compare to : $\frac{1}{2(l+h^\vee)} \sum_{a=1}^{d} \Big(\sum_{k\in\mathbb{Z}} J^a_k J_{a,-k} + \sum_{k\geq 0} kC \Big)$

naive definition

$\boxed{\text{Thm}}$ $[L_n, X_m] = -m X_{n+m}$

$[L_n, L_m] = (n-m) L_{n+m} + \frac{c_g(l)}{12}(n^3-n)\,\delta_{n,-m}$

where $c_g(l) = \frac{l \dim g}{2(l+h^\vee)}$ ... the central charge.

$\boxed{\text{Lemma}}$ $\sum_{a=1}^{d} \left([X, J^a]_m J_{a,n} + J_{a,m} [X, J_a]_n \right) = 0$, $X \in g$

Pf: $\sum_a [X, J^a]_m J_{a,m} = \sum_a \sum_{b=1}^{d} K_0([X,J^a], J^b) J_{b,m} J_{a,n}$

$[X, J^a] = \sum_b K_0([X,J^a], J^b) J_b$

$= -\sum_a \sum_b K_0(J^a, [X, J^b]) J_{b,m} J_{a,n} = -\sum_b J_{b,m} [X, J^b]_n$ $\boxed{\text{}}$

Pf of the Thm.:

$[X_m, 2(l+h^\vee) L_n] = \sum_a \Big(\underbrace{\sum_{k<0} [X_m, J^a_k] J_{a,n-k} + \sum_{k<0} J^a_k [X_m, J_{a,n-k}]}_{1} \Big) + \underbrace{\sum_{k\geq 0} (\ldots)}_{2}$

$1 = \ldots = \sum_a \sum_{-m\leq k<0} J^a_{k+m} [X, J_a]_{n-k} + \sum_{k<0} \left(m X_{n} k\, \delta_{m,-k} + m X_k \delta_{m,-n+k} \right) C$

$2 = \ldots = \sum_a \sum_{m>k\geq 0} [X, J_a]_{n+m-k} J^a_k + \sum_{k\geq 0} \left(m X_k \delta_{m,-n+k} + m X_{n-k} \delta_{m,-k} \right) C$

$1 + 2$ by the lemma

$\| $

$2mlX_{n+m} + m \sum_{a=1}^{d} [J^a, [J_a, X]]_{n+m} = \underset{2mlX_{n+m}+}{\,} \mathrm{ad}(C)\, X_{n+m} = 2mh^\vee X_{n+m} + 2ml X_{n+m} = 2m(l+h^\vee) X_{n+m}$

The Casimir C acts on irred. by mult. by a number

$(\lambda,\lambda) + 2(\lambda,\rho)$

$\rho = \frac{1}{2} \sum_{\alpha\in\Delta^+} \alpha$ $\boxed{\text{}}$

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Σ closed Riem. surface

$x_1, \dots, x_n \in \Sigma_n$ distinct

$t_1, \dots, t_n$ local coordinates $\quad t_i : \begin{matrix} \Sigma \\ \cup \\ U_i \\ \cup \\ x_i \end{matrix} \longrightarrow \mathbb{C} \qquad t_i(x_i) = 0$

$\mathfrak{g}$ ... a cpx simple Lie alg.

$\mathfrak{g}(x_1, \dots, x_n)$ ... the Lie alg. of $\mathfrak{g}$-valued meromorphic functions on Σ with poles at $x_1, \dots, x_n$ at most.

$\tau_j : \mathfrak{g}(x_1, \dots, x_n) \hookrightarrow \mathfrak{g}((t_j))$ ... via Laurent expansion at x_j in t_j

$\tilde{\bigoplus\limits_{i=1}} \mathfrak{g}((t_i)) \oplus \mathbb{C}c$ ... a diagonal central extension:

$$\left[(x_1 \otimes f_1, \dots, x_n \otimes f_n), (y_1 \otimes g_1, \dots, x_n \otimes g_n)\right] =$$
$$= \left([x_1, y_1] \otimes f_1 g_1, \dots, [x_n, y_n] \otimes f_n g_n\right) - \sum_i (x_i, y_i) \operatorname*{Res}_{t_i = 0} (f_i \, dg_i)$$

$$f \longmapsto (\tau_1(f), \dots, \tau_n(f))$$

$$\mathfrak{g}(x_1, \dots, x_n) \hookrightarrow \bigoplus_i \left(\mathfrak{g}((t_i)) \oplus \mathbb{C}c\right)$$
$$\cup$$
$$\bigoplus_i \mathfrak{g}((t_i))$$

(recall: use Residue thm) to prove it's Lie mor.

$\ell \in \mathbb{N}$

$\lambda_1, \dots, \lambda_n \in P_\ell = \{\lambda \in P_+ \mid 0 \leq (\lambda, \theta) \leq \ell\}$

$H_{\lambda_1}, \dots, H_{\lambda_n}$ the integrable highest weight modules of level ℓ with the highest weight $\lambda_1, \dots, \lambda_n$.

left action of $\mathfrak{g}(\lambda_1, \dots, \lambda_n)$ on $H_{\lambda_1} \otimes \dots \otimes H_{\lambda_n}$

The space of vacua (of conf. theories) $V^{+}_{\vec{\lambda}}(\Sigma)$ is the space of lin. forms $\varphi : H_{\lambda_1} \otimes \dots \otimes H_{\lambda_n} \longrightarrow \mathbb{C}$ invariant under $\mathfrak{g}(x_1, \dots, x_n)$.

$\Sigma = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$

$t_i = z - x_i \qquad x_i \neq \infty$

$t_i = \frac{1}{z} \qquad x_i = \infty$

$H_{\vec\lambda} := \bigotimes_{i=1}^{n} H^{k}_{\lambda_i}$ $\quad V_{\vec\lambda} := \bigotimes_i V_{\lambda_i}$ ... finite dimensional

recall:

$\mathcal{M}_\lambda = \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{g}[[t]] \oplus \mathbb{C}c)} V_\lambda$

$(\mathfrak{g} \otimes t\,\mathbb{C}[[t]])V_\lambda = 0$

$c \cdot V_\lambda = \ell \cdot V_\lambda$

$H_\lambda = \mathcal{M}_\lambda / J_\lambda$ ... max. ideal

$\boxed{\text{Thm}}$ The natural restriction mapping $\mathrm{Hom}_{\mathbb{C}}(H_{\vec\lambda}, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(V_{\vec\lambda}, \mathbb{C})$ induces an inj. mor.

$$j : V^{+}_{\vec\lambda}(\mathbb{CP}^1) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(V_{\vec\lambda}, \overset{\text{0-action}}{\mathbb{C}})$$

(and finite dim. of $V^{+}_{\vec\lambda}(\mathbb{CP}^1)$ follows)

Moreover, if none of the x_i is ∞, then $\psi \in \mathrm{Hom}_{\mathfrak{g}}(V_{\vec\lambda}, \mathbb{C})$ is $\psi \in \mathrm{Im}\, j$

iff $\forall k = 1, 2, \ldots, n$ $\quad \forall \varphi_k := w_1 \otimes \cdots \otimes w_{k-1} \otimes v_{\lambda_k} \otimes w_{k+1} \otimes \cdots \otimes w_n \in V_{\vec\lambda}$ (highest weight)

we have

$$(\ast) \qquad \sum_{\substack{\vec m_k \\ |\vec m_k| = l_k}} \prod_{j \neq k} \binom{l_k}{m_j} (x_j - x_k)^{-m_j} \psi\left(\prod_{i \neq k} \rho_i(X_\theta)^{m_i} \varphi_k \right) = 0$$

$l_k := \ell - (\theta, \lambda_k) + 1$

$\vec m_k = (m_1, \ldots, \widehat{m_k}, \ldots, m_n) \in \mathbb{N}_0^{n-1}$, $\quad |\vec m_k| := \sum_{i \neq k} m_i$

The "moreover" statement will be useful to prove the following

<u>Verlinde formula</u> (for $g = 0$, i.e. $\Sigma_g = \mathbb{CP}^1$):

$$\dim V^{+}_{\vec\lambda}(\Sigma_g) = \sum_{j=0}^{\ell} (S_0^j)^{2-2g} \prod_{a=1}^{n} \frac{S_{i_a}^g}{S_0^g} =$$

$$= \left(\frac{\ell+2}{2}\right)^{g-1} \sum_{j=0}^{\ell} \frac{\prod_{a=1}^{n} \sin \frac{(i_a+1)(j+1)\pi}{\ell+2}}{\left(\sin \frac{(j+1)\pi}{\ell+2}\right)^{2g-2+n}} \qquad i_a = ??$$

<u>PROOF of Thm</u>: $\overset{\psi}{\underset{\shortparallel}{}}\langle\psi| \in V^{+}_{\vec\lambda}(\mathbb{CP}^1)$

$X \otimes 1 \in \mathfrak{g}(x_1, \ldots, x_n)$ const. function, $X \in \mathfrak{g}$

$(X\otimes 1) \cdot \psi = \langle\psi|(X \otimes 1) = 0$ by def.

expand $X \otimes 1$ via $\mathfrak{g}(x_1, \ldots) \hookrightarrow \bigoplus_i \mathfrak{g}_i((t_i)) \oplus c\mathbb{C}$:

$$X \otimes 1 \longmapsto (X \otimes 1, \ldots, X \otimes 1)$$

$$0 = (X \otimes 1)\circ\psi = \sum_i \overset{\text{action on the } i\text{-th position}}{\rho_i(X)}\,\psi$$

$v \in V_{\vec\lambda} = V_{\lambda_1}\otimes \cdots \otimes V_{\lambda_m}$ with diag. action $X\cdot(v_1\otimes\cdots\otimes v_n) = \sum_i v_1\otimes\cdots\otimes \rho_i(X) v_i\otimes\cdots\otimes v_n$

$$0 = \big((X\otimes 1)\circ\psi\big)(v) = \psi\big((X\otimes 1)v\big) = \psi\Big(\sum_i \rho_i(X)\,v\Big)$$

$$= \psi(X\cdot v)$$

and $\psi(X\cdot v) = 0$ means that $\psi \in \mathrm{Hom}_{\mathfrak{g}}(V_{\vec\lambda}, \mathbb{C})$.

(*) is necessary:

assume $\psi = j(\phi)$

$h := \frac{1}{z - x_k}$, $\varphi \in V_{\vec\lambda}$

$$\phi\Big(\rho_k\big[\tau_k\underbrace{(X_\Theta\otimes h)}_{\mathfrak{g}(x_1,\dots,x_n)}{}^{l_k}\big]\varphi\Big) \rightsquigarrow \psi\ ??$$

$\tau_k(X_\Theta\otimes h)$

$$0 = \big((X_\Theta\otimes h)\cdot\phi\big)(\varphi) = -\phi\big((X_\Theta\otimes h)\varphi\big) = -\sum_{j=1}^{n}\phi\,\rho_j\,\tau_j(X_\Theta\otimes h)\,\varphi = (**)$$

$$\tau_j(X_\Theta\otimes h) = \begin{cases} j=k \ \dots\ X_\Theta\otimes t_k^{-1} \\ j\neq k\ \dots\ X_\Theta\otimes\tau_j\big(\frac{1}{z-x_k}\big) = \dots = X_\Theta\otimes\sum_{m=0}^{\infty}\frac{(-1)^m t_j^m}{(x_j-x_k)^{m+1}}\end{cases}$$

(i.e. holom. around x_i, $i\neq k$)

$t\,\mathfrak{g}[[t]]$ annihilates φ (since $\varphi\in V_{\vec\lambda}$)

hence only $m=0$ is relevant in the Laur. exp.

$$(**) = -\phi\big(\rho_k(X_\Theta\otimes t_k^{-1})\varphi\big) - \sum_{j\neq k}\frac{1}{x_j-x_k}\phi\big(\rho_j(X_\Theta\otimes 1)\varphi\big) = 0$$

Then it follows:

$$\phi\big(\rho_k\tau_k(X_\Theta\otimes h)^{l_k}\varphi\big) \overset{\text{next time?}}{=} (-1)^{l_k}\sum_{\substack{\vec m_k\\ |\vec m_k| = l_k}}\prod_{j\neq l}\binom{l_k}{m_j}(x_j-x_k)^{-m_j}\,\phi\Big(\prod_{j\neq k}\rho_j(X_\Theta)^{m_j}\varphi\Big)$$