5 Drinfeld–Jimbo Algebras and their Representations

In this lecture we introduce the Drinfeld–Jimbo quantised enveloping algebras and discuss their representation theory. This generalises our presentation of $U_q(\mathfrak{sl}_2)$ from previous lectures, and closely parallels the classical situation.

5.1 Drinfeld–Jimbo Quantised Enveloping Algebras

In this first section we give the definition of the Drinfeld–Jimbo algebras. We begin by recalling some relevant classical facts about the semi-simple Lie algebras, and then move onto the definition itself.

5.1.1 Semi-simple Lie Algebras

Let $\mathfrak{g}$ be a finite dimensional complex semi-simple Lie algebra, and $\mathfrak{h} \subseteq \mathfrak{g}$ a fixed Cartan subalgebra. As usual, we call the dimension of $\mathfrak{h}$ the *rank* of $\mathfrak{g}$. Let $R \subseteq \mathfrak{h}^*$ denote the root system associated with $(\mathfrak{g}, \mathfrak{h})$. Choose an ordered basis $\pi = \{\alpha_1, \ldots, \alpha_r\}$ of simple roots for R and let R^+ (respectively R^-) be the set of positive (respectively negative) roots with respect to π. Moreover, let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the corresponding triangular decomposition.

Identify $\mathfrak{h}$ with its dual via the Killing–Cartan form. The induced non-degenerate symmetric bilinear form on $\mathfrak{h}^*$ is denoted by $(\cdot, \cdot)$. The *root lattice* $Q := \mathbf{Z}R$ is contained in the weight lattice

$$P := \{\lambda \in \mathfrak{h}^* \,|\, (\lambda, \alpha_i)/d_i \in \mathbf{Z}, \text{ for all } \alpha_i \in \pi\}.$$

where $d_i := (\alpha_i, \alpha_i)/2$. The *fundamental weights* $\omega_i \in \mathfrak{h}^*$, for $i = 1, \ldots, r$ are characterized by $(\omega_i, \alpha_j)/d_j = \delta_{ij}$. Let us denote

$$P^+ := \mathrm{span}_{\mathbf{Z}}\{\omega_i \,|\, i = 1, \cdots, r\},$$

and call this the set of *integral dominant weights*. Recall that $(a_{ij}) := 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ is the Cartan matrix of $\mathfrak{g}$ with respect to π.

For $\mu \in P^+$, let $V(\mu)$ denote the uniquely determined finite-dimensional irreducible left $\mathfrak{g}$-module with highest weight μ. More explicitly, there exists a nontrivial vector $v_\mu \in V(\mu)$ satisfying $_\mu = 0, Hv_\mu = \mu(H)v_\mu$, (for all $H \in \mathfrak{h}, E \in \mathfrak{n}^+), and \mathrm{V}(\mu)$ is generated by this vector.

5.1.2 Drinfeld–Jimbo Algebras

We keep here the notation of the previous section, and introduce one of the central definitions in the theory of quantum groups:

Definition 5.1. The *Drinfeld–Jimbo quantised enveloping algebra* of $\mathfrak{g}$ is the algebra

$$U_q(\mathfrak{g}) := \mathbf{C}\left\langle E_i, F_i, K_i, K_i^{-1} \,|\, i = 1, \ldots, l \right\rangle / I_{\mathfrak{g}},$$

where l is the rank of $\mathfrak{g}$, and $I_{\mathfrak{g}}$ is the ideal generated by the elements

$$\begin{gathered} K_iK_i^{-1} = K_i^{-1}K_i = 1, \qquad K_iK_j = K_jK_i, \\ K_iE_j = q^{(\alpha_i,\alpha_j)}E_jK_i, \qquad K_iF_j = q^{-(\alpha_i,\alpha_j)}F_jK_i, \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} E_i^{1-a_{ij}}E_jE_i^k = 0, \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} F_i^{1-a_{ij}}F_jF_i^k = 0, \\ E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \end{gathered}$$

with $q_i := q^{d_i}$, and the q-deformed binomials defined as in Lecture 3.

Exercise: It is now natural to ask how this definition relates to the the classical Serre presentation of $U(\mathfrak{g})$ (see [1] for the exact definition of the classical Serre presentation). The following two exercises show there is a direct generalisation of situation for the $U_q(\mathfrak{sl}_2)$.

1cm **(i)** Obviously, the presentation of $U_q(\mathfrak{g})$ we have given above is not well-defined when $q = 1$. However, just as for the $U_q(\mathfrak{sl}_2)$ case, there exists a re-expression $\widetilde{U}_q(\mathfrak{g})$ of the algebra which is well-defined for the $q = 1$ case. This involves adding l additional generators G_i, and new set of relations replacing the offending expressions

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}.$$

Find these relations and explicitly describe the isomorphism.
(ii) In the $q = 1$ case, the algebra $\widetilde{U}_1(\mathfrak{g})$ has generators E_i, F_i, G_i, and K_i. Show that quotienting $\widetilde{U}_1(\mathfrak{g})$ by the ideal generated by the elements $K_i = 1$, for $i = 1, \ldots, l$, gives back the classical Serre presentation of $U(\mathfrak{g})$, and hence that $\widetilde{U}_1(\mathfrak{g})$ is an l-fold cover of $U(\mathfrak{g})$.

Now just as in the classical case, there exists a canonical vector space basis of $U_q(\mathfrak{g})$, as the following proposition demonstrates:

Proposition 5.2 *The following two sets are vector space bases for* $U_q(\mathfrak{g})$*:*

$$\{F_1^{r_1} \cdots F_l^{r_l} K_1^{s_1} \cdots K_l^{s_l} E_1^{t_1} \cdots E_l^{t_l} \,|\, r_i, t_i \in \mathbf{N}_0,\ s_i \in \mathbf{Z}\},$$

and

$$\{E_1^{r_1} \cdots E_l^{r_l} K_1^{s_1} \cdots K_l^{s_l} F_1^{t_1} \cdots F_l^{t_l} \,|\, r_i, t_i \in \mathbf{N}_0,\ s_i \in \mathbf{Z}\}.$$

We call them the PBW *bases.*

Clearly, this generalises the PBW-bases introduced for $U_q(\mathfrak{sl}_2)$ in Lecture 2. Moreover, it is clear that this result shows that the classical triangular decomposition of the universal enveloping algebra of a semi-simple Lie algebra generalises to the quantum setting in the form

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}_+) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_-).$$

The definition of $\mathfrak{g}$ is quite uninteresting from a deformation theory point of view. However, it is very interesting from a Hopf algebraic point from view:

Proposition 5.3 *For any compact semi-simple Lie algebra $\mathfrak{g}$, a Hopf algebra structure on $U_q(\mathfrak{g})$ is determined by*

$$\begin{aligned} \Delta(K_i) = K_i \otimes K_i, \quad & \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad & \Delta(F_i)F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i, \\ \varepsilon(K_i) = 1, \quad & \varepsilon(E_i) = 0, \quad & \varepsilon(F_i) = 0, \\ S(K_i) = K_i^{-1}, \quad & S(E_i) = -E_i K_i^{-1}, \quad & S(F_i) = -K_i F_i. \end{aligned}$$

The proof of this proposition is just a careful generalisation of the proof for the special case of $U_q(\mathfrak{sl}_2)$, and we leave it as an instructive exercise.

Exercise: Find an explicit description of the induced Hopf algebra structure on $\widetilde{U}_q(\mathfrak{g})$. Show that in the $q = 1$ case, this structure descends to a well-defined Hopf algebra on $\widetilde{U}_1(\mathfrak{g})/\langle K_i - 1\rangle$, and that it is isomorphic to the canonical Hopf algebra structure on $U(\mathfrak{g})$ discussed in Lecture 1.

5.2 Finite Dimensional Representations of Drinfeld–Jimbo Algebras

Now that we have given the definition of the Drinfeld–Jimbo algebras, we can move on to discussing their representations. While the theory closely parallels the classical case, there is an important difference: The fact that $U_q(\mathfrak{g})$ is an l-fold cover of $U(\mathfrak{g})$ when $q = 1$ gives rise to 2^l different representation *types*. As we shall see, it is the so-called *type* 1 representations that will be of most interest to us.

5.2.1 Type 1 Representations

Let T be a representation of a Drinfeld–Jimbo algebra $U_q(\mathfrak{g})$ on a vector space V. For a (not necessarily linear) functional Λ on the root lattice Q, we set

$$V_\Lambda = \{x \in V \mid T(K_\alpha)x = \Lambda(\alpha)x, \text{ for all } \alpha \in Q\},$$

where $K_\alpha = K_1^{r_1} \cdots K_l^{r_l}$, and $\alpha = r_1\alpha_1 + \cdots + r_l\alpha_l$. That is, each V_Λ is a joint eigenspace of the commuting operators $T(K_i)$, for $i = 1, 2, \ldots, l$.

If $V_\Lambda \neq \{0\}$, then we say that Λ is a *weight*, we call the number $m_\Lambda := \dim V_\Lambda$ the *multiplicity* of Λ, and we say that V_Λ is a *weight subspace* of the representation T. The non-zero vectors in V_Λ are called *weight vectors*. A representation is called a *weight representation* if its underlying space V decomposes into a direct sum of weight vectors. As we will see below, the weights of most interest to us are all of the form $(z_1, \ldots, z_l)$, for $z_i \in \mathbf{C}$, where

$$(z_1, \ldots, z_l)(\sum_{i=1}^{l} n_i \alpha_i) = (z_1)^{n_1} \cdots (z_l)^{n_l}.$$

A weight representation T of $U_q(\mathfrak{g})$ on a vector space V is called a *representation with highest weight* if there exists a weight vector $e_\Lambda \in V_\Lambda$ such that

$$T(U_q(\mathfrak{g}))e_\Lambda = V_\Lambda; \qquad \text{and} \qquad T(E_i)e_\Lambda = 0, \qquad (\text{for all } i = 1, 2, \cdots, l).$$

We then call the function Λ a *highest weight*, and the vector e_Λ a *highest weight vector* of the representation T.

We say that the a highest weight representation of $U_q(\mathfrak{g})$ is a *representation of type* 1 if it is of the form

$$\Lambda := (q^{n_1}, \ldots, q^{n_l}), \qquad (n_i \in \mathbf{N}_0).$$

Clearly, if $\lambda = \sum_{i=1}^{l} n_i \alpha_i$ is some element of P^+, and $(\cdot, \cdot)$ is the bilinear product on $\mathfrak{g}^*$ induced by the Cartan–Killing form of $\mathfrak{g}$, then

$$\Lambda(\alpha) = q^{(\alpha, \lambda)}, \qquad (\alpha \in Q),$$

in analogy with the classical situation.

5.2.2 Verma Modules and Type 1 Representations

We will recall that for a classical enveloping algebra $U(\mathfrak{g})$, the *Borel subalgebra* of $U(\mathfrak{g})$ is defined to be

$$U(\mathfrak{b}) := \mathbf{C} \langle H_i, F_i \,|\, i = 1, \ldots, l \rangle .$$

Moreover, for any $\lambda \in \mathfrak{h}^*$, and $\mathbf{C}_\lambda$ the corresponding one-dimensional $\mathfrak{h}$-module, the associated *Verma module* is defined as

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbf{C}_\lambda.$$

As one might guess, these definitions carry over directly to the quantum setting. Explicitly, the *Borel subalgebra* of a Drinfeld–Jimbo algebra $U_q(\mathfrak{g})$ is defined to be

$$U(\mathfrak{b}) := \mathbf{C}\left\langle K_i, K_i^{-1}, F_i \,|\, i = 1, \ldots, l \right\rangle.$$

Moreover, the *quantum Verma module* associated to $\lambda \in P^+$, is given by

$$M(\lambda) := U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b})} \mathbf{C}_\lambda.$$

Generalising the well-known classical result about Verma sub-modules, we have the following:

Lemma 5.4 *For any quantum Verma module $M(\lambda)$, there exists a maximal irreducible submodule M, and hence a $U_q(\mathfrak{g})$-representation*

$$L(\lambda) := M(\lambda)/M$$

Just as in the classical case, these representations are of central importance, as the following result demonstrates:

Proposition 5.5 *It holds that*

1. *L_λ is an irreducible representation;*
2. *For any $\lambda \in \mathfrak{h}^*$, there exists, up to equivalence, a unique irreducible type 1 representation of $U_q(\mathfrak{g})$ with highest weight λ. All type 1 representations arise in this way;*
3. *If $\lambda \in P^+$, then the representation L_λ is finite dimensional.*

5.2.3 Classifying Representations

Just as for the $U_q(\mathfrak{sl}_2)$ case, it turns out that all the representations of $U_q(\mathfrak{g})$ are of highest-weight type:

Theorem 5.6 *Any irreducible finite-dimensional representation of a Drinfeld–Jimbo algebra $U_q(\mathfrak{g})$ is a weight representation with highest-weight. Such a weight representation is uniquely determined, up to equivalence, by its highest weight.*

We would like to find a more concrete version of this description and relate it to our results on type 1 representations. To do this we must first look at the one-dimensional representations of $U_q(\mathfrak{g})$. As a little thought will confirm, these are classified by l-tuples $\omega = (\omega_1, \ldots, \omega_l)$, where each $\omega_i \in \{1, -1\}$, and the corresponding representation T_ω acts on $\mathbf{C}e$ according to

$$T_\omega(E_i) = T(F_i) = 0, \qquad\qquad T(K_i)e = \omega e.$$

With this fact in hand we are ready to give our next result:

Proposition 5.7 *If T is a finite dimensional irreducible highest weight representation of $U_q(\mathfrak{g})$, and T_ω is a one-dimensional representation of $U_q(\mathfrak{g})$, then the tensor product*
$T \otimes T_\omega$ is a finite dimensional irreducible highest weight representation of $U_q(\mathfrak{g})$. Moreover, every finite dimensional irreducible highest weight representation of $U_q(\mathfrak{g})$ is of this form.

We call a representations of the form $T \otimes T_\omega$ a *representations of type ω*. Clearly, a type 1 representation is the same thing as a representation of type $(1, \ldots, 1)$.

Finally, we come to the quantum analogue of the classical Weyl theorem, which states that all finite dimensional representations of a compact semi-simple Lie algebra are completely reducible:

Theorem 5.8 *Every finite dimensional representation of $U_q(\mathfrak{g})$ is completely reducible.*

References

[1] A. Klimyk, K. Schmüdgen, *Quantum Groups and their Representations*, Springer Verlag, Heidelberg–New York, 1997