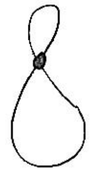
VASSILIEV KNOT INVARIANTS

sources: Dror Bar-Natan: On the Vassiliev knot invar.
Toshitake Kohno; Conformal field theory and topology

more active participants are welcome!

isotopy!!



def a *singular knot* is an immersion $S^1 \longrightarrow \mathbb{R}^3$ such that it's singularities are only transversal self-intersections.

ambient isotopy of sing. knots is similar to amb. is. of ordinary knots: f, g sing. knots are ambient isotopic $\equiv$ $\exists H: \mathbb{R}^3 \times [0,1] \longrightarrow \mathbb{R}^3$ s.t. $H_0 = 1_{\mathbb{R}^3}$, $H_t \circ f$ is sing. knot $\forall t$ $(H_t(x) := H(x,t),\ x \in \mathbb{R}^3,\ t \in [0,1])$ H_t is a homeo $\forall t$ and $H_1 \circ f = g$.

oriented sing. knot ... (oriented always understood in the seq.)

a map $V: \{\text{or. sing. knots}\}/_{\text{isotopy}} \longrightarrow \mathbb{C}$ (or whatever ...) is a *sing. knot invariant*

Given an non-sing. knots invariant
$V: \{\text{or. \underline{non}-sing. knots}\}/_{\text{isotopy}} \longrightarrow \mathbb{C}$,

we extend it to an inv. of sing. knots by a "local" formula:

V() = V() − V() (explain "local")

singular.

i.e. let K be a sing. knot with m-singularities in balls B_i, then

$$V(K) = \sum_{\varepsilon_1, \dots, \varepsilon_m = \pm 1} \varepsilon_1 \cdots \varepsilon_m K_{\varepsilon_1 \cdots \varepsilon_m}$$

where $K_{\varepsilon_1 \cdots \varepsilon_m}$ is a non-singular knot obtained by replacing the sing. in B_k by the positive crossing $\nearrow\!\!\!\!\nwarrow$ for $\varepsilon_k = +1$ and by the negative cr. $\nwarrow\!\!\!\!\nearrow$ for $\varepsilon_k = -1$ for all k.

Def. A (non-sing.) knot invariant V is called a (Vassiliev inv.) of (type) order m $\equiv$
V vanishes on all sing. knots w. $> m$ singularities

obv.: the set of all order m invariants is a vector space over $\mathbb{C}$, denoted $\mathcal{V}_m$, and

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \dots$$

Observation $\mathcal{V}_0 \cong \mathbb{C}$

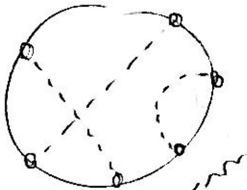
Pf: $V(\nearrow\!\!\!\!\nwarrow) - V(\nwarrow\!\!\!\!\nearrow) = V(\times\!\!\!\!\bullet) \overset{\text{order } 0}{=} 0$

hence V is invariant under exchange of pos./neg. cr.
By those exchanges, any knot can be made into an unknot (knot isotopic to O)
Thus V is characterized only by $V(O) \in \mathbb{C}$. ☒

Exercise $\mathcal{V}_0 \cong \mathcal{V}_1$.

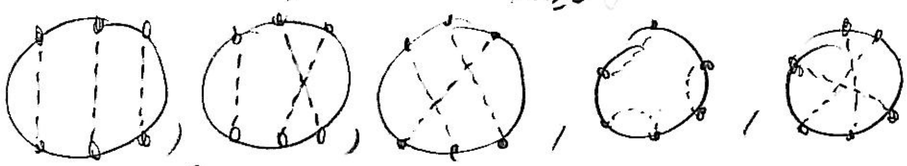
The spaces $\mathcal{V}_m$ can be described by so called "chord diagrams":

def. A chord diagram w. m chords is a standard circle (w. a stand. orient.) w. $2m$ distinct points on it which are divided into disjoint pairs.
Two chord diagrams are equal if there is a or. preserving homeo $S^1 \to S^1$ preserving the pairing.
The pairs are denoted by dashed "chords":



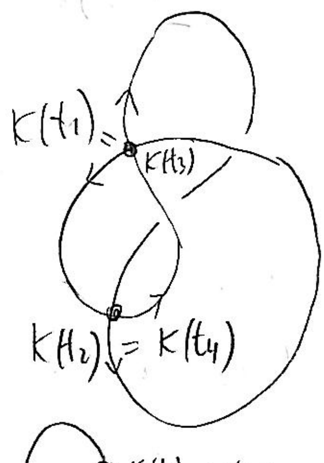
(oriented counter-clockwise)

Let $\mathcal{D}_m^{\mathbb{C}}$ denote the $\mathbb{C}$ $(\mathbb{R})$-lin. span of all chord diags w. m chords, $\mathcal{D}^{\mathbb{C}} := \bigoplus_{m \geq 0} \mathcal{D}_m^{\mathbb{C}}$

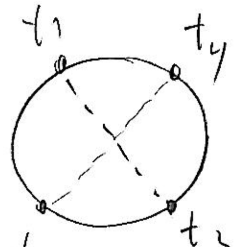
ex. $\mathcal{D}_3^{\mathbb{C}} = \mathbb{C}\{$  $\}$

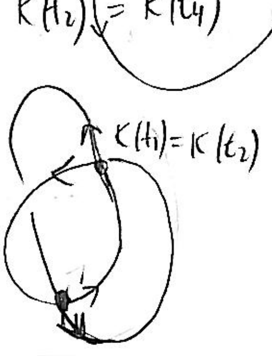
def. A sing. knot $K: S^1 \to \mathbb{R}^3$ is said to respect a chord diag. $C \in \mathcal{D}_m^{\mathbb{C}}$ $\equiv$ $\forall t, t' \in S^1_{=C}$ $K(t) = K(t') \Leftrightarrow$ t and t' are opposite ends of a chord in C or $t = t'$.

EV.



respects





also respects -"-

(4)

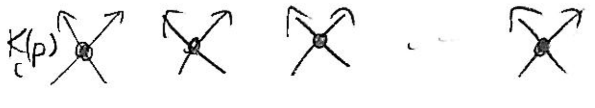
def. Let $\mathcal{V}_m \xrightarrow{\Lambda} \mathrm{Hom}_{\mathbb{C}}(\mathcal{D}^c_m, \mathbb{C})$ be given by

$$V \longmapsto (C \mapsto V(K_C))$$

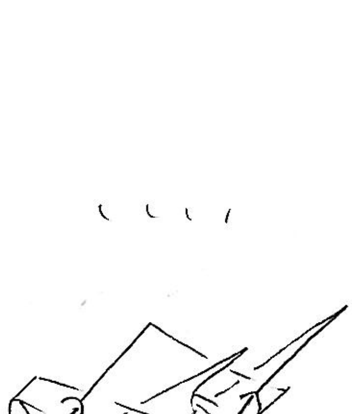
where K_C is any sing knot respecting C

observation: Λ is well defined, i.e. $\Lambda(V)(C)$ doesn't depend on the choice of K_C

Pf: fixing an endpoint p of a chord in C, there is a canonical knot $\tilde{K}_C$ w. m-singularities



the data of the chord tell us what first end of an arrow gets connected to what end of a different arrow.



now given arb. K_C respecting C, if we can change arbitrary +-cross. to --cr. or vice versa, then obviously K_C can be transformed to $\tilde{K}_C$.

Let $K_C^+ = K_C$ have + crossing in certain small ball, let K_C^- be the same as K_C except the crossing is − and K_C^S the crossing is singular.

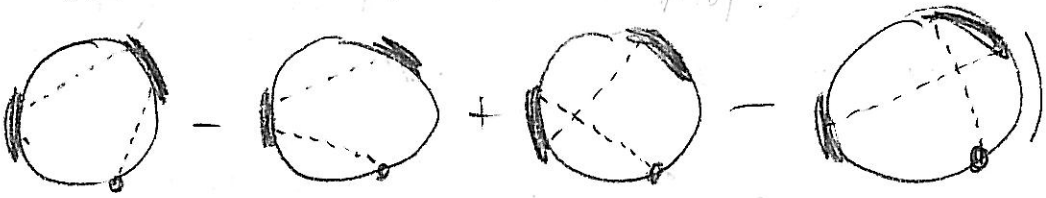
Since K_C^S has m+1 singularities and V is of order m,

$$0 = V(K_c^S) = V(K_c^+) - V(K_c^-) \quad \text{i.e.} \quad V(K_c^+) = V(K_c^-)$$

$\boxed{\text{proposition}}$

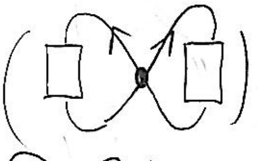
not intersected by any other chord

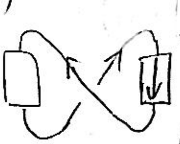
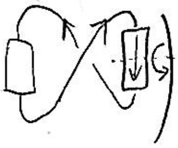
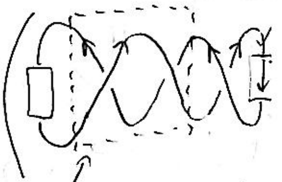
(1) $\Delta(V)$ (chord diagram w. an isolated chord) "framing independence"

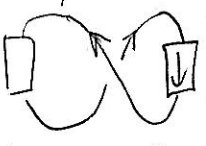
(2) $\Delta(V)$ () $= 0$

"4-terms relation" (4T)

no other chords emanating from " "

Pf: (1) $\Delta(V)\left(\begin{smallmatrix}\text{isol.}\\\text{chord}\end{smallmatrix}\right) = V($  $) =$

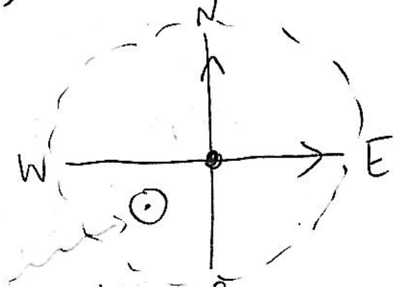
$= V($  $) - V($  $) = \dots - V($  $) = \dots - V($  $)$

$= \dots - V($  $) = 0$

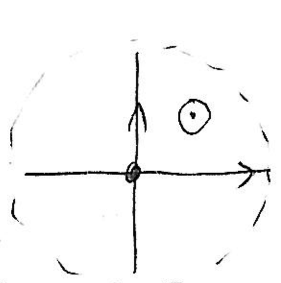
Reidemeister move 

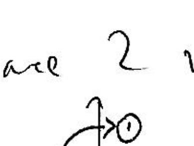
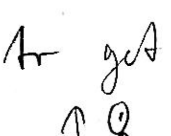
(2) Assume a knot K looks like

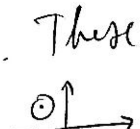
in a small 3D ball

K = SW := 

upwards going arc

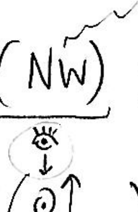
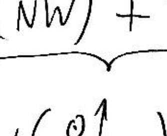
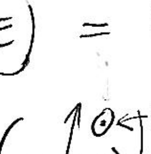
There are 2 ways to get from SW to NE: 

either  or  These 2 correspond

to the 2 eqs below: 

a) $V(SW) \underbrace{- V(SW) + V(NW)}\ \underbrace{- V(NW) + V(NE)} = V(NE)$

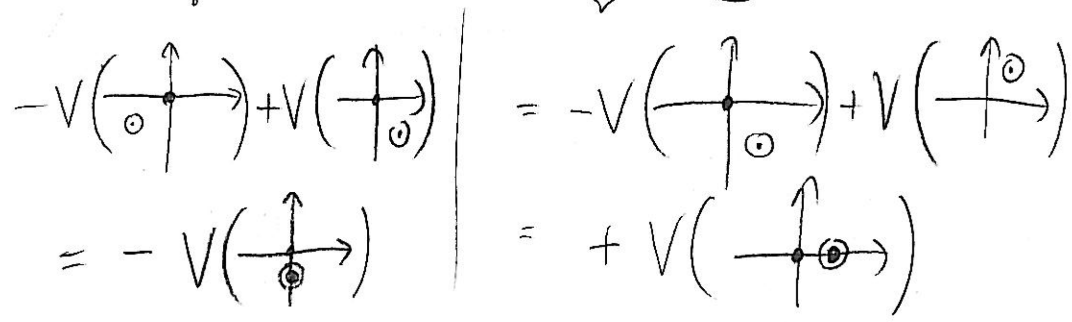
pos. cr.

$= -V($  $) + V($  $)$ | $= -V($  $) + V($  $)$ 

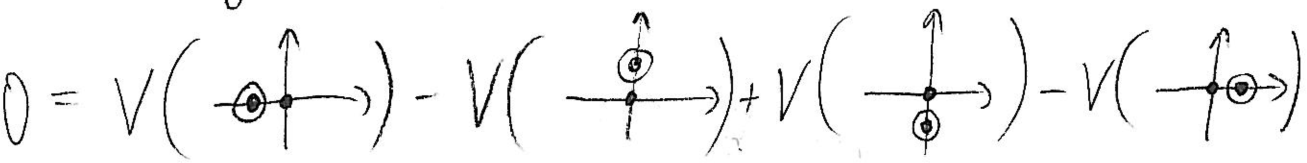
neg. cr.

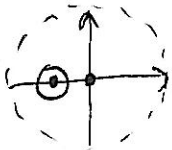
$= +V($  $)$ | $= -V($  $)$

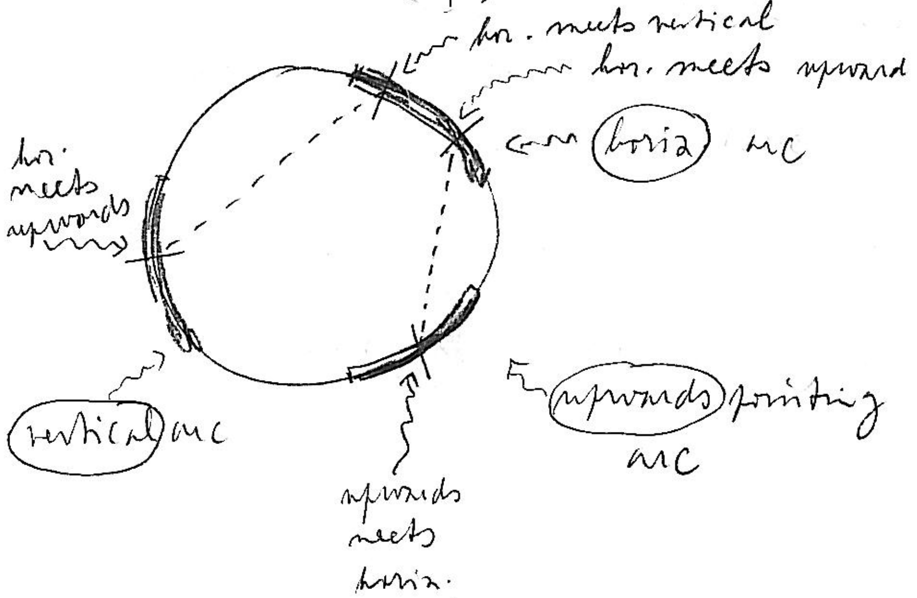
b) $$V(SW) - \underbrace{V(SW) + V(SE)} - \underbrace{V(SE) + V(NE)} = V(NE)$$



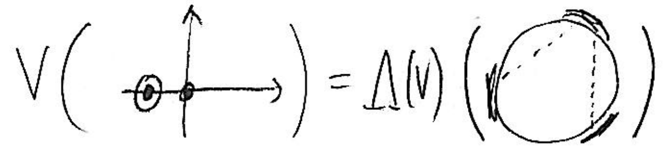
substracting a) − b) ⟹



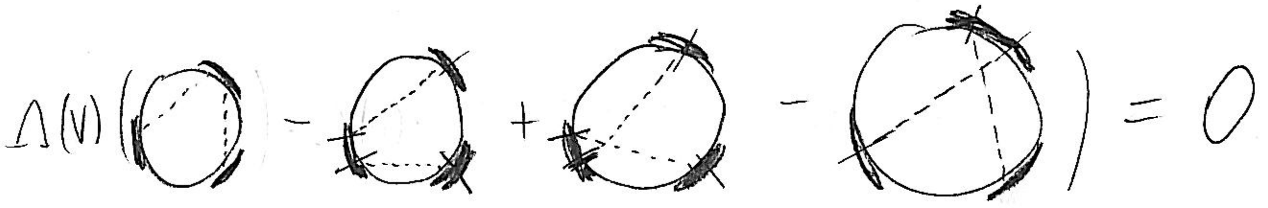
Knot  has a chord diagram



outside the bold arcs, there may be more chords, but there are no more chords on bold arcs other than those shown.

i.e. 

similarly for other knots:



more generally: $\Lambda(V)$  $= 0.$ isolated chord = not intersecting any other chord (generalises (1))

def. Let $\mathcal{A}_m^{C}$ (or chord) $= \mathcal{D}_m^C /$ all 4T relations and framing independence rel.

"algebra" of chord diagrams

linear maps ($\hookrightarrow$ later) $\mathcal{A}_m^C \longrightarrow \mathbb{C}$ is called a <u>weight system</u>.

Now Λ induces $V_m \xrightarrow{\Lambda} \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}_m^C, \mathbb{C})$

if $\Lambda(V) = 0$, then $V(K) = 0$ $\forall K$ knot w. m singularities

(every such knot respects some chord diag $C \in \mathcal{A}_m^C$ and $0 = \Lambda(V)(C) = V(K)$)

hence $\mathrm{Ker}\,\Lambda = V_{m-1}$, hence

$$V_m / V_{m-1} \xrightarrow{\Lambda} \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}_m^C, \mathbb{C}) \text{ is injective.}$$

Thm (Kontsevich, Bar-Natan) Λ is an iso.

The inverse is a nice <u>integral construction</u>!

consequence: any weight system produces an order m-invariant

can be shown: any <u>f.d. Lie alg. repr. produces a W.S.</u>

<u>Example of order m-invariant</u>

$n_+ = \#$ of $+$ crossings

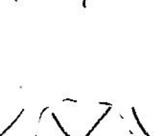
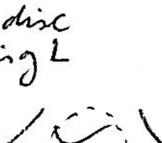
n_- $-$

recall the Jones <u>polynomial</u>:

$$J(\text{or. knot}) = (-1)^{n_- + n_+}\, t^{\frac{1}{2}(n_+ - 2n_-)} \langle \text{unor. knot} \rangle \underbrace{(-t^{1/2} - t^{-1/2})}_{\text{unimportant normalization}}$$

$\mathbb{C}[t^{?}]$ $\langle O \rangle = 1$, $\langle L \cup O \rangle = -(t^{1/2} + t^{-1/2}) \langle L \rangle$,

bounding a disc NOT intersecting L

$\langle$  $\rangle = \langle$  $\rangle + t^{-1/2} \langle$  $\rangle$ is computed inductively

(last time (Khovanov homology lecture): $q = -t^{1/2}$)

$\boxed{\text{Lemma}}$ $+t\,J(\nwarrow\!\!\!\!\nearrow) - t^{-1}J(\nwarrow\!\!\!\!\nearrow) = -(t^{1/2}-t^{-1/2})\,J(\uparrow\uparrow)$ 8

PF: $t\underbrace{(-1)^{n_++n_-}\,t^{-\frac{1}{2}(n_+-2n_-)}}_{=:\square}\langle\times\rangle - t^{-1}(-1)^{n_+^{-1}+n_-^{+1}}\,t^{-\frac{1}{2}(n_+^{-1}-2n_-^{-2(+1)})}\langle\times\rangle =$

$= \square\left(t\langle)(\rangle + \cancel{t^{1-\frac{1}{2}}\langle\asymp\rangle} - \cancel{t^{-1+\frac{3}{2}}\langle\asymp\rangle} - t^{-1+\frac{3}{2}-\frac{1}{2}}\langle)(\rangle\right) =$

$= \square\, t^{1/2}(t^{1/2}-t^{-1/2})\langle)(\rangle = (-1)^{n_+-1+n_-}\,t^{-\frac{1}{2}(n_+-1-2n_-)}\langle)(\rangle(-1)(t^{1/2}-t^{-1/2})$

$= J(\uparrow\uparrow)(-1)(+t^{1/2}-t^{-1/2})$ $\blacksquare$

Set $t=e^x$ and expand in x, thus $J(K)\in\mathbb{C}[[x]]$

Obv. $K \xmapsto{J_m}$ coefficient at x^m in $J(K)$ is an isotopy invariant of knots.

$\boxed{\text{observation}}$ J_m is an inv. of order m.

PF: rewrite lemma in terms of x:

$t^2 J(\nwarrow\!\!\!\!\nearrow) - J(\nwarrow\!\!\!\!\nearrow) = (-t^{3/2}+t^{1/2})\,J(\uparrow\uparrow)$

$(1+2x+\frac{(2x)^2}{2!}+\dots)\,J(\nwarrow\!\!\!\!\nearrow) - J(\nwarrow\!\!\!\!\nearrow) = (-\cancel{1}-x^{3/2}-\frac{x^{2\cdot\frac{3}{2}}}{2!}+\dots+\cancel{1}+x^{1/2}+\frac{x^{2\cdot 1/2}}{2}+\dots)\,J(\uparrow\uparrow)$

$J(\nwarrow\!\!\!\!\nearrow) - J(\nwarrow\!\!\!\!\nearrow) = O(x)$

Hence $J(\text{knot w. } m\text{-singularities}) = O(x^m)$

$J_k(\;-\shortparallel-\;) =$ coef. at x^k in $O(x^m) \overset{\text{if } k<m}{=} 0$ $\blacksquare$

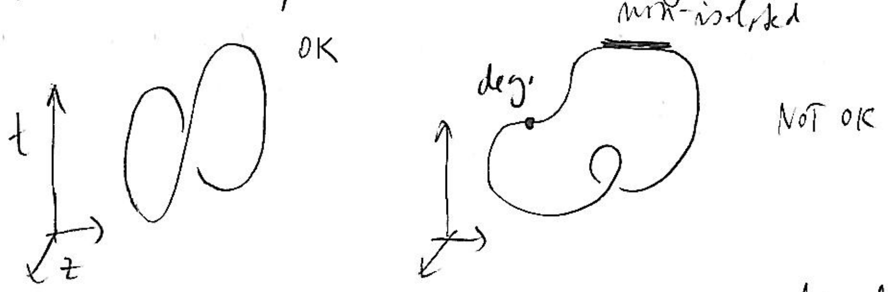
Hence J_k is of order k.

Kontsevich knot integral

$K: S^1 \to \mathbb{R}^3$ a oriented knot

$\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$
$\quad\quad (z, t)$

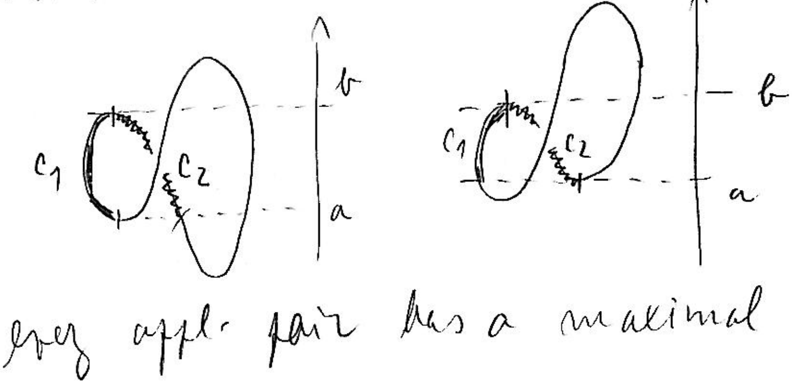
Let $(z,t) \mapsto t$ be a Morse function on $K(S^1)$, i.e. it's critical point are nondegenerate and isolated



$\boxed{\text{def.}}$ • a curve $[a,b) \to \mathbb{C} \times \mathbb{R}$ is applicable $\equiv$ it has the form $t \mapsto (z(t), t)$ and $(z(t), t) \in K(S^1)\ \forall t \in [a,b)$

obvious: Every applicable curve's domain can be extended to maximal domain $[a_M, b_M)$ (if $[a', b') \to \mathbb{C} \times \mathbb{R}$ satisfies 1) $\forall t \in [a', b')$, then $[a', b') = [a_M, b_M)$)
$\quad \underset{[a_M, b_M)}{\cup\!\!\!\,|}$

• a $\overbrace{\text{pair}}^{\text{(unordered)}}$ of applicable curves $c_1, c_2 : [a,b) \to \mathbb{C} \times \mathbb{R}$ is an applicable pair $\equiv$ a is the minimum of the maximal domains of c_1 or c_2 and b is the maximum of the maximal domain of c_1 or c_2



are applicable pairs for respective knots

Obr.: every appl. pair has a maximal common domain def.

$$Z(K) := \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \sum_{\substack{m\text{-tuple } P \\ \text{of applicable pairs} \\ (c_i, c_i')}} (-1)^{P_\downarrow} C_P \int_{\substack{t_1 < \dots < t_m \\ \forall i \; t_i \in [a_i, b_i]}} \bigwedge_{i=1}^{m} \frac{dz_i(t_i) - dz_i'(t_i)}{z_i(t_i) - z_i'(t_i)}$$

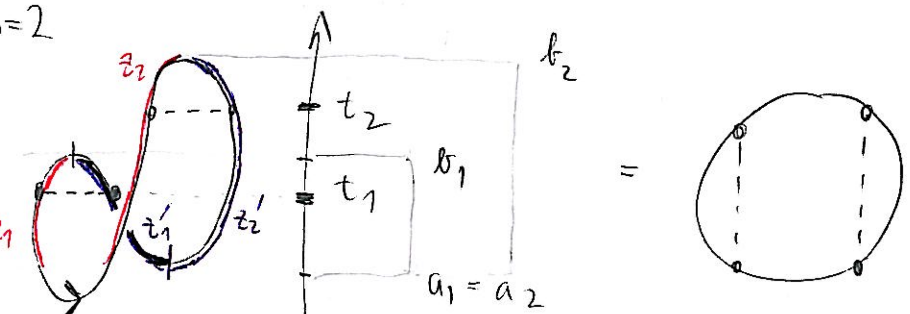
if $C_P = 0$ in $\mathcal{A}^c_1$ then the integral is not even considered (it diverges!)

$c_i, c_i' : \underbrace{[a_i, b_i]}_{\text{max. domain for the pair}} \longrightarrow \mathbb{C} \times \mathbb{R}$

$c_i(t) = (z_i(t), t)$, $c_i'(t) = (z_i'(t), t)$

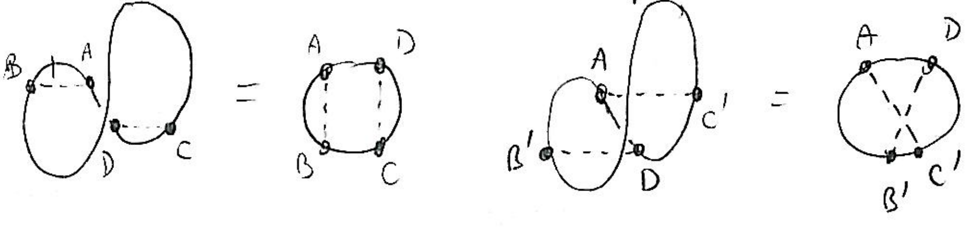
$C_P :=$ chord diag. w. circle $K(S^1)$ and chords connecting $(z_i(t_i), t_i)$ with $(z_i'(t_i), t_i)$, $\forall i = 1, \dots, m$

ex. $m=2$

$C_P =$ 

obvious: C_P doesn't depend on the choice of t_i's in their maximal common domains of $t_1 < \dots < t_m$

(remark: different ordering of t_i's give diff. chord diags.:



)

we are summing over all these different orderings via $\sum_P$, since P is ordered m-tuple

$P_\downarrow$ is the number of curves amongst $c_1, c_1', \dots, c_m, c_m'$ going upwards (recall K is oriented!)

$Z : \{\text{knots}\} \longrightarrow \mathcal{A}^{C} := \prod_{m\geq 0} \mathcal{A}^{C}_{m}$ (over $\mathbb{C}$) 11

• some integrals might diverge, but we will show that then $C_p = 0$ in $\mathcal{A}^C$ and then $C_p \cdot \infty := 0$.

remarks • $\bigwedge_{i=1}^{m} \frac{dz_i(t_i) - dz_i'(t_i)}{z_i(t_i) - z_i'(t_i)} = \Phi^{*}\left(\bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'} \right)$ ← Gibor's lecture

assume

$z_i, z_i' : [a_i, b_i) \longrightarrow \mathbb{C}$

→ $[a, b]$ common for all i's

$\Phi : [a,b] \longrightarrow \mathbb{C}^{2m} \supset \mathrm{Conf}_{2m}(\mathbb{C})$ a curve in $\mathrm{Conf}_{2m}\mathbb{C}$

$\boxed{\text{lemma}}$ The integrals are convergent

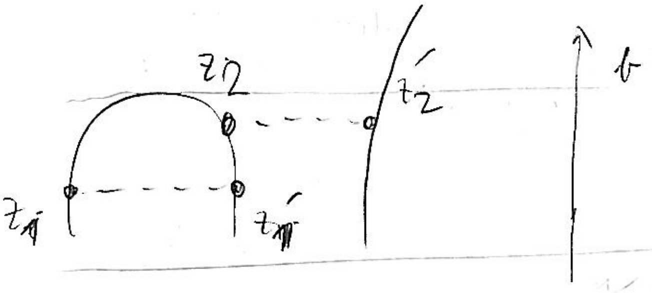
PF: 2 "dangerous" cases:

1)

There is no chord w. index $> i$ s.t. its domain intersects $[a, b]$

Then i-th chord is isolated and $C_p = 0$ in $\mathcal{A}^C$

2)

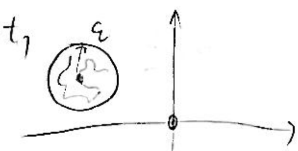


idea: $\int_{b \geq t_2 > t_1 \geq a} \frac{dz_1 - dz_1'}{z_1 - z_1'} \wedge \underbrace{\frac{dz_2 - dz_2'}{z_2 - z_2'}}_{}$

↓ as $t_1 \to 0$
0

this suffices to compensate the LHS singularity

$\int_{b \geq t_2 > t_1} \frac{dz_2 - dz_2'}{z_2 - z_2'} = \int_{b \geq t_2 \geq t_1} \frac{dw}{w}$ = index of



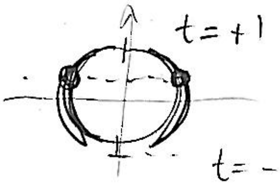
given ε, $\forall t_1$ small $\forall t_2$ $w(t_2)$ is contained in an ε-ball

$\Rightarrow$ index $\to 0$ as $t_1 \to 0$

[? more details]

<u>convention</u>: $m=0$ term of $Z(K)$ is $\bigcirc \in \mathcal{A}^c$

$\boxed{\text{Ex. 1}}$ $K = \bigcirc$



$$z_i(t) = -\sqrt{1-t^2} \qquad z_i'(t) = +\sqrt{1-t^2}$$

$m \leftarrow$ number of "downwards" endpoints of chords

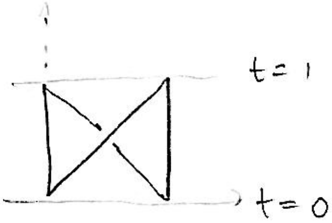
$$Z(K) = \sum_{m\geq 0} \frac{1}{(2\pi i)^m} \sum_{\text{appl. pairings}} (-1)^m \underbrace{\bigcirc}_{\substack{m \\ \text{parallel} \\ \text{chords}}} \int_{-1\leq t_1 < \dots < t_m \leq 1} \bigwedge_{k=1}^{m} \frac{\cancel{\delta}\left(+\frac{-\cancel{\delta} t_k}{\cancel{\delta}\sqrt{1-t_k^2}}\right)dt_k}{\cancel{\delta}\left(+\sqrt{1-t_k^2}\right)}$$

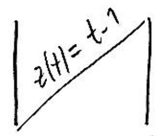
$= 0$ in $\mathcal{A}^c$

$$\int_{-1}^{1} \frac{-t_k}{1-t_k^2}\,dt_k \ \underline{\text{diverges}}$$

$$= \underbrace{\bigcirc}_{m=0} + \underset{m=1}{0} + \underset{m=2}{0} + \dots$$

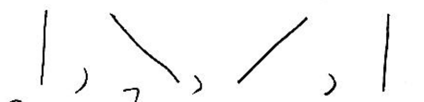
$\boxed{\text{Ex. 2}}$ $K =$ (piecewise linear knot)



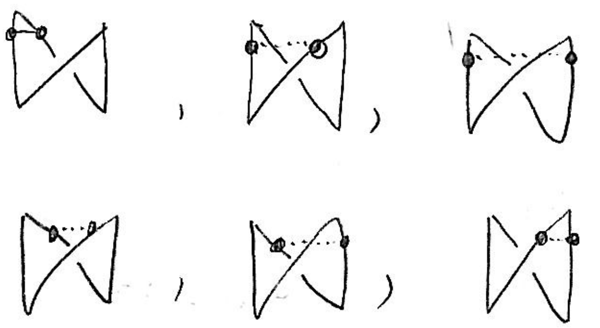


The missing segment is param. by t close to 0 or 1, but close to $\frac{1}{2}$ it's different: to avoid self-intersection

4 applicable curves :



$\binom{4}{2} = 6$ appl. pairs :

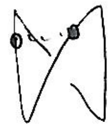


6^m m-tuples of appl. pairs

1-forms corr. to appl. pairs :

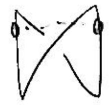
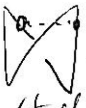
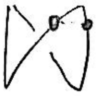
$$z_i(t_i) = 0 \qquad \frac{dz_i - dz_i'}{z_i - z_i'} = \frac{dt}{1-t} \qquad \begin{pmatrix} t \text{ close to} \\ 0 \text{ or } 1 \end{pmatrix}$$

$$z_i'(t_i) = 1-t$$

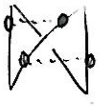


$z_i(t) = 0$ $\qquad \dfrac{dz_i - dz_i'}{z_i - z_i'} = -\dfrac{dt}{t}$

$z_i'(t) = t$

 $\mapsto 0$,  $\longmapsto \dfrac{dt}{t-1}$,  $\mapsto \dfrac{+dt}{+t}$,  $\longmapsto \dfrac{dt}{t-1}$

(t close to 0 or 1) (t close to 0 or 1)

e.g. $m=2$ -term: $\left(\dfrac{1}{2\pi i}\right)^2$  $\displaystyle\int_{0\le t_1<t_2\le 1} \frac{dt_1}{t_1-1} \wedge \left(-\frac{dt_2}{t_2}\right)$

$= -\dfrac{1}{4\pi^2}$  $\displaystyle\int_{0\le t_2\le 1} \frac{\ln(1-t_2)}{t_2}\, dt_2$

"dilogarithm"

$=$  $\left(-\dfrac{1}{4\pi^2}\right)\cdot\left(-\dfrac{\pi^2}{6}\right)$

$m\ge 3$ terms are expressed using iterated integrals of

$\omega_0 = \dfrac{dt}{t}$, $\omega_1 = \dfrac{dt}{1-t}$ $\qquad \displaystyle\int_{0\le t_1<\dots<t_m\le 1} \omega_{I_1}(t_1)\wedge\dots\wedge\omega_{I_m}(t_m)$ $\qquad I_k\in\{0,1\}$

$\boxed{\text{def}}$ Polylogarithm $\displaystyle L_{k_1,\dots,k_n}(z) := \sum_{\substack{0<m_1<\dots<m_n\\ m_k\text{'s}\in\mathbb{N}}} \frac{z^{m_n}}{m_1^{k_1}\cdots m_n^{k_n}}$, $z\in\mathbb{C}$, $|z|<1$

$k_1,\dots,k_n\in\mathbb{N}$, $k_n\ge 2$

$L_{k_1,\dots,k_n}(1) =: \zeta(k_1,\dots,k_n)$ multiple zeta value

$\boxed{\text{PROPOSITION}}$ $\displaystyle L_{k_1,\dots,k_n}(z) = \int_0^z \omega_0^{k_n-1}\omega_1\,\omega_0^{k_{n-1}-1}\omega_1\cdots\omega_0^{k_1-1}\omega_1 =: \int_0^z \tilde\omega_0(z_1)\int_0^{z_1}\tilde\omega_0(z_2)\int_0^{z_2}\cdots dz_N\cdots dz$

where $\omega_0(z) = \underbrace{\tfrac{1}{z}}_{\tilde\omega_0(z)}dz$, $\omega_1(z) = \underbrace{\tfrac{1}{1-z}}_{\tilde\omega_1(z)}dz$ $\qquad N = \sum_{\ell=1}^{n} k_\ell$

integr. along straight lines

PF: (1) if $k_n\ge 2$ $\dfrac{d}{dz}L_{k_1\dots k_n}(z) = \dfrac{1}{z}L_{k_1,\dots,k_n-1}$

(2) $\dfrac{d}{dz}L_{k_1,\dots,k_{n-1},1} = \dfrac{1}{1-z}L_{k_1,\dots,k_n}$

LHS in (1) is $\displaystyle\sum_{m_1<\dots<m_n} \frac{z^{m_n-1}}{m_1^{k_1}\cdots m_{n-1}^{k_{n-1}} m_n^{k_n-1}} = \frac{1}{z}L_{k_1,\dots,k_n-1}$

(14)

LHS in (2) is $\displaystyle\sum_{m_1<\dots<m_{n-1}<m_n} \frac{z^{m_n-1}}{m_1^{k_1}\cdots m_{n-1}^{k_{n-1}}} = \sum_{m_1<\dots<m_{n-1}} \frac{z^{m_{n-1}}}{m_1^{k_1}\cdots m_{n-1}^{k_{n-1}}} \cdot \sum_{\substack{m_n \\ m_n>m_{n-1}}} \frac{z^{m_n-1}}{z^{m_{n-1}}}$

$$= L_{k_1,\dots,k_{n-1}} \cdot \frac{1}{1-z}$$

integrating (1), (2) yields the proposition, e.g.

(1) $\frac{d}{dz}L_2(z) = \frac{1}{z}L_1(z) \Rightarrow L_2(z) = \int_0^z \frac{1}{z_1}\int_0^{z_1}\frac{1}{1-z_2}\,dz_2\,dz_1 =: \int_0^z \omega_0\omega_1$

$$L_3(z) = \int_0^z \frac{1}{z_1}\int_0^{z_1}\frac{1}{z_2}\int_0^{z_3}\frac{1}{1-z_3}\,dz_3\,dz_2\,dz_1$$
$$=: \int_0^z \omega_0\omega_0\omega_1$$

(2) $\frac{d}{dz}L_{2,1}(z) = \frac{1}{1-z}L_2(z) \Rightarrow L_{2,1}(z) = \int_0^z \frac{1}{1-z_1}\int_0^{z_2}\frac{1}{z_2}\int_0^{z_3}\frac{1}{1-z_3}\,dz_3\,dz_2\,dz_1$

$$= \int_0^z \omega_1\omega_0\omega_1 \qquad \blacksquare$$

$$L_{k_1\dots k_n}(z=1) = \zeta(k_1,\dots,k_n) = \int_0^1 \omega_0(t_1)\int_0^{t_1}\omega_0(t_2)\cdots\int_0^{t_{k_1-2}}\omega_0(t_{k_1-1})\int_0^{t_{k_1-1}}\omega_1(t_{k_1})$$
$$\dots\ dt_N\cdots dt_{k_1}\cdots dt_1 \quad \|$$

and these integral appear in $Z_N(\bowtie)$ $\quad \int^{k_{n-1}} \omega_0\omega_1\cdots$

<u>consequence of the integral expression of $\zeta(k_1,\dots,k_n)$</u>: $\quad 1\ge t_1>t_2>\dots t_N\ge 0$

$\varepsilon_k\in\{0,1\}$

$$\int_{1\ge t_1>\dots>t_N\ge 0} \omega_{\varepsilon_1}(t_1)\cdots\omega_{\varepsilon_N}(t_N) = \int_{\substack{0 \\ 1\ge t_N'>\dots>t_1'\ge 0}}^1 \omega_{1-\varepsilon_1}(t_1')\cdots\omega_{1-\varepsilon_N}(t_N') = \int_{1\ge t_N'>\dots>t_1\ge 0} \omega_{1-\varepsilon_N}(t_N')\cdots\omega_{1-\varepsilon_1}(t_1')$$

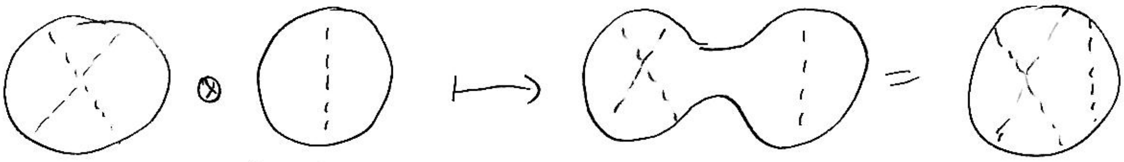
subst. $t_e' = 1-t_e$
$|\det \mathrm{Jac}| = 1$

e.g. $\underline{\zeta(1,2)} = \displaystyle\int_{t_1>t_2>t_3}\omega_0(t_1)\omega_1(t_2)\omega_1(t_3) = \int_{t_3'>t_2'>t_1'}\omega_0(t_3')\omega_0(t_2')\omega_1(t_1') = \underline{\zeta(3)}$

Euler

Horizontal isotopy invariance of $K(Z)$

Def. $\cdot : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ is defined by connected sum at any point other than endpoint of a chord



and extend by $(a_n)_{n\geq 0} \cdot (b_n)_{n\geq 0} = \left(\sum_{k=0}^{n} a_k \cdot b_{n-k}\right)_{n\geq 0}$

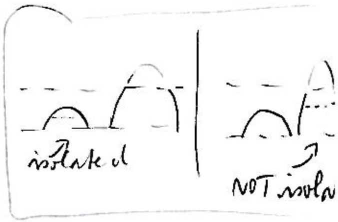
pancake surgery

Thm $\cdot$ on $\mathcal{A}$ is well defined. It is ass., comm. and unital w. unit ○.

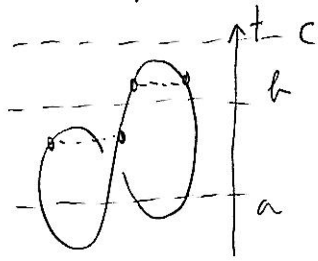
[Proof of the well definedness is quite difficult – omitted]
an example is on p. 29

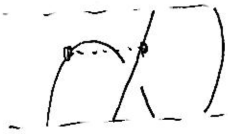
Assume the notation as in the def of $K(Z)$:

$\mathcal{A}^{K,[a;b]} := \mathrm{Span}_{\mathbb{C}}\{$ chord diagrams w. chords only in the arcs corr. to the points $(z,t) \in K([0;1])$ satisfying $a \leq t \leq b$ $\}$ / 4T and "partial" framing independence ↳



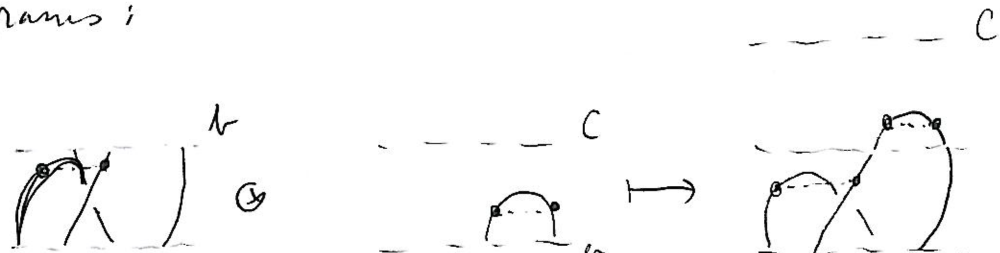
ex. $K =$





$\in \mathcal{A}^{K,[a;b]}$

There is a product $\mathcal{A}^{K,[a;b]} \otimes \mathcal{A}^{K,[b;c]} \longrightarrow \mathcal{A}^{K,[a;c]}$ given by merging the diagrams:

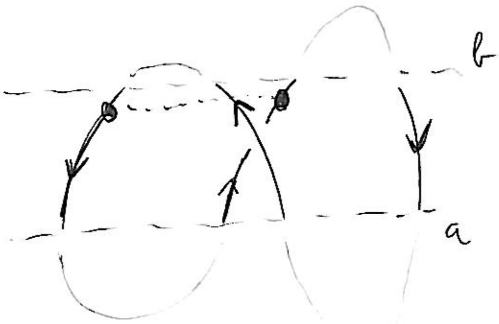


$\boxed{\text{def.}}$ $Z(K;[a;b]) :=$ as $Z(K)$, but all integration domains are intersected with $[a;b]$

$\boxed{\text{observation}}$ $Z(K;[a;b]) \cdot Z(K;[b;c]) \longrightarrow Z(K;[a;c])$

the product $\mathcal{A}^{K;[a;b]} \otimes \mathcal{A}^{K;[b;c]} \to \mathcal{A}^{K;[a;c]}$

Let $[a;b]$ be chosen such that there is no critical point with $a \leq t \leq b$



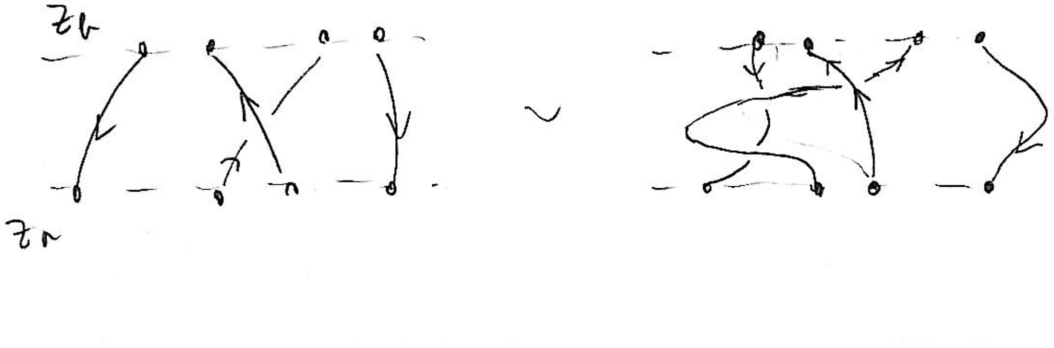
$\in_{\mathcal{A}} K;[a;b]$ n arcs going upwards
n downwards

This is a braid on n points, thus corresponds to a curve Φ in $\mathrm{Conf}_n(\mathbb{C})$ connecting z_a to $z_b \in \mathrm{Conf}_n(\mathbb{C})$

We will show that $Z(K;[a;b])$ computes "holonomy" of certain connection ω on $\mathrm{Conf}_n(\mathbb{C})$ with values in $\mathcal{A}^{K;[a;b]}$ along the curve Φ.
(will be given as a product)

"ω flat" $\Rightarrow$ the holonomy is the same along all curves (connecting z_a, z_b) homotopic to Φ

Homotopy of curves in $\mathrm{Conf}_n(\mathbb{C}) \leftrightarrow$ isotopy of braids, i.e. "horizontal deformations"



Formal connections - overview
(Chen - Iterated integrals)

X a manifold, $A := \prod_{n\geq 0} A_n$ ← f.d. over $\mathbb{C}$ — completion of graded ass. algebra over $\mathbb{C}$

<u>A-valued connection</u> := A-valued 1-form Ω on X
(classical conn. is only locally defined 1-form sat. some compatibility and is $\mathfrak{g}$-valued.
We will use $X = \mathrm{Conf}(n, \mathbb{C})$, which is covered by a single chart, hence Ω is globally defined)

A-valued forms on X ... $\Omega^*(X) \otimes A$
has differential $d(\omega \otimes a) := d\omega \otimes a$
ass. multiplication $(\omega \otimes a) \wedge (\omega' \otimes a') := \omega \wedge \omega' \otimes aa'$
(signs?)

<u>curvature</u> of Ω is $d\Omega + \Omega \wedge \Omega$.

Let $\Phi : I \to X$ be a smooth curve in X.
Holonomy of Ω along Φ is the function $h_{\Phi,\Omega} : I \to A$
satisfying 1) $h_{\Phi,\Omega}(0) = 1 \in A$

2) $\forall t \in I$ $\quad \frac{d}{dt} h_{\Phi,\Omega}(t) = \overbrace{\Omega_{\Phi(t)}\left(\frac{d}{dt}\Phi(t)\right)}^{\in A} \cdot h_{\Phi,\Omega}(t)$

if $h_{\Phi,\Omega}$ exists and is unique.

$\boxed{\text{Thm}}$ Let Ω be homog. of degree 1 in A.
(Chen) "Often" $h_{\Phi,\Omega}(t) = 1 + \sum_{m\geq 1} \int_{0\leq t_1 < \dots < t_m \leq t} (\Phi^*\Omega)_{(t_m)} \wedge \dots \wedge (\Phi^*\Omega)_{(t_1)}$

If Ω is <u>flat</u>, i.e. $d\Omega + \Omega \wedge \Omega = 0$, then $h_{\Phi,\Omega}$ (and also each m-summand) doesn't depend on the homotopy class of Φ (among curves w. fixed endpoints) idea of the proof of the integral formula:

correctness: we are integrating functions w. values in A, but $\underbrace{(\Phi^*\Omega)\wedge\cdots\wedge(\Phi^*\Omega)}_{m}$ has values in A_m, which is f.d., hence we integrate every component separately.

$\sum\limits_{m\geq 0}$ makes sense since each summand is in diff. deg

integrating 2) $\Rightarrow$ $\overbrace{h_{\Phi,\Omega}(1) - h_{\Phi,\Omega}(0)}^{1} = \int_0^t \Omega(\Phi'(s))\cdot h_{\Phi,\Omega}(s)\,ds$

approximation h_n for $h_{\Phi,\Omega}$:

$$h_{n+1}(t) := 1 + \int_0^t \Omega_{\Phi(s)}(\Phi'(s))\cdot h_n(s)\,ds = 1 + \int_0^t h_n\,\Phi^*\Omega$$

$$h_0(t) := 1$$

$$h_1(t) = 1 + \int_0^t \Phi^*\Omega$$

$$h_2(t) = 1 + \int_0^t \Big(1 + \int_0^{t_1} \overbrace{\Phi^*\Omega}^{\dots dt_2}\Big)\, \overbrace{\Phi^*\Omega}^{\dots dt_1} =$$

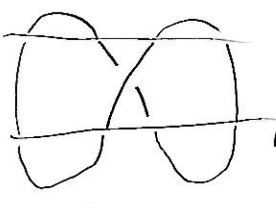
$$= 1 + \int_0^t \Phi^*\omega + \int_0^t\Big(\int_0^{t_1} \underbrace{\Phi^*\Omega}_{\substack{\text{for } t_2\\ 0\leq t_2\leq t_1}}\Big)\, \underbrace{\Phi^*\Omega}_{\substack{\text{for } t_1\\ 0\leq t_1\leq 1}}$$

$\vdots$

our application: given the nicely embedded knot $K: I\to\mathbb{R}^3$

$K(t) = (z(t), t)$ locally.

Let there be no critical point for $a\leq t\leq b$

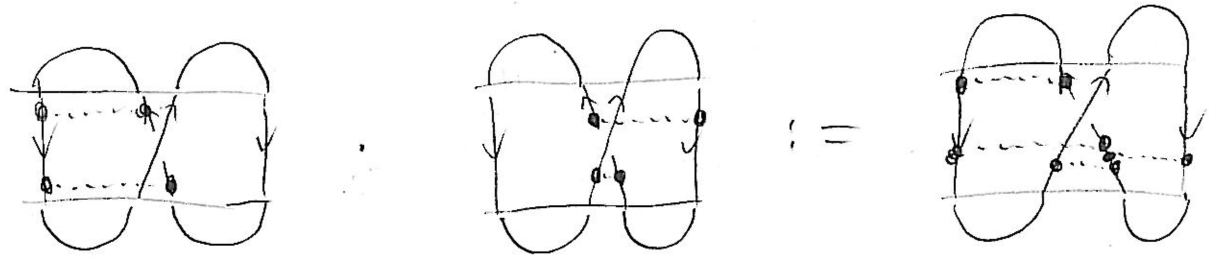


b

a

$A_m := \{ C \in \mathcal{A}^{K_1[a;b]} \mid C$ has m chords $\}$

$A = \prod_{m \geq 0} A_m$ has the following ass. product;

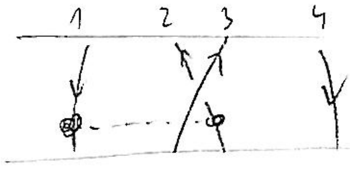


"putting the left chords above the right chords".

Define A-valued connection Ω on $\mathrm{Conf}_n(\mathbb{C})$,
where n is the number of strands in $K([0;1])$ with
$a \leq t \leq b$: [observe: n is even, half of the strands goes up, the other half down)
fix a numbering $1, 2, \ldots, n$ of the strands

$$\Omega := \sum_{1 \leq i < j \leq n} \Omega_{ij}\, \omega_{ij}\, (-1)^{N(i,j)}$$

$\Omega_{ij} \in A_1$ is the diag. with 1 chord connecting i-th and j-th strand

e.g.  $= \Omega_{12}$

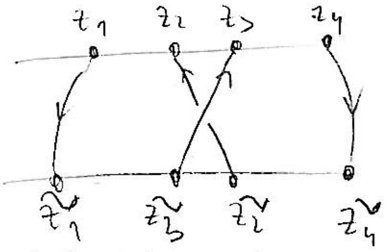
$\omega_{ij} \in \Omega^1(\mathrm{Conf}_n(\mathbb{C}))$

$$\shortparallel\quad d \log(z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j}$$

$$N(i,j) = \begin{cases} +1 & \ldots \text{ else,} \\ -1 & \ldots \text{ iff } i\text{-th and } j\text{-th strand are oriented in opposite way} \end{cases}$$

Ω is called formal Knizhnik-Zamolodchikov conn.

The part of $K([a;1])$ with $a \le t \le b$ determines a curve $\Phi : I \longrightarrow \mathrm{Conf}_n(\mathbb{C})$:



$$\Phi(0) = (\tilde{z}_1, \dots, \tilde{z}_4)$$

$$\Phi(1) = (z_1, \dots, z_4)$$

[orientation is not important now]

Compute holonomy $h_{\Phi,\Omega} : I \longrightarrow A$:

$$h_{\Phi,\Omega}(t) = 1 + \sum_{m \ge 1} \int_{0 \le t_1 < \dots < t_m \le 1} \Phi^* \Omega \wedge \dots \wedge \Phi^* \Omega =$$

$$= 1 + \sum_{m \ge 1} \int_{0 \le t_1 < \dots < t_m \le 1} \sum_{\substack{1 \le i_1 < j_1 \le n \\ \vdots \\ 1 \le i_m < j_m \le n}} \Omega_{i_1 j_1} \cdots \Omega_{i_m j_m} \, \omega_{i_1 j_1} \wedge \dots \wedge \omega_{i_m j_m}$$

?? up to $\frac{1}{(2\pi i)^m}$

$$= 1 + \sum_{m \ge 1} \int_{0 \le t_1 < \dots < t_m \le 1} \sum_{\substack{m \text{ tuples of} \\ \text{appl. curves} \\ ((z_i(t_i), t_i) \\ (z_i'(t_i), t_i))}} (-1)^{P_{\downarrow\uparrow}} C_P \bigwedge_{i=1}^{m} \frac{d z_i(t_i) - d z_i'(t_i)}{z_i(t_i) - z_i'(t_i)} = Z(K;[a;t]$$

where C_P is $K([a;1])$ with $a \le t \le b$ with chords connecting i_1 to j_1, i_2 to j_2, ... (listed from top to bottom)

$P_{\downarrow\uparrow}$ is the number of pairs (c_i, c_i') of appl. curves, where c_i is differently oriented than c_i'

observe: $(-1)^{P_{\downarrow\uparrow}} = (-1)^{P_\downarrow}$ (recall: $P_\downarrow$ is the number of curves going down).

$\boxed{\text{claim}}$ Ω is flat: $d\Omega + \Omega \wedge \Omega = 0$

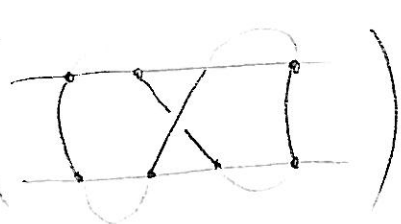
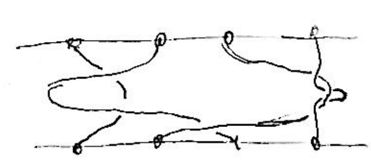
Consequently (by Chen thm), $Z(K;[a;b])$ depends on the choice of Φ only up to homotopy fixing endpoints

... horizontal isotopy invariance of $Z(K)$:

$$Z(K) = \left[Z(K;[t_{MIN},a]) \cdot Z(K;[a,b]) \cdot Z(K;[b;t_{MAX}]) \right]$$

no critical points in $K[0,1]$ with $a \leq t \leq b$

equiv. class by the full framing indep. relation (without it,  isolated)

$Z\Big($  $\Big) = Z\Big($  $\Big)$

(this is NOT yet the FULL isotopy invariance of $K(Z)$!)

Pf of the claim:

$d\omega_{ij} = 0 \Rightarrow d\Omega = 0$

$\Omega \wedge \Omega \overset{?}{=} 0$ $\qquad \Omega \wedge \Omega = \sum\limits_{\substack{1 \leq i<j \leq m \\ 1 \leq i'<j' \leq m}} (-1)^{N(i,j)+N(i',j')} \Omega_{ij} \Omega_{i'j'} \, \omega_{ij} \wedge \omega_{i'j'} = (*)$

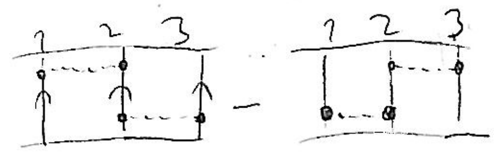
for simplicity, let $N(i,j) = N(i',j') = +1$:

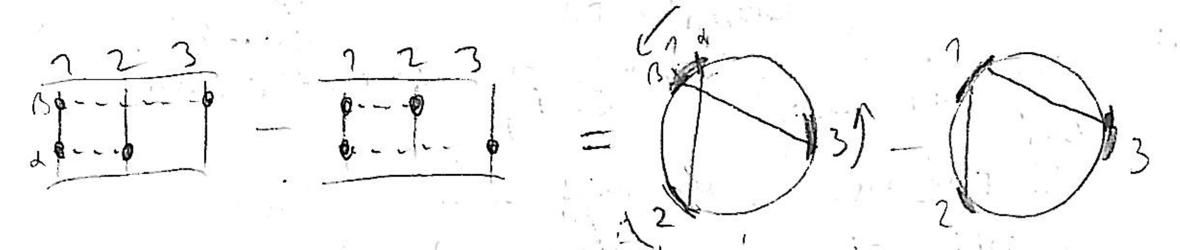
1) if $\{i,j,i',j'\}$ contains 2 or 4 distinct elts, then $\Omega_{ij}\Omega_{i'j'} = \Omega_{i'j'}\Omega_{ij}$ and $\omega_{ij} \wedge \omega_{i'j'} = -\omega_{i'j'} \wedge \omega_{ij}$ hence these terms contribute 0 to $(*)$.

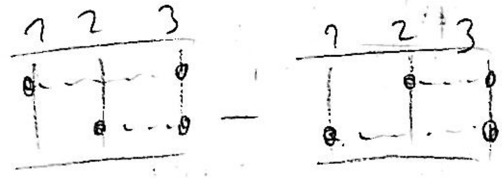
2) let $\{i,j,i',j'\} \overset{WLOG}{=} \{1,2,3\}$ contributes

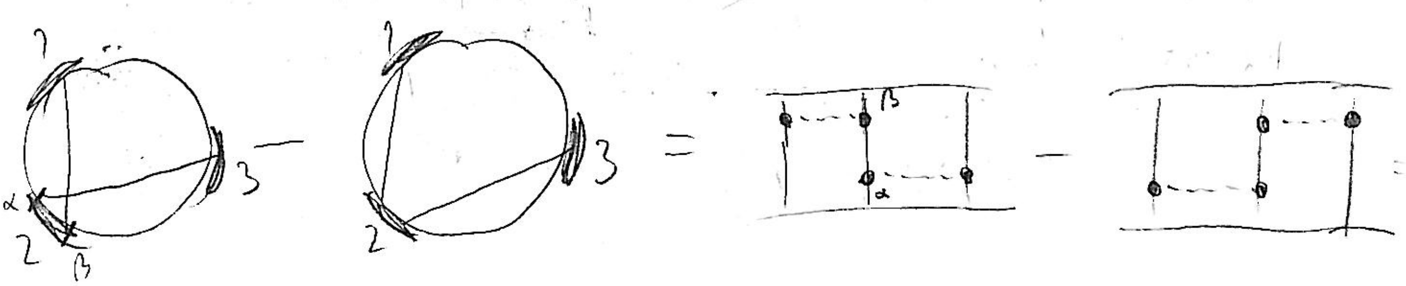
$$(**)\begin{cases} (\Omega_{12}\Omega_{23} - \Omega_{23}\Omega_{12})\, \omega_{12} \wedge \omega_{23} + \\ + (\Omega_{12}\Omega_{13} - \Omega_{13}\Omega_{12})\, \omega_{12} \wedge \omega_{13} \\ (\Omega_{13}\Omega_{23} - \Omega_{23}\Omega_{13})\, \omega_{13} \wedge \omega_{23} \end{cases}$$

12 23
12 13
13 12
13 23
23 12
23 13

$E_1 =$ 

$E_2 =$ 

$E_3 =$ 

$=$ 

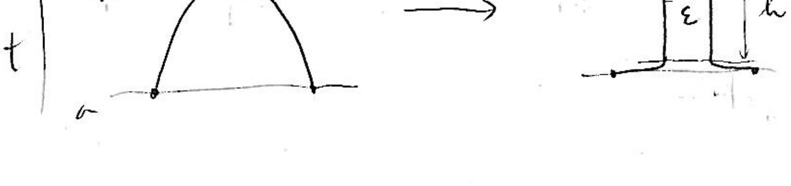
Hence $E_2 = E_1$, $E_3 = E_1$.

Thus $(**) = (\Omega_{12}\Omega_{23} - \Omega_{23}\Omega_{12})\underbrace{(\omega_{12}\wedge\omega_{23} + \omega_{12}\wedge\omega_{13} + \omega_{13}\wedge\omega_{23})}_{=0 \text{ Arnold's identity for } \mathrm{Conf}_n(\mathbb{C})}$

[direct verification: Zibor] $\blacksquare$

Moving critical points

$\forall$ crit. point can be transformed by horizontal transf- as foll.:

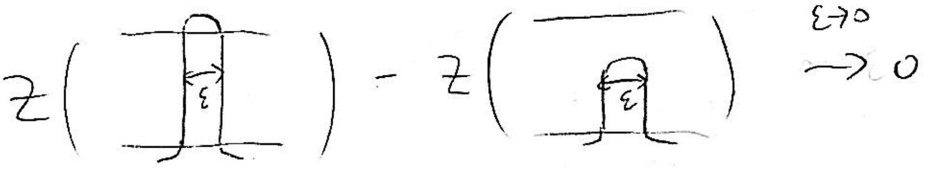


needle of width ε and height h

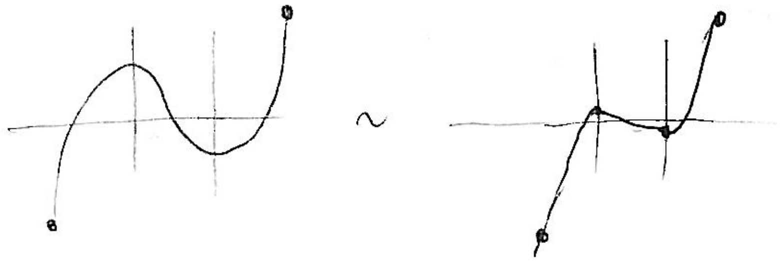
Lemma If two knot embeddings K_1, K_2 differ only by height of a needle of width ε, then

$$\lim_{\varepsilon \to 0+} \| Z_m(K_1) - Z_m(K_2) \| = 0$$

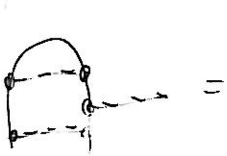
Here $\|\cdot\|$ is any norm on $\mathcal{A}_m^C$ (fixed from now on), since $\mathcal{A}_m^C$ is finite dim, all norms are equivalent and the claim is indep. on the choice.
Z_m is the piece of Z with m chords.

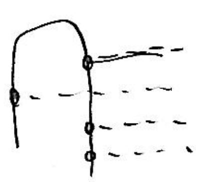


Conseq.: ε can be made arb. small by horizontal defs, so crit. points can be "almost removed":

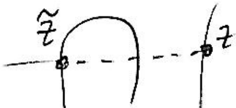


PF: $Z_m(K_1)$ and $Z_m(K_2)$ differ only by integrals corr. to chords with one or both endpoints contained on the needle.
We will show: these terms in $Z_m(K) \to 0$ as $\varepsilon \to 0$.

1)  $= 0$ in $\mathcal{A}_m^C$ — top chord in the needle is isolated

2)  no chord with both endpoints on the needle

For every chord diag. D 

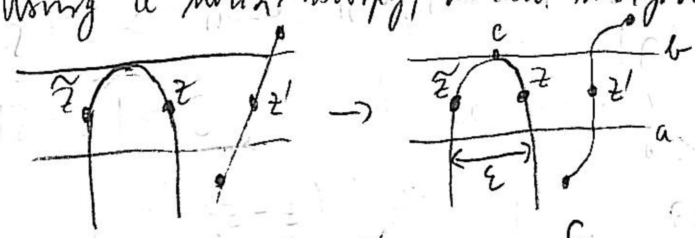
there is a chord diag $\tilde{D}$ coinciding w. D everywhere except 

In $Z_m(K)$, we pair D to $\tilde{D}$

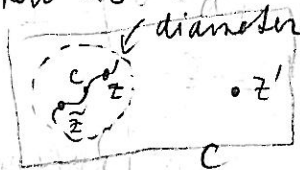
D contributes … $\displaystyle\int_a^b \frac{dz - dz'}{z - z'}$ $\quad \tilde{D}$ … $\displaystyle -\int_a^b \frac{d\tilde{z} - dz'}{\tilde{z} - z'}$

due to the sign $(-1)^{P_\downarrow}$ in $Z_m(K)$

Using a horiz. isotopy, we can straighten the arc on which z' lies so that $z'(t) = \text{const.}$



Then $dz' = 0$



$\Rightarrow \ln(\cdot - z')$ has domain on which the foll. holds:

$$\int_a^b \frac{dz}{z-z'} - \int_a^b \frac{d\tilde{z}}{\tilde{z}-z'} = \ln(\overbrace{\cancel{z(b)}}^{c} - z') - \ln(z(a) - z') - \ln(\overbrace{\cancel{\tilde{z}(b)}}^{c} - z') + \ln(\tilde{z}(a) - z') =$$

$$= \int_{z(a)}^{\tilde{z}(a)} \frac{dz''}{z''-z'}$$

This is independent of the curve connecting $z(a)$ and $\tilde{z}(a)$ as long as it stays within the domain of $\ln(\cdot - \tilde{z})$

$$\left| \int_{z(a)}^{\tilde{z}(a)} \frac{dz''}{z''-z'} \right| \leq \underbrace{\sup_{z'' \in [z(a), \tilde{z}(a)]} \left(\frac{1}{|z''-z'|} \right)}_{\text{bounded since}} \cdot \overbrace{|\tilde{z}(a) - z(a)|}^{\leq \varepsilon} \leq \frac{\varepsilon}{K_1}$$

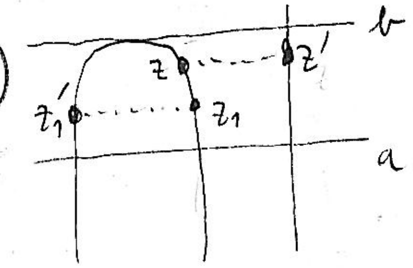
the line segment

bounded since

$|z'' - z'| \geq K_1$ since

z'' is within the ε-ball

Hence the contributions of D and $\tilde{D}$ are $\to 0$ as $\varepsilon \to 0+$.

3) 

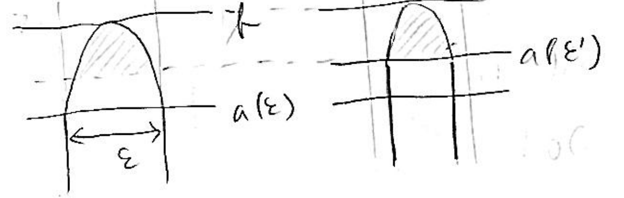
Any chord of the needle is not isolated and there are chords w. both endpoints on the needle.

$$\int_{a \leq t_1 \leq t \leq b} \frac{dz_1 - dz_1'}{z_1 - z_1'} \wedge \frac{dz - dz'}{z - z'} = \int_{a \leq t_1 \leq b} \frac{dz_1 - dz_1'}{z_1 - z_1'}(t_1) \cdot \overbrace{\int_{t_1 \leq t \leq b} \frac{dz - dz'}{z - z'}(t)}^{f(t_1)} = \int_{a \leq t_1 \leq b} f(t_1) \frac{dz_1 - dz_1'}{z_1 - z_1'}(t_1)$$

is convergent by lemma on p. 11

Shrinking the needle (ε) corresponds to raising a, i.e. $a = a(\varepsilon)$

$a \to b$ as $\varepsilon \to 0$:



By continuous dependence of the integral on the integration param. a, the int. $\to 0$ as $\varepsilon \to 0$.

A^c has algebra str., see p. 15

$$Z(\infty) = 1 + \sum_{n\geq 1} a_n , \quad a_n \in A_n^c$$

hence $Z(\infty)$ can be inverted in A^c!

$$\Big(1+\sum_{n\geq 1} a_n\Big)\Big(\sum_{n\geq 0} b_n\Big) = 1$$

$1 = b_0$
$0 = a_1 b_0 + b_1$
$0 = a_2 b_0 + a_1 b_1 + b_2$
$\vdots$

$\boxed{\text{def.}}$ $\tilde{Z}(K) := \dfrac{Z(K)}{Z(\infty)^{\frac{c}{2}-1}}$, where c is the number of crit. points on K

<u>Observe</u>: c is even

$\boxed{\text{Thm}}$ $\tilde{Z}(K)$ is invariant under all isotopies of K.

PF: it remains to check invariance of $\tilde{Z}$ under $\sim \leftrightarrow /$

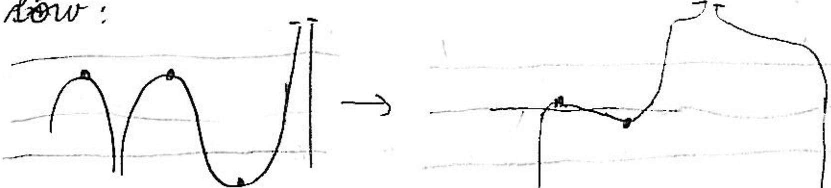
claim: $Z(\sim) = Z(\infty)\, Z(/)$

identical embedding except for the pictured area

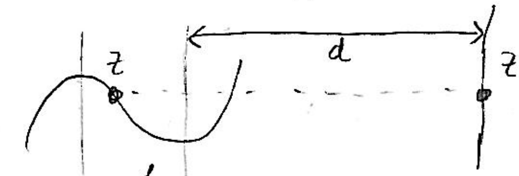
consequently: $\tilde{Z}(\sim) = \tilde{Z}(/)$

verification of the claim: use horiz. defs and needle shrinking to get the bumpy part $\sim$ far from the rest of the knot and make the bumps low:

e.g.



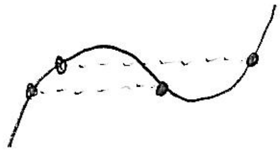
contributions to $Z(K)$ from chords whose 1 endpoint is not in the bumpy part:

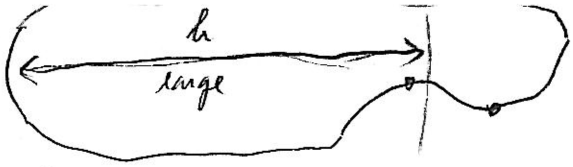


$$\lim_{d\to\infty} \int_a^b \frac{dz-dz'}{z-z'} = 0$$

hence these contribs can be neglected
(contrib. to chords w. at least 1 endpoint in the bumpy part)

nontriv. contr. to $Z(K)$ can come only from chords with <u>both</u> endpoints in the bumpy part; the contribs w. both endpoints outside the bumpy part are the same for $\sim$ and $/$.





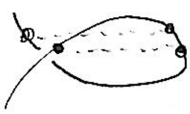
$\underbrace{\text{these contrib.}}_{C}$ are the same as in ↗ and $C - Z(\ \ ) \xrightarrow{h\to\infty} 0$

by horiz. isotopy: $Z(\ \ ) = Z(\ \ )$

(or contributions from  isolated chord)

hence $C = Z(\ \ )$ for any h.

claim C is the same as contributions



I CAN'T VERIFY THIS !
what does it mean precisely? (limit?)
how is ∞ precisely constructed from $\sim$?

then $C = Z(\ \ ) = Z(\infty)$ as before.

why don't we use $Z(\ \ )^{\frac{c}{2}-1}$?

M

Recall the original problem:

$$V_m \xrightarrow{\Delta} \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}_m^C, \mathbb{K})$$

$$\Delta(V)(C) := V(K_C) \quad \text{knot respecting } C$$

V_m $\mathcal{A}_m^C$

$\mathrm{Ker}\,\Delta = V_{m-1}$ and we want to show that Δ is surjective

$$V_m \xleftarrow{\Xi} \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}_m^C, \mathbb{K})$$

$$\Xi(W)(K) := W\tilde{Z}_m(K)$$

$$W: \mathcal{A}_m^C \to \mathbb{K}$$

$\boxed{\text{Theorem}}$ (Kontsevich) $\tilde{Z}(K)$ has rational coefficients

[hard; $\mathbb{R}$ coef. is easier]

$\boxed{\text{Thm}}$ For $\mathbb{K} = \mathbb{R}$,

2) $\Xi\Lambda(V) = V + \tilde{V}_{m-1}$ for any $V \in \mathcal{V}_m$, there is $\tilde{V} \in \mathcal{V}_{m-1}$ such that this holds!

1) $\Lambda\Xi(W) = W$ for any $W \in \mathcal{A}^C_{\mathbb{R}, m} \longrightarrow \mathbb{R}$

i.e. $\mathcal{V}_m / \mathcal{V}_{m-1} \underset{\Xi}{\overset{\Lambda}{\rightleftarrows}} \mathrm{Hom}_{\mathbb{R}}(\mathcal{A}^C_{\mathbb{R}, m}, \mathbb{R})$ is iso of v.s.

PROOF: 1) $\Lambda\Xi(W)(\underset{\mathcal{A}^C_m}{C \in}) = \Xi(W)(K_C) = W\tilde{Z}_m(K_C) \overset{?}{=} W(C)$

suffices $\tilde{Z}(K_C) = C + $ (homogen. chord diagrams with $> m$ chords)

~~suffices $Z(K_C) = C + \dots$~~

since $\tilde{Z}(K_C) = Z(K_C) \cdot \underbrace{Z(\infty)^{1-\frac{c}{2}}}_{\bigcirc + \text{ terms in } \mathcal{A}^C_{\geq 1}}$

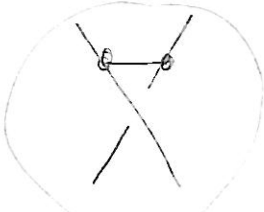
K_C is a sing. knot with m self-intersections respecting C

$\Xi(K_C) = \sum_{\varepsilon_1, \dots, \varepsilon_m = \pm 1} \varepsilon_1 \cdots \varepsilon_m K_{C, \varepsilon_1 \dots \varepsilon_m}$

i-th self-int. is replaced by $\nearrow$ $(\varepsilon_i = +1)$ or by $\nwarrow$ $(\varepsilon_i = -1)$

$\boxed{\text{claim}}$

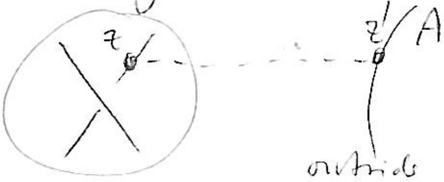
$Z(K_{C, \varepsilon_1 \dots (\varepsilon_i = +1) \dots \varepsilon_m}) - Z(K_{C, \varepsilon_1 \dots (\varepsilon_i = -1) \dots \varepsilon_m})$ has nonzero contrib. only from chords of the type



resolution of i-th singularity (i-th crossing)

PF: obv. only the chords w. at least one end attached to the i-th crossing contribute.

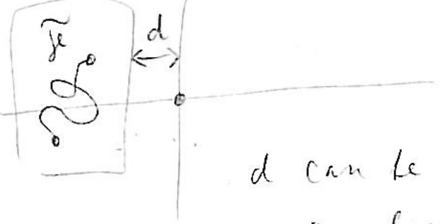
For



outside i-th crossing

use horizontal deformation of $K_{C, \varepsilon_1 \dots \varepsilon_m}$ to move the crossing horizontally far from the rest of the knot.

Then $\int_\gamma \frac{dz-dz'}{z-z'} \overset{!}{=} \int_{\tilde\gamma} \frac{d\lambda}{\lambda}$ $\quad (\lambda = z-z')$



big d

$\| \quad \ln\tilde\gamma(1) - \ln\tilde\gamma(0) \xrightarrow[d\to\infty]{} 0$

d can be arbitrarily big using horizont. defs.!

So contributions from

are arb. small, hence 0. $\boxed{}$ (claim)

if there is no chord near i-th crossing, then obv.

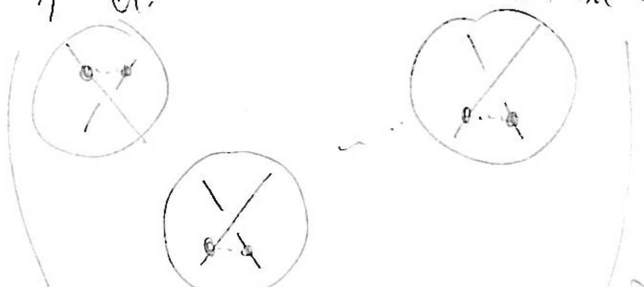
$\left(\text{such contributions to } Z(K_c, \varepsilon_1, \dots, \varepsilon_i=+1, \dots, \varepsilon_m)\right) = \left(\text{such contributions to } Z(K_c, \varepsilon_1, \dots, \varepsilon_i=-1, \dots, \varepsilon_m)\right)$

and thus $\left(\text{such contributions to } Z(K_c)\right) = 0$.

in particular $Z(K_c)$ = terms w. $\geq m$ chords.

$\boxed{\text{claim}}$ $Z_m(K_c) = \alpha C$ for some $\alpha \in \mathbb{R}$

Pf: 1st cr. $\quad$ m-th cr.



2nd cr.

exactly 1 chord at every crossing

term in $Z(K_c, \varepsilon_1, \dots, \varepsilon_m) \in A_m^C$

This chord diag is precisely C (recall every chord in C corr. to singularity in K_c)

(no matter whether "$\nwarrow$" or "$\nearrow$" is chosen at each vertex)

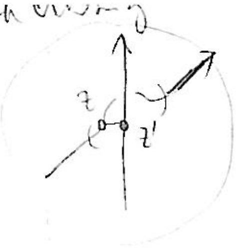
$\boxed{}$ claim

$\boxed{\text{claim}}$ $\alpha = 1$

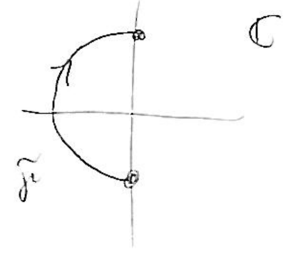
use hor. defs to turn all of the m crossings into

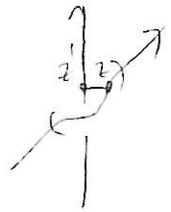
the vertical one is a straight line

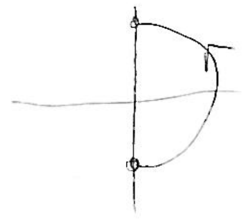
k-th crossing

1) 

$$\int \frac{dz - dz'}{z - z'} = \int_{\tilde{\gamma}} \frac{d\lambda}{\lambda}$$



2) 



$-$ contr. from 1) $+$ contr. from 2) $= \int_{\text{circle}} \frac{d\lambda}{\lambda} = 2\pi i$

(part of $Z(K_{C,\varepsilon_1,\dots,\varepsilon_k=+1,\dots,\varepsilon_m})$)

... $Z(K_{C,\varepsilon_1,\dots,\varepsilon_k=-1,\dots,\varepsilon_m})$

These contrib. in $Z_m(K_C)$ are $(2\pi i)^m$ and these cancel w. normal. fact. $\left(\frac{1}{2\pi i}\right)^m$ (p. 10)

■ claim

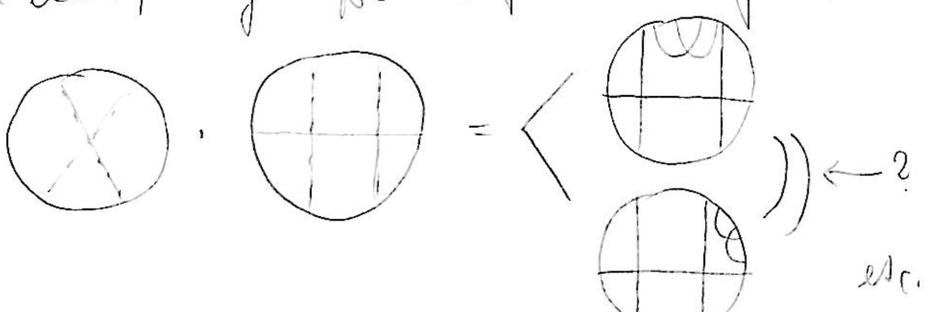
2) $$\Lambda(V - \Xi\Lambda(V)) = \Lambda(V) - \overbrace{\Lambda\Xi}^{1 \text{ by } 1)}\Lambda(V) = 0$$

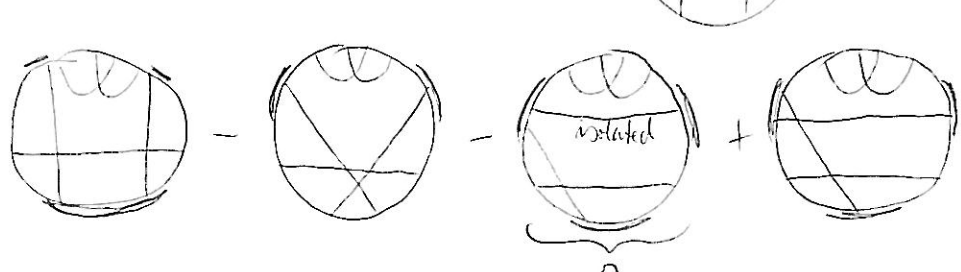
$$V - \Xi\Lambda(V) \in \ker\Lambda = V_{m-1}$$

■

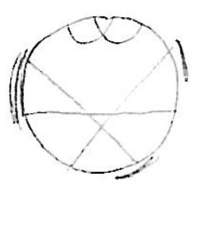
Remarks

an example of well definedness of mult. on $\mathcal{A}^c$:





$= 0$



$= 0$