}$,

where

$$\psi_{\alpha,\beta} : U_\alpha \times \tilde{U}_\beta \longrightarrow V_\alpha \times \tilde{V}_\beta \subseteq \mathbb{R}^{m+n}$$

$$\psi_{\alpha,\beta}(x,y) \longmapsto (\varphi_\alpha(x), \tilde{\varphi}_\beta(y)).$$

The coordinate changes are

$$\left(\psi_{\gamma,\delta}^{-1} \circ \psi_{\alpha,\beta}\right)(x,y) = \left((\varphi_\gamma^{-1}\circ\varphi_\alpha)(x), \tilde{\varphi}_\delta^{-1}\circ\tilde{\varphi}_\beta(y)\right),$$

diff. fction!

It is easy to check ~~the~~

M, N - Hausdorff $\Rightarrow$ $M\times N$ Hausdorff ~~top.~~ space

Ex: Let

$\mathbb{W}^2 = \{ x \in \mathbb{R}^3 \mid q(x,x) = -1, x_3 > 0 \}$ with quadratic form induced by bilinear form

$$q(x,y) = x_1y_1 + x_2y_2 - x_3y_3$$

and

$\mathbb{B}^2 = \{ x\in\mathbb{R}^3 \mid x_1^2 + x_2^2 < 1, x_3 = 0\}$ a 2-dim ball.

For any $p \in \mathbb{W}^2$ let L_p denote the Eucl straight line through p and $(0,0,-1)$.

9) Let $f : \mathbb{W} \to \mathbb{B}^2$ be the stereographic projection defined by
$$f(p) = L_p \cap \mathbb{B}^2.$$

1/ Calculate explicitly the maps $f(X, Y, Z)$ for $(X, Y, Z) \in \mathbb{W}^2$ and $f^{-1}(x, y, 0) \in \mathbb{B}^2$.

2/ A coordinate chart $\varphi : U \to V \subseteq \mathbb{R}^2$ for $U \subseteq \mathbb{W}^2$ is given by $\varphi^{-1}(x_1, x_2) = (\cos x_1 \sinh x_2, \sin x_1 \sinh x_2, \cosh x_2)$ for $(x_1, x_2) \in V = (0, 2\pi) \times (0, \infty) \subseteq \mathbb{R}^2$. Let $\psi = \varphi \circ f^{-1}$ be a coordinate chart of $\mathbb{B}^2$ with coordinate fions y_1, y_2. $\psi = (y_1, y_2) : \mathbb{B}^2 \to \mathbb{R}^2$. Calculate ψ^{-1}.

Solut.

1/ Let $p = (X, Y, Z) \in \mathbb{W}^2$. Then
$$L_p = \{ (X, Y, Z) + t(X, Y, Z+1) \mid t \in \mathbb{R} \}$$
and
$$L_p \cap \mathbb{B}^2 = (X, Y, Z) - \frac{Z}{Z+1}(X, Y, Z+1) = \left(\frac{X}{Z+1}, \frac{Y}{Z+1}, 0 \right).$$
We conclude that
$$f(X, Y, Z) = \left(\frac{X}{Z+1}, \frac{Y}{Z+1}, 0 \right) \in \mathbb{B}^2.$$
Conversely, let
$$f^{-1}(x, y, 0) = (X, Y, Z) \text{ with } X^2 + Y^2 - Z^2 = -1.$$
Since $x = \dfrac{X}{Z+1}$, $y = \dfrac{Y}{Z+1}$, we conclude
$$x^2 + y^2 = \frac{X^2 + Y^2}{(Z+1)^2} = \frac{Z^2 - 1}{(Z+1)^2} = \frac{Z-1}{Z+1} \Rightarrow$$
$$Z = \frac{1 + x^2 + y^2}{1 - x^2 - y^2}, \quad Z + 1 = \frac{2}{1 - x^2 - y^2}. \text{ Then}$$
$$f^{-1}(x, y, 0) = (X, Y, Z) = ((Z+1)x, (Z+1)y, Z) =$$
$$= \left(\frac{2x}{1 - x^2 - y^2}, \frac{2y}{1 - x^2 - y^2}, \frac{1 + x^2 + y^2}{1 - x^2 - y^2} \right).$$

(10) 2/ We have

$$\psi^{-1}(y_1, y_2) = (f \circ \varphi^{-1})(y_1, y_2) =$$

$$= \left(\frac{\sinh y_2}{1+\cosh y_2} \cos y_1, \frac{\sinh y_2}{1+\cosh y_2} \sin y_1, 0 \right)$$

Ex: Let

$$\mathbb{H}^2 = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$$

be the upper half plane $\subseteq \mathbb{C}$ as a model for hyperbolic geometry in dim $= 2$. The group

$$SL(2, \mathbb{R}) = \{ A \in M(2 \times 2, \mathbb{R}) \mid \det A = 1 \}$$

acts on $\mathbb{H}^2$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$,

$$f_A : \mathbb{H}^2 \to \mathbb{H}^2$$

$$z \mapsto f_A(z) = \frac{az+b}{cz+d}$$

Calculate $(f_A)_*(v)$ for $v \in T_z \mathbb{H}^2$;

$$(f_A)_*(v) = \frac{d}{dt}\Big|_{t=0} f_A(z+tv) = \frac{d}{dt}\Big|_{t=0} \frac{a(z+tv)+b}{c(z+tv)+d} =$$

$$= \frac{av(cz+d) - (az+b)cv}{(cz+d)^2} = \frac{(ad-bc)v}{(cz+d)^2} = \frac{1}{(cz+d)^2}$$

(1) **Ex** : $\forall x \in M$, M a smooth diff. manifold, X a vector field on M, there exists a diff map (locally on $\mathbb{R}$):

1/ $\sigma(0,x) = x$,

2/ $t \to \sigma(t,x)$ is a solution of $\frac{d}{dt}\sigma(t,x) = X(\sigma(t,x))$,

3/ $\sigma(t, \sigma(s,x)) = \sigma(t+s, x)$.

" flow of X throug x ".

$M = \mathbb{R}^2$, $X(x,y) = -y\,\partial_x + x\,\partial_y$ a vector field

check that

$$\sigma(t,(x,y)) = (x\cos t - y\sin t,\ x\sin t + y\cos t)$$

is a flow gen. by X; the flow lines are circles around the origin + the origin itself

Ex: $M = \mathbb{R}^2$, $X'(x,y) = y\,\partial_x + x\,\partial_y$, find the flow generated by X.

(11.5)

Exercise

Let M be a smooth manifold, $\mathfrak{X}(M)$ the space of smooth vector fields, ∇ an affine connection (~~not~~ torsion-less or Levi-Civita connection condition)

We say a map

$$A : \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \to C^\infty(M) \text{ or } \mathfrak{X}(M)$$

is a tensor field if it is $C^\infty(M)$-linear in each argument:

$$A(X_1, \dots, fX_i + gY_i, \dots, X_r) = fA(X_1, \dots, X_i, \dots, X_r) + gA(X_1, \dots, Y_i, \dots X_r)$$

$\forall$ $X, Y \in \mathfrak{X}(M)$, $f, g \in C^\infty(M)$.

1/ Show that the torsion $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$X, Y \mapsto T(X,Y) = [X,Y] - (\nabla_X Y - \nabla_Y X)$$

is a tensor.

Pf: $T(X,Y) = -T(Y,X) \Rightarrow$ linearity in 1st argument,

$\therefore$ $T(X_1 + X_2, Y) = T(X_1, Y) + T(X_2, Y)$ clear,

$T(fX, Y) \overset{?}{=} fT(X,Y)$

$$\begin{aligned} T(fX,Y) &= [fX,Y] - (\nabla_{fX} Y - \nabla_Y fX) \\ &= f[X,Y] - (Yf)X - (f\nabla_X Y - (Yf)X - f\nabla_Y X) \\ &= f([X,Y] - (\nabla_X Y - \nabla_Y X)) - (Yf)X + (Yf)X \\ &= fT(X,Y) \end{aligned}$$

2/ Let $A : \underbrace{\mathfrak{X}(M) \times \dots \mathfrak{X}(M)}_{r \text{ times}} \to C^\infty(M)$ be a tensor.

The covariant derivative of A is a map

$$(\nabla A) : \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{r+1} \to C^\infty(M)$$

defined by

$$(\nabla A)(X_1, \dots, X_r, Y) = Y(A(X_1, \dots, X_r)) - \sum_{j=1}^{r} A(X_1, \dots, \nabla_Y X_j, \dots, X_r).$$

Show that ∇A is a tensor!

Pf:

↘

Exercises

$\sigma: I \times M \to M$, M ... diff. manifold
$I \subseteq \mathbb{R}$

i.e. $\sigma_t : M \to M$ 1-par. group of diff. of M,
$t \in (-\epsilon, \epsilon)$ infin. gen. of σ_t for $t \to 0$
is $X \in TM$.

Under σ_ϵ, $\epsilon \to 0$, $x \in M$ with coordinates $x = (x^1, \dots, x^m)$ in a given chart
is mapped to
$$x^i \to \sigma_\epsilon^i(x) = \sigma^i(\epsilon, x) = x^i + \epsilon X^i(x),$$
$i = 1, \dots, m$

where $X^i(x)$ is i-th component
of $X|_p = \sum_{i=1}^m X^i(x) \left(\frac{\partial}{\partial x^i}\right)_x$

Assuming local analyticity of $\sigma(t,x)$ in t, we get
$$\sigma^i(t,x) = x^i + t \frac{d}{ds}\Big|_{s=0} \sigma^i(s,x) + \frac{t^2}{2}\left(\frac{d}{ds}\right)^2 \sigma^i(s,x) + \dots$$
$$= \left(1 + t\frac{d}{ds} + \frac{t^2}{2}\left(\frac{d}{ds}\right)^2 + \dots\right)\Big|_{s=0} \sigma^i(s,x)$$
$$= \exp\left(t \frac{d}{ds}\Big|_{s=0}\right) \sigma^i(s,x) \equiv \exp(tX)x,$$
satisfying

1/ $\sigma(0,x) = x = \exp(0 \cdot X)x$,

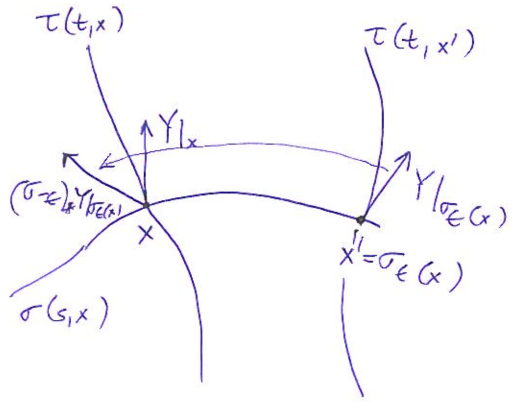
2/ $\frac{d}{dt}\sigma(t,x) = X \exp(tX)x = \frac{d}{dt}\left(\exp(tX)x\right)$,

3/ $\sigma(t, \sigma(r,x)) = \sigma(t, \exp(rX)x) = \exp(tX)\exp(rX)x$
$= \exp((t+r)X)x = \sigma(r+t, x)$.

Now we consider
$$X \leftrightarrow \sigma(s,x) \quad : \quad \frac{d}{ds}\sigma^i(s,x) = X^i(\sigma(s,x)),$$
$$Y \leftrightarrow \sigma(t,x) \quad : \quad \frac{d}{dt}\sigma^i(t,x) = Y^i(\sigma(t,x)).$$

Let us evaluate the (infinitesimal) change of Y along $\sigma(s,x)$:
compare Y at x with $(\sigma_{-\epsilon})_* : T_{\sigma_\epsilon(x)} M \to T_x M$ acting
$Y \in T_x M$

on Y at $\sigma_\epsilon(x)$. Then it make sense to compare $(\sigma_{-\epsilon})_* Y|_{\sigma_\epsilon(x)}, Y|_x$ as two vectors in $T_x M$. Then the Lie derivative of Y along the flow generated by X (or, the vector flow of X) is
$$\mathcal{L}_X Y = \lim_{\epsilon \to 0} \left((\sigma_{-\epsilon})_* Y|_{\sigma_\epsilon(x)} - Y|_x\right).$$



This definition is equivalent to

$$\mathcal{L}_X Y = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(Y|_x - (\sigma_\epsilon)_* Y|_{\sigma_{-\epsilon}(x)}\right)$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(Y|_{\sigma_\epsilon(x)} - (\sigma_\epsilon)_* Y|_x\right)$$

(U,φ) be a chart with coordinates $x=\varphi^{-1}$, $X=\sum X^i\frac{\partial}{\partial x^i}$, $Y=\sum_j Y^j\frac{\partial}{\partial x^j}$ be vector fields on U. Then $\sigma_\epsilon(x)$ has coordinates $x^i+\epsilon X^i(x)$ and

$$Y|_{\sigma_\epsilon(x)} = Y^i\left(x^j+\epsilon X^j(x)\right)\left(\frac{\partial}{\partial x^i}\right)_{x+\epsilon X} \overset{\epsilon\to 0}{\sim} \left[Y^i(x)+\epsilon X^j(x)\frac{\partial}{\partial x^j}Y^i(x)\right]\left(\frac{\partial}{\partial x^i}\right)_{x+\epsilon X} =$$

and because $(\sigma_{-\epsilon})_*$ maps $\left(\frac{\partial}{\partial x^i}\right)_{x+\epsilon X}$ to T_xM by Jacobian of $(\sigma_{-\epsilon})_*$, we get

$$= \left[Y^i(x)+\epsilon X^j(x)\frac{\partial}{\partial x^j}Y^i(x)\right]\frac{\partial}{\partial x^i}\left[x^l-\epsilon X^l(x)\right]\left(\frac{\partial}{\partial x^l}\right)_x$$
$$= \left[Y^i(x)+\epsilon X^j(x)\frac{\partial}{\partial x^j}Y^i(x)\right]\left[\delta_i^l-\epsilon\frac{\partial}{\partial x^i}X^l(x)\right]\left(\frac{\partial}{\partial x^l}\right)_x$$
$$= Y^i(x)\left(\frac{\partial}{\partial x^i}\right)_x + \epsilon\Big[\underbrace{X^j(x)\left(\frac{\partial}{\partial x^k}Y^k(x)\right) - Y^j(x)\left(\frac{\partial}{\partial x^k}X^k(x)\right)}_{\text{"}\mathcal{L}_X Y\ (=[X,Y]:=X\cdot Y-Y\cdot X)}\Big]\left(\frac{\partial}{\partial x^j}\right)_x + O(\epsilon^2)$$

$\Rightarrow$

Geometrically - the Lie bracket shows non-commutability of two flows:

$\sigma(s,x) \leftrightarrow X$; • move by $\epsilon\to 0$ along σ
$\tau(t,x) \leftrightarrow Y$; —"— $\delta\to 0$ —"— τ $\Rightarrow$

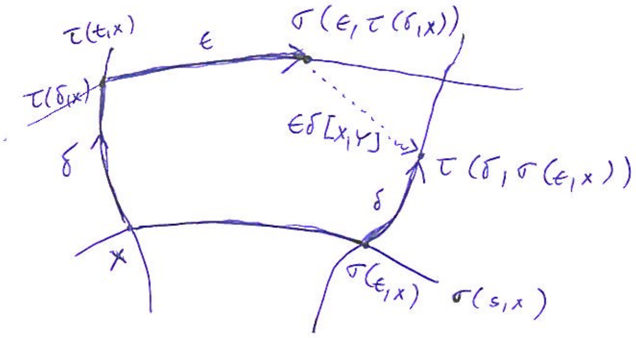
$x\in M$
$$\tau^i(\delta,\sigma(\epsilon,x)) \overset{\epsilon\to 0}{\sim} \tau^i(\delta, x^j+\epsilon X^j(x)) \overset{\delta\to 0}{\sim} x^i+\epsilon X^i(x)+\delta Y^i(x^j+\epsilon X^j(x))$$
$$\sim x^i+\epsilon X^i(x)+\delta Y^i(x)+\epsilon\delta X^j(x)\frac{\partial}{\partial x^j}Y^i(x)+O(\epsilon^2,\delta^2)$$

• move by δ along τ first ($\delta\to 0$)
—"— ϵ, $\epsilon\to 0$, along σ $\Rightarrow$

$$\sigma^i(\epsilon,\tau(\delta,x)) \overset{\delta\to 0}{\sim} \sigma^i(\epsilon, x^j+\delta X^j(x)) \sim x^i+\delta Y^i(x)+\epsilon X^i(x^j+\delta Y^j(x))$$
$$\sim x^i+\delta Y^i(x)+\epsilon X^i(x)+\epsilon\delta Y^j(x)\frac{\partial}{\partial x^j}X^i(x)+O(\epsilon^2,\delta^2)$$

$$\Rightarrow \tau^i(\delta,\sigma(\epsilon,x)) - \sigma^i(\epsilon,\tau(\delta,x)) = \epsilon\delta[X,Y]^i + O(\epsilon^2,\delta^2),$$

$$\Rightarrow \mathcal{L}_X Y = [X,Y]=0 \quad \text{iff} \quad \sigma(s,\tau(t,x)) = \tau(t,\sigma(s,x))$$



Another example: let $\omega \in T^*M$, $X \in TM$:

$$\mathcal{L}_X \omega := \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left((\sigma_\epsilon)^* \omega|_{\sigma_\epsilon(x)} - \omega|_x \right), \quad \omega|_x \in T_x^*M.$$

In the local chart (U, φ) a coordinates $x = \varphi^{-1}$: $\omega = \omega_i \, dx^i$ $\quad (dx^i)(\frac{\partial}{\partial x^j}) = \delta^i{}_j$

$$(\sigma_\epsilon^*)\, \omega|_{\sigma_\epsilon(x)} \approx \omega_i(x) dx^i + \epsilon \left[X^j(x) \frac{\partial}{\partial x^j} \omega_i(x) + \frac{\partial}{\partial x^i} X^j(x) \omega_j(x) \right] dx^i$$

$$\Rightarrow \mathcal{L}_X \omega = \left(X^j \frac{\partial}{\partial x^j} \omega_i + \omega_j \frac{\partial}{\partial x^i} X^j \right) dx^i \quad \text{is an element of } T^*M$$

Yet another example: $\quad Y \in \mathfrak{X}(M)$, $\omega \in T^*M$

consider $Y \otimes \omega \in TM \otimes T^*M$; then $Y \otimes \omega|_{\sigma_\epsilon(x)}$, $\sigma_\epsilon(x) \in M$ is mapped to $Y \otimes \omega$ at $x \in M$ by $(\sigma_{-\epsilon})_* \otimes (\sigma_\epsilon)_*$,

$$\left[(\sigma_{-\epsilon})_* \otimes (\sigma_\epsilon)^* \right] (Y \otimes \omega)|_{\sigma_\epsilon(x)} = \left[(\sigma_{-\epsilon})_* Y \otimes (\sigma_\epsilon)^* \omega \right]\Big|_x \qquad X \in TM$$

by leibnitz rule $\Rightarrow$

$$L_X (Y \otimes \omega) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\left((\sigma_{-\epsilon})_* Y \otimes (\sigma_\epsilon)^* \omega \right)\Big|_x - (Y \otimes \omega)|_x \right]$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[(\sigma_{-\epsilon})_* Y \otimes \left((\sigma_\epsilon)^* \omega - \omega \right) + \left((\sigma_{-\epsilon})_* Y - Y \right) \otimes \omega \right]$$

$$= Y \otimes (L_X \omega) + (L_X Y) \otimes \omega.$$

Exercises

Example: Let $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ denote the upper half space model of the hyperbolic space with its standard Riemm. metric

$$g_x(v_1, v_2) = \frac{\langle v_1, v_2 \rangle_0}{x_n^2}, \quad v_1, v_2 \in T_x \mathbb{H}^n \simeq \mathbb{R}^n$$

$\langle v_1, v_2 \rangle_0$ denotes the standard Euclidean scalar product.

Calculate all Chistoffel symbols Γ_{ij}^k w.r to the global coordinate chart $\varphi: \mathbb{H}^n \to V \subseteq \mathbb{R}^n$, $\varphi(x) = x$.

Sol:

$$\Gamma_{ij}^k = \frac{1}{2} \cancel{g^{kk}} \sum_l g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l}),$$

and for hyperbolic space

$$g_{ij}(x) = \frac{1}{x_n^2}\delta_{ij},$$
$$g^{ij}(x) = x_n^2 \delta_{ij}.$$

Therefore,

$$\Gamma_{ij}^k = \frac{1}{2} g^{kk}(g_{ik,j} + g_{jk,i} - g_{ij,k}),$$

where

$$g_{ab,c} = -\frac{2}{x_n^3}\delta_{ab}\delta_{cn}.$$

Then $\Gamma_{ij}^k = 0$ if $i,j,k \leq n-1$. For symmetry reasons, $\Gamma_{ij}^k = \Gamma_{ji}^k$, have to consider $i \leq j$ only

Let $k = n$. Then

$$\Gamma_{ij}^n = \frac{1}{2} x_n^2 (g_{in,j} + g_{jn,i} - g_{ij,n}).$$

If $i,j \leq n-1$, we conclude

$$\Gamma_{ij}^n = -\frac{1}{2} x_n^2 g_{ij,n} = \frac{1}{x_n}\delta_{ij}$$

If $i \leq n-1$, $j = n$, we have

$$\Gamma_{ij}^n = 0.$$

If $i,j = k$, we obtain

$$\Gamma_{nn}^n = \frac{1}{2} x_n^2 g_{nn,n} = -\frac{1}{x_n}$$

Exercises

Ex. Given a curve $c : [a,b] \to \mathbb{R}^3$
$$c(t) = (f(t), 0, g(t)) \quad (\text{without self-intersection and } f(t) > 0 \ \forall t \in [a,b])$$

$M \subseteq \mathbb{R}^3$ ... surface of revolution (around vertical Z-axis)

∇ ... Levi-Civita connection of the metric induced from $M \hookrightarrow \mathbb{R}^3$ by Euclidean metric on $\mathbb{R}^3$, $g|_M$.

An (almost global) coordinate chart is
$$\varphi : \underset{M}{\underset{\cap}{U}} \longrightarrow V = (a,b) \times (0, 2\pi) \ni (x_1, x_2)$$
$$\varphi^{-1}(x_1, x_2) = (f(x_1)\cos x_2, f(x_1)\sin x_2, g(x_1))$$

1/ Calculate the Christoffel symbols of this chart, and
$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

2/ let $\gamma_1(t) = \varphi^{-1}(x_1 + t, x_2)$; calculate
$$\frac{D}{dt} \gamma_1'(t)$$
and discuss the vanishing of this vector field.

3/ let $\gamma_2(t) = \varphi^{-1}(x_1, x_2 + t)$; calculate
$$\frac{D}{dt} \gamma_2'(t)$$
and discuss — " — $\quad (f'(x_1) = 0\ .)$

Pf:
$$\frac{\partial}{\partial x_1}\Big|_{\varphi^{-1}(x_1,x_2)} = (f'(x_1)\cos x_2, f'(x_1)\sin x_2, g'(x_1)),$$
$$\frac{\partial}{\partial x_2}\Big|_{\varphi^{-1}(x_1,x_2)} = (-f(x_1)\sin x_2, f(x_1)\cos x_2, 0),$$
they are basis of $T_{\varphi^{-1}(x_1,x_2)} M$.

(ii)

The induced metric is

$$g_{ij} = \left\langle \frac{\partial}{\partial x_i}\Big|_{\varphi^{-1}(x_1,x_2)}, \frac{\partial}{\partial x_j} \right\rangle_{\mathbb{R}^3} \Rightarrow g_{ij} = \begin{pmatrix} (f'(x_1))^2 + (g'(x_1))^2 & 0 \\ 0 & f^2(x_1) \end{pmatrix}$$

and

$$g^{ij} = \begin{pmatrix} \frac{1}{f'(x_1)^2 + g'(x_1)^2} & 0 \\ 0 & \frac{1}{f^2(x_1)} \end{pmatrix} \Rightarrow$$

Christoffel symbols are

$$\Gamma^1_{11} = \frac{1}{2} g^{11}(g_{11,1} + g_{11,1} - g_{11,1}) = \frac{f'(x_1)f''(x_1) + g'(x_1)g''(x_1)}{f'(x_1)^2 + g'(x_1)^2},$$

$$\Gamma^2_{11} = \frac{1}{2} g^{22}(g_{12,1} + g_{12,1} - g_{11,2}) = 0,$$

$$\Gamma^1_{12} = \frac{1}{2} g^{11}(g_{11,2} + g_{12,1} - g_{12,1}) = 0 = \Gamma^1_{21},$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22}(g_{12,2} + g_{22,1} - g_{12,2}) = \frac{f'(x_1)}{f(x_1)} = \Gamma^2_{21},$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}(g_{12,2} + g_{12,2} - g_{22,1}) = -\frac{f(x_1)f'(x_1)}{f'(x)^2 + g'(x_1)^2},$$

$$\Gamma^2_{22} = \frac{1}{2} g^{22}(g_{22,2} + g_{22,2} - g_{22,2}) = 0.$$

We have

$$\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1} = \frac{f'(x_1)f''(x_1) + g'(x_1)g''(x_1)}{f'(x_1)^2 + g'(x_1)^2} \frac{\partial}{\partial x_1},$$

$$\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_2} = \frac{f'(x_1)}{f(x)} \frac{\partial}{\partial x_2},$$

$$\nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_1} = \frac{f'(x_1)}{f(x_1)} \frac{\partial}{\partial x_2},$$

$$\nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_2} = -\frac{f(x_1)f'(x_1)}{f'(x_1)^2 + g'(x_1)^2} \frac{\partial}{\partial x_1}.$$

(18) 2/ Now $\gamma_1'(t) = \frac{\partial}{\partial x_1}\Big|_{\gamma_1(t)}$, and this implies

$$\left(\frac{D}{dt}\gamma_1'\right)(t) = \nabla_{\gamma_1'(t)}\frac{\partial}{\partial x_1} = \left(\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_1}\right)(\gamma_1(t))$$

$$= \frac{f'(x_1+t)f''(x_1+t) + g'(x_1+t)g''(x_1+t)}{f'(x_1+t)^2 + g'(x_1+t)^2}\frac{\partial}{\partial x_1}\Big|_{\gamma_1(t)} \in T_{\gamma_1(t)}M.$$

The condition $\frac{D}{dt}\gamma_1' = 0$ is equivalent to

$$f'(t)f''(t) + g'(t)g''(t) = 0 \qquad \forall t \in (a,b),$$

equivalent to

$$(f'(t))^2 + (g'(t))^2 = \text{constant}.$$

Since $\|c'(t)\|^2 = f'(t)^2 + g'(t)^2$, $\frac{D}{dt}\gamma_1' = 0$ iff c is parametrized by its arclength.

3/ We have $\gamma_2'(t) = \frac{\partial}{\partial x_2}\Big|_{\gamma(t)}$,

$$\left(\frac{D}{dt}\gamma_2'\right)(t) = \nabla_{\gamma_2'(t)}\frac{\partial}{\partial x_2} = \left(\nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_2}\right)(\gamma_2(t)) =$$

$$= -\frac{f(x_1)f'(x_1)}{(f'(x_1))^2 + (g'(x_1))^2}\frac{\partial}{\partial x_1}\Big|_{\gamma_2(t)} \in T_{\gamma_2(t)}M;$$

Since $f > 0$ $\forall t \in (a,b)$ $\Rightarrow$ $f'(x_1) = 0$.

5

Exercise

Example: A coordinate chart of the sphere $S^2 \subseteq \mathbb{R}^3$, of radius $r>0$, is given by

$$\varphi^{-1}(x_1,x_2) = (r\cos x_1 \cos x_2,\ r\cos x_1 \sin x_2,\ r\sin x_1).$$

1/ Calculate $\nabla_{\partial_1}\partial_1$, $\nabla_{\partial_1}\partial_2$, $\nabla_{\partial_2}\partial_1$, $\nabla_{\partial_2}\partial_2$. $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$

2/ Let R be the Riemannian curvature tensor. Calculate $R(\partial_1,\partial_2)\partial_2$.

Pf: 1/ $\partial_1 = \frac{\partial}{\partial x_1} = (-r\sin x_1 \cos x_2,\ -r\sin x_1 \sin x_2,\ r\cos x_1)$,

$\partial_2 = \frac{\partial}{\partial x_2} = (-r\cos x_1 \sin x_2,\ r\cos x_1 \cos x_2,\ 0)$.

Then $g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\cos^2 x_1 \end{pmatrix}$, $g^{ij} = \begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{r^2\cos^2 x_1} \end{pmatrix}$.

The Christoffel symbols are given by

$\Gamma^1_{11} = \frac{1}{2} g^{11}(g_{11,1} + g_{11,1} - g_{11,1}) = 0$,

$\Gamma^2_{11} = 0$,

$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2} g^{11}(g_{11,2} + g_{21,1} - g_{12,1}) = 0$,

$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22}(g_{12,2} + g_{22,1} - g_{12,2}) =$

$= \frac{1}{2r^2\cos^2 x_1}(-2r\cos x_1 \sin x_1) = -\tan x_1$,

$\Gamma^1_{22} = \frac{1}{2} g^{11}(g_{21,2} + g_{21,2} - g_{22,1}) = \frac{1}{2r^2}(2r^2\cos x_1 \sin x_1)$

$\Gamma^2_{22} = 0$ $\qquad = \sin x_1 \cos x_1$,

$\Rightarrow \nabla_{\partial_1}\partial_1 = \Gamma^1_{11}\partial_1 + \Gamma^2_{11}\partial_2 = 0$,

$\nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = \Gamma^1_{12}\partial_1 + \Gamma^2_{12}\partial_2 = -\tan x_1 \partial_2$,

$\nabla_{\partial_2}\partial_2 = \Gamma^1_{22}\partial_1 + \Gamma^2_{22}\partial_2 = \sin x_1 \cos x_1 \partial_1$.

(6)

Exercises

Ex: An almost global coordinate chart of the sphere $S^2 \subseteq \mathbb{R}^3$ is given by

$\varphi : U \to V = \underset{x_1}{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)} \times \underset{x_2}{(0, 2\pi)}$, $U \subseteq S^2$:

$$\varphi^{-1}(x_1, x_2) = (\cos x_1 \cos x_2 , \cos x_1 \sin x_2 , \sin x_1)$$

1/ Determine Christoffel symbols with respect to this coordinate system.

2/ Let X be the parallel vector field along $c_1 : (-\pi, \pi) \to S^2$

$c_1(t) = \varphi^{-1}(0, \pi + t)$

$X(0) = \frac{\partial}{\partial x_1}\Big|_{c_1(0)}$

and calculate X explicitly in terms of the basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$.

3/ Let Y — " — $c_2 : (-\pi, \pi) \to S^2$

$c_2(t) = \varphi^{-1}(\pi/4, \pi + t)$,

$Y(0) = \frac{\partial}{\partial x_1}\Big|_{c_2(0)}$

and calculate Y — " —

Pf: 1/ It follows from the last exercise: $\left.\begin{array}{l} f(x_1) = \cos x_1, \\ g(x_1) = \sin x_1, \\ f'(x_1)^2 + g'(x_1)^2 = 1. \end{array}\right\} \Rightarrow$

$$\Gamma^1_{11} = \frac{1}{2} g^{11} (g_{11,1} + g_{11,1} - g_{11,1}) = \frac{f'(x_1) f''(x_1) + g'(x_1) g''(x_1)}{(f'(x_1))^2 + (g'(x_1))^2} = 0,$$

$$\Gamma^2_{11} = \frac{1}{2} g^{22} (g_{12,1} + g_{12,1} - g_{11,2}) = 0,$$

$$\Gamma^1_{12} = \frac{1}{2} g^{11} (g_{11,2} + g_{12,1} - g_{12,1}) = 0 = \Gamma^1_{21},$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22} (g_{12,2} + g_{22,1} - g_{12,2}) = \frac{f'(x_1)}{f(x_1)} = -\tan x_1 = \Gamma^2_{21},$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11} (g_{12,2} + g_{12,2} - g_{22,1}) = \frac{-f(x_1) f'(x_1)}{(f'(x_1))^2 + (g'(x_1))^2} = \sin x_1 \cos x_1,$$

$$\Gamma^2_{22} = \frac{1}{2} g^{22} (g_{22,2} + g_{22,2} - g_{22,2}) = 0.$$

(7) 2/ We have

$$R(\partial_1, \partial_2)\partial_2 = \nabla_{\partial_1}\nabla_{\partial_2}\partial_2 - \nabla_{\partial_2}\nabla_{\partial_1}\partial_2 - \nabla_{[\partial_1,\partial_2]}\partial_2$$

$$= \nabla_{\partial_1}\left(\cos x_1 \sin x_1 \partial_1\right) - \nabla_{\partial_2}\left(-\tan x_1 \partial_2\right) - \nabla_0 \partial_2$$

$$= \left(\cos^2 x_1 - \sin^2 x_1\right)\partial_1 + \tan x_1 \sin x_1 \cos x_1 \partial_1$$

$$= \cos^2 x_1 \partial_1$$

(8) 2/ We are interested in the ∥ vector field X along equator, parametrized by $c: (-\pi, \pi) \to S^2$

$$c(t) = \varphi^{-1}(0, \pi + t)$$

with initial condition $X(0) = \frac{\partial}{\partial x_1}\Big|_{c(0)}$.

This implies

$$\varphi \circ c_1(t) =: (c_{11}(t), c_{12}(t)) = (0, \pi + t),$$

hence

$$(c_{11}'(t), c_{12}'(t)) = (0, 1),$$

and so the property to be parallel for $X = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ implies

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} + \begin{pmatrix} \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{12} & \Gamma^2_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0, \quad \begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$\left.\begin{array}{l} a_1 = a_1(t), \\ a_2 = a_2(t) \end{array}\right\}$ vzhledem k bázi $\frac{\partial}{\partial x_1}\Big|_{c_1(t)}, \frac{\partial}{\partial x_2}\Big|_{c_1(t)}$.

For $x_1 = 0$ we have

$$-\begin{pmatrix} \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{12} & \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} 0 & -\sin x_1 \cos x_1 \\ \tan x_1 & 0 \end{pmatrix} = 0,$$

~~3/ $\varphi \circ c_2(t) = (\frac{\pi}{4}, \pi + t) = (c_{21}(t), c_{22}(t))$,~~

so the diff. equation implies $a_1' = 0$, $a_2' = 0$, which together with $a_1(0) = 1$, $a_2(0) = 0$ $\Rightarrow$ $\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\Rightarrow$

$$X(t) = \frac{\partial}{\partial x_1}\Big|_{c_1(t)}$$

3/ $\varphi \circ c_2(t) = (\frac{\pi}{4}, \pi + t) = (c_{21}(t), c_{22}(t))$, $Y(0) = \frac{\partial}{\partial x_1}\Big|_{c_2(0)}$.

Then $\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} 0 & -1/2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$$\begin{pmatrix} a_1'(t) \\ a_2'(t) \end{pmatrix} = \exp\left(t \begin{pmatrix} 0 & -1/2 \\ 1 & 0 \end{pmatrix}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{t}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \sin\frac{t}{\sqrt{2}} \\ \sqrt{2}\sin(\frac{t}{\sqrt{2}}) & \cos\frac{t}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{t}{\sqrt{2}} \\ \sqrt{2}\sin\frac{t}{\sqrt{2}} \end{pmatrix} \Rightarrow Y(t) = \cos(\tfrac{t}{\sqrt{2}}) \frac{\partial}{\partial x_1}\Big|_{c_2(t)} + \sin\tfrac{t}{\sqrt{2}} \sqrt{2} \frac{\partial}{\partial x_2}\Big|_{c_2(t)}$$

Exercises

Ex: Let $\mathbb{H}^2 \subseteq \mathbb{R}^2$ be the upper half-space (flat model of hyperbolic geometry)
$\| \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ ↙ gives the global coordinate chart on $\mathbb{H}^2$
with Riemannian metric

$$g_{\mathbb{H}^2}(x,y) = \frac{dx^2 + dy^2}{y^2}.$$

We already computed Christoffel symbols; we would like to compute geodesics.

$c : I \to \mathbb{H}^2$ is geodesic curve provided

$$\nabla_{\dot{c}(t)} \dot{c}(t) = 0. \quad \forall t \in I$$

The geodesic equations are $\qquad c(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$$\ddot{x}(t) - \frac{2}{y(t)} \dot{x}(t) \dot{y}(t) = 0$$

Ex: The curvature tensor of $g_{\mathbb{H}^2}$?

$$g_{ij} = y^{-2} \delta_{ij} \,,\; g^{ij} = y^2 \delta^{ij}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2} y^2 \left(\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12} \right)$$

$$= \frac{1}{2} y^2 \frac{\partial}{\partial y} y^{-2} = -y^{-1}$$

$$-\Gamma^2_{11} = \Gamma^2_{22} = -y^{-1}, \quad \text{and the rest is zero.}$$

$$R^1_{212} = -\partial_2 \Gamma^1_{21} + \partial_1 \Gamma^1_{22} + \sum_k \left(\Gamma^k_{22} \Gamma^1_{k1} - \Gamma^k_{21} \Gamma^1_{k2} \right)$$

$$= -y^{-2} + 0 + (y^{-2} - y^{-2}) = -y^{-2},$$

$$R^2_{121} = -y^2,$$

$$R^1_{111} = R^2_{222} = 0.$$

(10) Examples

Ex: $\mathbb{H}^2 \subseteq \mathbb{R}^2$

$\{(x,y)\in\mathbb{R}^2 \mid y>0\}$

Geodesic equation $\ddot{c}^k(t) + \Gamma^k_{ij}\, \dot{c}^i_t \dot{c}^j_t = 0$

where $c = \begin{pmatrix} x(t) \\ y(t)\end{pmatrix}$; it is the system
$$\ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0,$$
$$\ddot{y} + \frac{1}{y}(\dot{x})^2 - \frac{1}{y}(\dot{y})^2 = 0.$$

This is equivalent to
$$y\ddot{x} - 2\dot{x}\dot{y} = 0,$$
$$y\ddot{y} + (\dot{x})^2 - (\dot{y})^2 = 0.$$

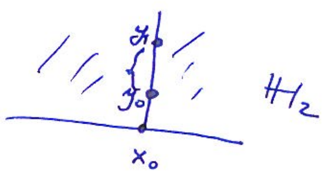
We consider two cases:

1/ $\dot{x} = 0 \Rightarrow$
$$y\ddot{y} - (\dot{y})^2 = 0$$
$$\Rightarrow \frac{d}{dt}\left(\frac{\dot{y}}{y}\right) = \frac{y\ddot{y} - (\dot{y})^2}{y^2} = 0 \Rightarrow \frac{\dot{y}}{y} = a, \quad a\in\mathbb{R}$$

and because $y>0$, integration again gives $y = e^{a(t-t_0)}$, $t_0\in\mathbb{R}$.

If $\underline{t}$ represents the length of the curve, the constant a is
$$1 = g(\dot{c},\dot{c}) = g_{ij}\dot{c}^i\dot{c}^j = \frac{1}{y^2}(\dot{x}^2 + \dot{y}^2) = 0 + a^2 \Rightarrow a = \pm 1$$
$t \mapsto (x_0, e^{\pm(t-t_0)})$ is geodesic.

$\Rightarrow$ the length of geodesic is



$(x_0, y_0) \leftrightarrow (x_0, y_1)$
$$t = \left|\ln\left(\frac{y_0}{y_1}\right)\right|.$$

2/ Assume $\dot{x}\neq 0$; then
$$\frac{d}{dt}\left(\frac{y\dot{y}}{\dot{x}}\right) = \frac{\dot{x}y\ddot{y} + \dot{x}(\dot{y})^2 - \ddot{x}y\dot{y}}{\dot{x}^2}$$
$$= \frac{\dot{x}\left(y\ddot{y} + (\dot{x})^2 - (\dot{y})^2\right] - \dot{y}\left[y\ddot{x} - 2\dot{x}\dot{y}\right] - (\dot{x})^3}{(\dot{x})^2}$$
$$= \frac{0 - 0 - (\dot{x})^3}{(\dot{x})^2} = -\dot{x}$$

(11) and so $\frac{d}{dt}\left(\frac{y\dot{y}}{\dot{x}} + x\right) = 0$,

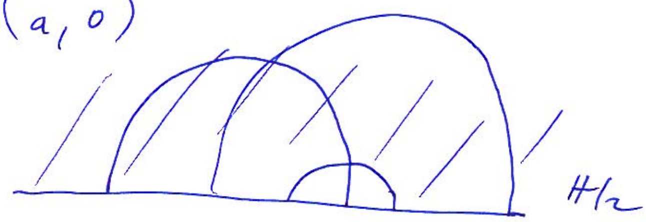
$$\frac{y\dot{y}}{\dot{x}} + x = a, \quad a \in \mathbb{R}$$

$$x\dot{x} + y\dot{y} = a\dot{x} \Rightarrow \frac{1}{2}(x^2+y^2) = ax + b, \quad a, b \in \mathbb{R}$$

$$\Rightarrow (x-a)^2 + y^2 = a^2 + 2b$$

and the non-emptyness of the geodesic $\Rightarrow \underbrace{a^2+2b}_{= r^2} > 0$

$\Rightarrow$ geodesic is upper semi-circle in $H/2$ centered on x-axis - $(a, 0)$



Exercises

Ex: All about the sphere S^2:

3-dim Euclidean metric in spherical coordinates

$$g = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \quad , \quad \begin{aligned} x_1 &= x_1(r,\theta,\varphi) \\ x_2 &= x_2(r,\theta,\varphi) \\ x_3 &= x_3(r,\theta,\varphi) \end{aligned}$$

So restricting to $r \equiv R = \text{const}$,

get $g = R^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$

$\parallel$

$\{g_{ij}\}_{ij}$, $g_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} \Rightarrow g_{\theta\theta} = R^2$,

$g^{ij} = \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2\sin^2\theta} \end{pmatrix}$ $\quad g_{\theta\varphi} = g_{\varphi\theta} = 0$, $g_{\varphi\varphi} = R^2\sin^2\theta$

Let us consider, in the local chart given by (θ,φ) of S^2, the curves (θ_0, φ), θ_0 fixed and φ variable. The tangent vector is, at (θ_0,φ), $\frac{\partial}{\partial\varphi}$. Its length is

$$\left[\left(0\cdot\frac{\partial}{\partial\theta} + 1\cdot\frac{\partial}{\partial\varphi}\right) g^{ij} \left(0\frac{\partial}{\partial\theta} + 1\cdot\frac{\partial}{\partial\varphi}\right)\right]^{1/2} = \frac{1}{R\sin\theta_0}$$

so unit tangent vector field to the curve $\theta = \theta_0$ is $\frac{1}{R\sin\theta_0}\frac{\partial}{\partial\varphi}$.

1) Christoffel symbols: $\sum_{i=1}^{m} \Gamma^i_{jk} \, g_{il} = \Gamma_{ljk}$

$$\Gamma_{\varphi\varphi\theta} = \frac{1}{2}(g_{\varphi\varphi,\theta} + g_{\varphi\theta,\varphi} - g_{\varphi\theta,\varphi}) = \frac{1}{2} g_{\varphi\varphi,\theta} = R^2 \sin\theta\cos\theta$$

$$\Gamma_{\theta\varphi\varphi} = \frac{1}{2}(g_{\theta\varphi,\varphi} + g_{\theta\varphi,\varphi} - g_{\varphi\varphi,\theta}) = -R^2\sin\theta\cos\theta$$

$$\Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = g^{\varphi\varphi}\Gamma_{\varphi\varphi\theta} = \frac{1}{R^2\sin^2\theta}R^2\sin\theta\cos\theta,$$

$$\Gamma^{\theta}_{\varphi\varphi} = g^{\theta\theta}\Gamma_{\theta\varphi\varphi} = -\frac{1}{R^2}R^2\sin\theta\cos\theta = -\sin\theta\cos\theta.$$

(13)

2/ Solve equation of ‖ transport for the curve $\theta = \theta_0$.

$$\nabla_{\frac{1}{R\sin\theta_0}\frac{\partial}{\partial\varphi}} X = 0 \qquad X = X^\varphi \frac{\partial}{\partial\varphi} + X^\theta \frac{\partial}{\partial\theta}$$

$\Leftrightarrow$

$$\frac{1}{R\sin\theta_0}\left(\partial_\varphi X^j + X^k \Gamma^j_{k\varphi}\right) = 0 \qquad \begin{array}{l} j\in\{\theta,\varphi\} \\ k\in\{\theta,\varphi\}\end{array}$$

$\Leftrightarrow$

$$\begin{cases} j=\theta: & 0 = \frac{1}{R\sin\theta_0}\left(\partial_\varphi X^\theta + X^k\Gamma^\theta_{k\varphi}\right) = \frac{1}{R\sin\theta_0}\left(\partial_\varphi X^\theta - X^\varphi \sin\theta_0\cos\theta_0\right) \\ j=\varphi: & 0 = \frac{1}{R\sin\theta_0}\left(\partial_\varphi X^\varphi + X^k\Gamma^\varphi_{k\varphi}\right) = \frac{1}{R\sin\theta_0}\left(\partial_\varphi X^\varphi + X^\theta \frac{\cos\theta_0}{\sin\theta_0}\right)\end{cases}$$

$\Leftrightarrow$

$$0 = \partial_\varphi X^\theta - X^\varphi \sin\theta_0\cos\theta_0 ,$$

$$0 = \partial_\varphi X^\varphi + X^\theta \frac{\cos\theta_0}{\sin\theta_0} .$$

$\Leftrightarrow$ taking second φ-derivative of 1st eq., subst. 2nd:

$$0 = \partial^2_\varphi X^\theta - \partial_\varphi X^\varphi \sin\theta_0\cos\theta_0 = \partial^2_\varphi X^\theta + X^\theta \frac{\cos\theta_0}{\sin\theta_0}\sin\theta_0\cos\theta_0$$

$$= \partial^2_\varphi X^\theta + X^\theta \cos^2\theta_0 .$$

taking second φ —‖— 2nd eq., subst. 1st :

$$0 = \partial^2_\varphi X^\varphi + \partial_\varphi X^\theta \frac{\cos\theta_0}{\sin\theta_0} = \partial^2_\varphi X^\varphi + X^\varphi \cos^2\theta_0$$

and its solution is

$$X^\theta(\varphi) = A\cos\alpha\varphi + B\sin\alpha\varphi$$

$$X^\varphi(\varphi) = C\cos\alpha\varphi + D\sin\alpha\varphi$$

with $\alpha = \cos\theta_0$.

(14)

When $X(0) = (X^\theta(0), X^\varphi(0))$ for $\varphi = 0$,

$$(\partial_\varphi X^\theta)|_{\varphi=0} = X^\varphi(0) \sin\theta_0 \cos\theta_0 ,$$

$$(\partial_\varphi X^\varphi)|_{\varphi=0} = - X^\theta(0) \frac{\cos\theta_0}{\sin\theta_0} ,$$

which determine the constants $A, B, C, D \Rightarrow$

$$\boxed{\begin{aligned} X^\theta(\varphi) &= X^\theta(0)\cos\alpha\varphi + X^\varphi(0) \frac{\sin\theta_0\cos\theta_0}{\alpha} \sin\alpha\varphi , \\ X^\varphi(\varphi) &= X^\varphi(0)\cos\alpha\varphi - X^\theta(0)\frac{\sin\alpha\varphi}{\sin\theta_0} \end{aligned}}$$

$$\frac{\sin\theta_0\cos\theta_0}{\alpha} = \sin\theta_0$$

3/ Norm of the vector is independent on // transport:

$$X(0) = (X^\theta(0), X^\varphi(0)) = X^\theta(0)\frac{\partial}{\partial\theta} + X^\varphi(0)\frac{\partial}{\partial\varphi}$$

$$\| X(0) \|^2 = X^\theta(0)^2 R^2 + R^2 \sin^2\theta_0 \, X^\varphi(0)^2$$

$$X(\varphi) = \begin{pmatrix} X^\theta(0)\cos\alpha\varphi + X^\varphi(0) \sin\theta_0 \sin\alpha\varphi \\ X^\varphi(0)\cos\alpha\varphi - X^\theta(0)\frac{\sin\alpha\varphi}{\sin\theta_0} \end{pmatrix}$$

$$\begin{aligned} \| X(\varphi) \|^2 &= R^2 \left(X^\theta(0)\cos\alpha\varphi + X^\varphi(0)\sin\theta_0 \sin\alpha\varphi \right)^2 \\ &\quad + R^2 \sin^2\theta_0 \left(X^\varphi(0)\cos\alpha\varphi - X^\theta(0)\frac{\sin\alpha\varphi}{\sin\theta_0} \right)^2 \\ &= \dots \\ &= X^\theta(0)^2 R^2 + R^2 \sin^2\theta_0 \, X^\varphi(0)^2 = \| X(0) \|^2 \end{aligned}$$

(15) $\boxed{\text{Exercise}}$

Ex. How an affine connection transforms with respect to a coordinate change?

(U, φ) $\quad U \cap V \neq 0 \quad x = (x_1, \dots, x_m)$ $\quad \left\{\frac{\partial}{\partial x_i}\right\}_{i=1,\dots,m}$ basis of tangent space

(V, ψ) $\quad y = (y_1, \dots, y_m)$ $\quad \left\{\frac{\partial}{\partial y_j}\right\}_{j=1,\dots,m}$

∇: Γ^i_{jk} ... for coordinate chart x : $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^m \Gamma^k_{ij} \frac{\partial}{\partial x_k}$

$\tilde{\Gamma}^i_{jk}$ ... ——— " ——— y : $\nabla_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial y_j} = \sum_{k=1}^m \tilde{\Gamma}^k_{ij} \frac{\partial}{\partial y_k}$

Then $\frac{\partial}{\partial y_j} = \sum_{i=1}^m \left(\frac{\partial x^i}{\partial y_j}\right) \frac{\partial}{\partial x^i}$ $\quad \left(\left\{\frac{\partial x^i}{\partial y^j}\right\}_{i,j=1}^m \text{ is the Jacobian of } \cancel{x} \to x(y)\right)$,

so

$$\nabla_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial y_j} = \nabla_{\frac{\partial}{\partial y_i}} \left(\sum_{k=1}^m \underbrace{\left(\frac{\partial x^k}{\partial y_j}\right)}_{\text{fion}} \underbrace{\frac{\partial}{\partial x^k}}_{\text{vector}} \right) =$$

$$= \sum_{k=1}^m \nabla_{\frac{\partial}{\partial y_i}} \left(\frac{\partial x^k}{\partial y_j} \frac{\partial}{\partial x^k} \right) =$$

use standar def. formulas
$\nabla_{fX} Y = f \nabla_X Y$
$\nabla_X (fY) = X(f) + f \nabla_X Y$

$$= \sum_{k=1}^m \left(\frac{\partial^2 x^k}{\partial y^i \partial y^j} \frac{\partial}{\partial x^k} + \frac{\partial x^l}{\partial y^i} \frac{\partial x^k}{\partial y^j} \underbrace{\left(\nabla_{\frac{\partial}{\partial x^l}} \frac{\partial}{\partial x^k}\right)}_{\Gamma^r_{lk} \frac{\partial}{\partial x^r}} \right)$$

$$= \sum_{k=1}^m \left(\frac{\partial^2 x^k}{\partial y^i \partial y^j} + \frac{\partial x^l}{\partial y^i} \frac{\partial x^s}{\partial y^j} \Gamma^k_{ls} \right) \frac{\partial}{\partial x^k},$$

and so

$$\sum_k \tilde{\Gamma}^k_{ij} \frac{\partial}{\partial y^k} = \sum_k \left(\frac{\partial^2 x^k}{\partial y^i \partial y^j} + \frac{\partial x^l}{\partial y^i} \frac{\partial x^s}{\partial y^j} \Gamma^k_{ls} \right) \frac{\partial}{\partial x^k},$$

which is

$$\tilde{\Gamma}^k_{ij} = \sum_{l=1}^m \left(\frac{\partial^2 x^l}{\partial y^i \partial y^j} \frac{\partial y^k}{\partial x^l} + \sum_{r,s=1}^m \frac{\partial x^r}{\partial y^i} \frac{\partial x^s}{\partial y^j} \frac{\partial y^k}{\partial x^l} \Gamma^l_{rs} \right),$$

so Γ is not a tensor.

Exercise

Ex: Geodesic normal coordinate system (coordinate chart):

(M, g) Riem man., $p \in M$, $\epsilon > 0$ such that

$$\exp_p : B_\epsilon(0_p) \to B_\epsilon(p) \subset M$$

is a diff.; $v_1, \dots, v_n$ ON-base of T_pM, $\varphi = (x_1, \dots, x_n)$ local coord. chart

$$\varphi : B_\epsilon(p) \to V := \{w \in \mathbb{R}^m \mid |w| < \epsilon\}$$

defined by

$$\varphi^{-1}(x_1, \dots, x_m) = \exp_p\left(\sum_{i=1}^m x_i v_i\right)$$

$(x_1, \dots, x_m) \equiv$ geodesic normal coordinates

1/ Show that

$$g_{ij}(p) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

in the geodesic normal coordinates.

Pf:

Because $\varphi(p) = 0$, we have

$$\frac{\partial}{\partial x_i}\Big|_p = \frac{d}{dt}\Big|_{t=0} \varphi^{-1}(0 + te_i) = \frac{d}{dt}\Big|_{t=0} \exp_p(tv_i) = v_i.$$

$$\Rightarrow \quad g_{ij}(p) = g_p\left(\frac{\partial}{\partial x_i}\Big|_p, \frac{\partial}{\partial x_j}\Big|_p\right) = \langle v_i, v_j \rangle_p = \delta_{ij}$$

2/ $w = (w_1, \dots, w_m) \in \mathbb{R}^m$ arbitrary, $c(t) = \varphi^{-1}(tw)$, $t \in \mathbb{R}$.

Explain why $c(t)$ is a geodesic, and deduce

$$\sum_{i,j=1}^m w_i w_j \Gamma_{ij}^k(c(t)) = 0.$$

Pf: $c(t) = \varphi^{-1}(tw_1, \dots, tw_m) = \exp_p\left(t \sum_{j=1}^m w_j v_j\right)$.

Denote $v = \sum_{j=1}^m w_j v_j$, so we have shown c is a geodesic with initial vector v.

Let $(c_1, \dots, c_m) = \varphi \circ c$, i.e. $c_j(t) = tw_j$

$c_j'(t) = w_j$

and $\frac{D}{dt}$ covariant derivative along c. $c_j''(t) = 0$, $\searrow$

since $\underline{c}$ is a geodesic,

$$0 = \frac{D}{dt} c' = \frac{D}{dt} \sum_j c_j' \left(\frac{\partial}{\partial x_j} \circ c \right) = \sum_j w_j \nabla_{c'} \frac{\partial}{\partial x_j} =$$

$$= \sum_{i,j} w_j \nabla_{c'} \frac{\partial}{\partial x_j} = \sum_{i,j} w_i w_j \left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \circ c \right) =$$

$$= \sum_k \left(\sum_{i,j} w_i w_j \left(\Gamma_{ij}^k \circ c \right) \right) \left(\frac{\partial}{\partial x_k} \circ c \right)$$

$\frac{\partial}{\partial x_k}$ is a basis $\Rightarrow$ $\sum_{i,j} w_i w_j \Gamma_{ij}^k (c(t)) = 0 \quad \forall k = 1, \dots, m.$

3/ $\forall \Gamma_{ij}^k$ in the chart φ vanish at $p \in M$.

Pf: Evaluating at $t = 0$ $\Rightarrow$

$$\sum_{i,j} w_i w_j \Gamma_{ij}^k (p) = 0 \quad \forall w \in \mathbb{R}^m.$$

The choice $w = e_i + e_j \Rightarrow 2 \Gamma_{ij}^k (p) = 0$,

so $\forall$ Christ symbols vanish at $p \in M$, $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} (p) = 0$

~~Exercises~~

~~Ex: let $(G, \langle -, - \rangle)$ be a cpt Lie group with left-invariant metric, $d\mathrm{vol}_g$ - left inv. volume form. Cpt ness $\Rightarrow \mathrm{Vol}(G) < \infty$~~

~~Define $\langle \, , \, \rangle_{e \in G}$~~

~~$\langle v_1, v_2 \rangle_e = \int_G \langle \mathrm{Ad}_g$~~