14.3.2014 | Zibor KOHNO (1)

KZ-rovnice a monodromie

$\mathfrak{g}$ kompl. polojednoduchá Lie a.

$U(\mathfrak{g})$ univ. obal., $\Delta: U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$

$\Delta(x) = x \otimes 1 + 1 \otimes x$ pro $x \in \mathfrak{g}$

a rozšíř. až Δ je hom. alg.

K_0 ... nedeg. $\mathfrak{g}$-invariantní bilin. forma na $\mathfrak{g}$ (násobek Killinga)

($\mathfrak{g} = sl(n, \mathbb{C})$ $\quad K_0(X,Y) = \mathrm{Tr}\, XY$)

$\{x^a\}_{a=1}^{\dim \mathfrak{g}}$ a basis of $\mathfrak{g}$, $\{x_a\}$ dual basis w.r. to K_0

$C = \frac{1}{2} \sum_{a=1}^{\dim \mathfrak{g}} x^a x_a$ the Casimir elt., $C \in Z(U(\mathfrak{g}))$

($\mathfrak{g} = sl(2,\mathbb{C})$ $\quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$K_0(a,b) = \mathrm{Tr}\, ab$

$C = \frac{1}{2}(ef + fe + \frac{1}{2}h^2)$)

$$\Omega := \frac{1}{2} \sum_{a=1}^{\dim \mathfrak{g}} x^a \otimes x_a \in \mathfrak{g} \otimes \mathfrak{g} \subset U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

Lemma
1) $\Omega = \frac{1}{2}(\Delta(C) - C \otimes 1 - 1 \otimes C)$
2) $[\Delta(x), \Omega] = 0 \quad \forall x \in \mathfrak{g}$
3) $(1 \otimes \Delta)(\Omega) = \Omega_{12} + \Omega_{13}$ in $U(\mathfrak{g})^{\otimes 3}$
$(\Delta \otimes 1)(\Omega) = \Omega_{23} + \Omega_{13}$

$\Omega_{12} = \frac{1}{2} \sum_{a=1}^{\dim \mathfrak{g}} x^a \otimes x_a \otimes 1$

$\Omega_{13} = \frac{1}{2} \sum_a x^a \otimes 1 \otimes x_a$

PF: 1) $\Delta(C) = \frac{1}{2} \sum \Delta(x^a x_a) = \frac{1}{2} \sum \Delta(x^a)\Delta(x_a) = \frac{1}{2} \sum (1 \otimes x^a + x^a \otimes 1)(1 \otimes x_a + x_a \otimes 1) =$

$= \frac{1}{2} \sum (1 \otimes x^a x_a + x_a \otimes x^a + x^a \otimes x_a + x^a x_a \otimes 1)$

$= 1 \otimes C + 2\Omega + C \otimes 1$

2) $[\Delta(x), \Omega] = \frac{1}{2}[\Delta(x), \Delta(C) - C \otimes 1 - 1 \otimes C] = \frac{1}{2}\left([\Delta(x), \Delta(C)] - [\Delta(x), C \otimes 1] - [\Delta(x), 1 \otimes C]\right)$

$= \frac{1}{2}\left(\Delta[x,C] - [\Delta(x), 1 \otimes C] - [\Delta(x), C \otimes 1]\right) = 0$

($[x,C] = 0$, $C \in Z(U(\mathfrak{g}))$; $\Delta(x) = 1 \otimes x + x \otimes 1$)

3) easy.

$Y_n := \mathrm{Conf}_n(\mathbb{C}) :=$ config. space of n points in $\mathbb{C}$

$= \{(z_1 \cdots z_n) \in \mathbb{C}^n \mid z_i \neq z_j \Leftarrow i \neq j\}$

$z_0 \in \mathrm{Conf}_n(\mathbb{C}) \qquad \pi_1(Y_n, z_0) = PB_n$ ... the pure braid group

$Y_n \times S_n \longrightarrow Y_n$ (↙ symm. gr.) action by $(z_1 \cdots z_n) \times \sigma := (z_{\sigma(1)}, \cdots, z_{\sigma(n)})$

$Y_n / S_n =: X_n \qquad p : Y_n \to X_n$ Galois covering (???) principal S_n-bundle

$\pi_1(X_n, p(z_0)) = B_n$ ... the braid group

↳ B_n is generated by σ_i's s.t.

$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \dots \quad |i-j| > 1$

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \dots \quad i = 1, \dots, n-2$

$1 \to PB_n \to B_n \to S_n \to 0$

$\pi_1 F \qquad \pi_1 E \qquad \pi_1 B \qquad \pi_0 F$

V_i finite dim. repr. of $\mathfrak{g}$, $1 \le i \le n$

$W = V_1 \otimes \cdots \otimes V_n$

$E = Y_n \times W$ trivial holom. v. b. over Y_n

$\downarrow$

Y_n

$\omega_{ij} = d \log(z_i - z_j) = \dfrac{dz_i - dz_j}{z_i - z_j} \qquad i \neq j$ is hol. 1-form on Y_n

Lemma ω_{ij}'s satisfy Arnold relations:

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ik} + \omega_{ik} \wedge \omega_{ij} = 0 \qquad \forall i<j<k$$

(PF: easy)

$\omega := \dfrac{1}{K} \sum_{i<j} \Omega_{ij} \omega_{ij} \qquad K \in \mathbb{C}^{\times} \qquad \rho_i : \mathfrak{g} \to \mathrm{End} V_i$

End(W) valued hol. 1-form on Y_n

$\Omega_{ij} = \frac{1}{2} \sum_a 1 \otimes \cdots \rho_i(X^a) \cdots \rho_j(X_a) \otimes \cdots \otimes 1$ (ⓘ, ⓙ)

$\nabla_{\frac{\partial}{\partial z_i}} = \dfrac{\partial}{\partial z_i} - \dfrac{1}{K} \sum_{i<j} \Omega_{ij} \omega_{ij} \quad \Leftarrow \quad \nabla = d - \omega$ ($d \cdot 1$ is a conn. on triv. bundle)

is called KZ-connection

[holom. conn. ⇒ not enough to take global sections!

$\nabla : \mathcal{E} \to \Omega^1_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \mathcal{E}$

$\Omega^1_{Y_n} = \mathcal{H}om_{\mathcal{O}_{Y_n}}(\Theta_{Y_n}, \mathcal{O}_{Y_n})$]

Thm KZ connection is flat.

Lemma
1) $\Omega_{ij} = \Omega_{ji}$
2) $[\Omega_{ij} + \Omega_{jk}, \Omega_{ik}] = 0 \qquad i, j, k$ distinct
3) $[\Omega_{ij}, \Omega_{kl}] = 0 \qquad$ -"-
4) $[\tilde{\Omega}, \Omega_{ij}] = 0, \quad \tilde{\Omega} := \sum_{i<j} \Omega_{ij}$

PF: 1), 3) easy

$$2)\begin{pmatrix} X^b X^a_a & X_a & X_b \\ -X^b X^a & X_b & X_a \\ X^b & X^a & X_a X_b \\ -X^a & X^b & X_a X_b \end{pmatrix}$$

suffices $[\Omega_{12}, \Omega_{23} + \Omega_{13}]$ (since $\Omega_{ij} = \Omega_{ji}$)

$\Omega_{12} = \Omega \otimes 1$

$\Omega_{23} = 1 \otimes \Omega$

$\Omega_{13} = \dots$

$\Omega = \frac{1}{2}(\Delta(C) - C \otimes 1 - 1 \otimes C)$

prove: $[\Delta(C), \Delta(x)] = 0$

$[\Delta(C) \otimes 1, \frac{1}{2} \sum_a \Delta(x^a) \otimes x_a] = 0$

$k < i < j$

$i < k < j$

$i < j < k$

4) ...

$[\Omega_{ik}, \Omega_{ij}] + [\Omega_{kj}, \Omega_{ij}] \overset{2)}{=} 0$

lemma $\omega \wedge \omega = [\omega, \omega]_1 = 0$

[PF: $[\omega, \omega]_1 = \frac{1}{K^2} \sum_{\substack{i<j \\ k<l}} [\Omega_{ij}, \Omega_{kl}]\, \omega_{ij} \wedge \omega_{kl} = \dots$

KZ equation: $\frac{\partial}{\partial z_i} f - \frac{1}{K} \sum_{\substack{j=1..n \\ j \neq i}} \frac{\Omega_{ij}}{z_i - z_j} f$ $\quad \left(\Omega_{ij} = \frac{1}{2} \sum_a 1 \otimes \dots \otimes X^a \otimes \dots \otimes X_a \dots \otimes\right)$

lemma Let f be a solution of KZ_n-eq.

(1) $\left(\sum_{i=1}^{n} \partial_{z_i}\right) f = 0$, hence f is transl. invariant $f(z_1 + c_1, z_2 + c_2, \dots) = f(z_1, \dots, z_n)$

(2) $K\left(\sum_{i=1}^{n} z_i \partial_{z_i}\right) f = \tilde{\Omega} f$, where $\tilde{\Omega} = \sum_{i<j} \Omega_{ij}$

hence $f(e^c z_1, \dots, e^c z_n) = e^{\frac{c}{K}\tilde{\Omega}} f(z_1, \dots, z_n)$ for some $c \in \mathbb{C}$.

PF: (1) $\exp\left(\sum_i c_i \partial_{z_i}\right) f \overset{\text{comm.}}{=} \prod_i \exp(c_i \partial_{z_i}) f = \prod_i \sum_k \frac{c_i^k \partial_{z_i}^k}{k!} f(z) = f(z_1 + c_1, \dots)$

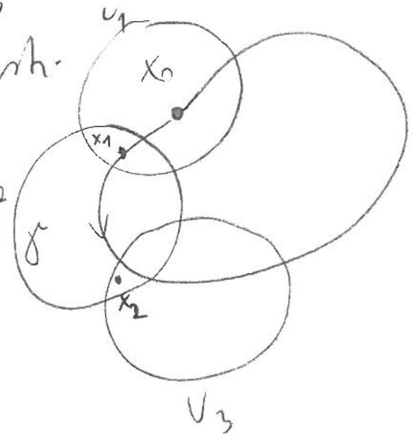
$= \exp(0) f = f$

(2) ...

def Let X be a top. sp., $\mathbb{C}_X$-mod $\mathcal{L}$ is a local system on X (sheaf of locally const. fctions ⇝ $\mathbb{C}_X$)
$\equiv \mathcal{L}$ is locally free $\mathbb{C}_X$-mod of finite rank
(i.e. $\forall x\in X\ \exists U \ni x$ open $\mathcal{L}|_U \cong (\mathbb{C}_X|_U)^n$)

Let X has a univ. cover (X is conn., loc. path conn., semi-loc. simply conn.)
$\mathcal{L}$ a local system on X of rank $n \longmapsto \rho : \pi_1(X,x_0) \longrightarrow GL(n,\mathbb{C})$
monodromy repr. of $\mathcal{L}$.

↓ constr.



$\gamma : [0,1] \longrightarrow X$, $\gamma(0)=\gamma(1)=x_0$
$U_1,\dots,U_m$ a finite cover of $\gamma[0,1]$
U_i's conn. & simply conn.
$x_i \in U_i \cap U_{i+1} \neq \emptyset$
$\mathcal{L}|_{U_i} \cong (\mathbb{C}_X|_U)^n$

$$\underbrace{\mathbb{C}^n \cong \mathcal{L}_{x_0} \cong \mathcal{L}(U_1) \cong \mathcal{L}_{x_1} \cong \mathcal{L}(U_2) \cong \dots \cong \mathcal{L}(U_m) \cong \mathcal{L}_{x_0} \cong \mathbb{C}^n}_{\psi_\gamma}$$

Thus $\gamma \longmapsto \psi_\gamma \in GL(n,\mathbb{C})$

- ψ_γ is indeed iso – take γ^{-1}
- does depend only on the homotopy class of γ
- doesn't depend on the choice of the cover.

in fact: local systems of rank $n /\sim \overset{1-1}{\longleftrightarrow} \mathrm{Hom}(\pi_1(X,x_0), GL(n,\mathbb{C})) /$ ⇝ conj.

conjugation-corr. to the choice of isos $\mathbb{C}^n \cong \mathcal{L}_{x_0}$ at the beg. and end of the chain ψ_γ

example $X = \mathbb{C} - \{0\}$

(1) $\nabla_{\frac{\partial}{\partial z}} = \frac{\partial}{\partial z} - \frac{\alpha}{z}$ ... flat conn on $X \times \mathbb{C} \to X$

$\mathcal{L}(U) = \{ f : U \to \mathbb{C} \mid \frac{\partial f}{\partial z} - \frac{\alpha}{z} f = 0 \}$ loc. sys. of rank 1

$[\gamma] \in \pi_1(X,1) \cong \mathbb{Z}$ $\rho : \pi_1(X,x_0) \longrightarrow GL(1,\mathbb{C})$

↳ $\gamma(t) = e^{2\pi i t}$, $t \in [0,1]$

U simply connected: $\mathcal{L}(U) = \mathbb{C} z^\alpha$ $\qquad z^\alpha = e^{\alpha \log z}$ (need $z \in$ simply conn.)

monodromy along γ:

$$z^\alpha = e^{2\pi i t \alpha} \xrightarrow{t \to 1} e^{2\pi i \alpha}$$

i.e. $\rho([\gamma]) = e^{2\pi i \alpha} \in GL(1, \mathbb{C})$ (the gen.)

$\rho(n[\gamma]) = e^{2\pi i n \alpha}$, $n \in \mathbb{Z}$

spec. case: $\alpha \in \mathbb{Z}$ $\quad \rho(\gamma) = 1$ $\quad$ <u>triv.</u> monodromy repr.

$\mathcal{L} \cong \mathbb{C}_X$

$\mathcal{L}(U) = \mathbb{C} z^n$

(2) $\nabla_{\frac{\partial}{\partial z}} = \frac{\partial}{\partial z} - \frac{\alpha}{z^2}$, $\qquad \mathcal{L}(U) = \mathbb{C} e^{-\frac{\alpha}{z}}$

$$\left(e^{-\frac{\alpha}{z}}\right)\Big|_{t = e^{2\pi i t}} \xrightarrow{t \to 1} e^{-\alpha}$$

$$\left(e^{-\frac{\alpha}{z}}\right)\Big| \xrightarrow{t \to 0} e^{-\alpha}$$

Hence $\rho([\gamma]) = 1 \in GL(1, \mathbb{C})$ and the monodromy repr. is <u>trivial</u>

<u>monodromy repr. of KZ connection</u> on $Y_n \times W \longrightarrow Y_n$ ($W = V_1 \otimes \cdots \otimes V_n$)

$\mathcal{L}(U) = \{ f : U \longrightarrow W \mid f \text{ solves } KZ_n \}$ is a loc. sys. of rank $\dim W$

$\rho^{KZ_n} : \pi_1(Y_n, x_0) \longrightarrow GL(W)$ $\qquad$ (KZ_n depends on a parameter $\kappa \in \mathbb{C}^\times$)

$\pi_1(Y_n, x_0) \cong PB_n$ pure braids

<u>special case</u>: $V_1 = \cdots = V_n = V$, $W = V^{\otimes n}$

$V^{\otimes n} = W$ has a left S_n-action $\quad (\sigma, v_1 \otimes \cdots \otimes v_n) \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$ (?? inverses)

Y_n right S_n-ac. $\quad ((z_1 \cdots z_n), \sigma) \mapsto (z_{\sigma^{-1}(1)} \cdots z_{\sigma^{-1}(n)})$

$Y_n / S_n = X_n$ ... principal S_n-bundle $Y_n \longrightarrow X_n$

associated vec. bundle

$$F := Y_n \underset{S_n}{\times} V^{\otimes n} := Y \times V^{\otimes n} \Big/ (y\sigma, \vec{v}) \sim (y, \sigma \vec{v})$$

KZ_n - conn. is S_n-invariant (check!)

hence descends to a flat conn. on $F \to X_n$

and this has a monodromy representation

$$\rho^{KZ_n} : \pi_1(X_n, x_0) \longrightarrow GL(V^{\otimes n})$$

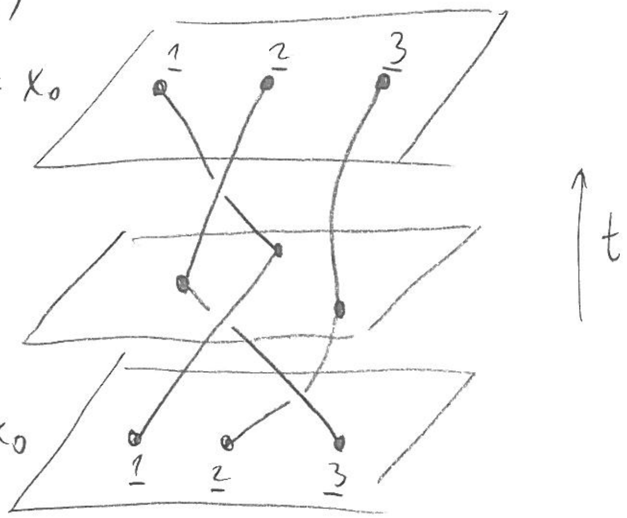
$\quad\quad\quad\quad\quad \cong$

$\quad\quad\quad\quad B_n$ ... braids (all!)

$\gamma(1) = x_0$

realisation of PB_n

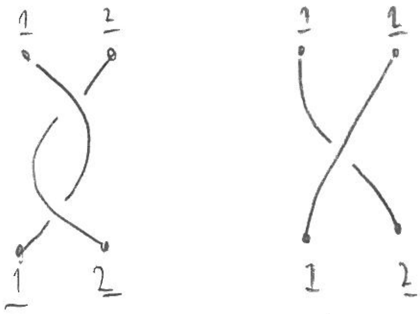
$\gamma(t) = (z_1(t) \cdots z_n(t)) \in Y_n$



$\gamma(0) = x_0$

generators of PB_n: $\gamma(1)(\underline{1}) = \underline{1}$, $\gamma(1)(\underline{2}) = \underline{2}$ ...

e.g.



$\in PB_2$ $\quad\quad \notin PB_2$, $\in B_2$

since $\gamma(0) = (\underline{1}, \underline{2})$
$\gamma(1) = (\underline{2}, \underline{1})$
$\gamma(0) = \gamma(1)$ in X_2
BUT NOT in Y_2!

$\boxed{n=2}$ $\quad \frac{\partial}{\partial z_1} f = \frac{1}{K} \frac{\Omega_{12}}{z_1 - z_2} f \quad\quad Y_2 \supset U \xrightarrow{f} V^{\otimes 2}$

$\quad\quad \frac{\partial}{\partial z_2} f = \frac{1}{K} \frac{\Omega_{21}}{z_2 - z_1} f$

$f(z_1, z_2) = f(0, z_2 - z_1)$ by the translation inv. earlier

$f(e^c z_1, e^c z_2) = e^{\frac{c}{K}\Omega_{12}} f(z_1, z_2)$ ..

hence, for $e^c(z_2 - z_1) = 1$, we have $f(0, \overbrace{e^c(z_2 - z_1)}^{1}) = e^{\frac{c}{K}\Omega_{12}} f(0, z_2 - z_1) =$ $\quad$ [$e^{\frac{c}{K}\Omega_{12}} = (z_2 - z_1)^{\frac{\Omega_{12}}{K}}$]

$\Rightarrow f(0, z_2 - z_1) = e^{-\frac{c}{K}\Omega_{12}} \underbrace{f(0,1)}_{=: v}$, $v \in V^{\otimes 2}$

$f(z_1, z_2) = (z_2 - z_1)^{\frac{\Omega_{12}}{K}} v = e^{\frac{\Omega_{12}}{K} \underline{\log}(z_2 - z_1)} v$ $\quad$ hence that's only local sol.

$\gamma(t) = \left(\frac{1}{2}(3 - e^{2\pi i t}), \frac{1}{2}(3 + e^{2\pi i t})\right) \qquad t \in [0;1]$

$\gamma(0) = (\underline{1}, \underline{2}) = \gamma(1)$. $\qquad \gamma(\frac{1}{2}) = (\underline{2}, \underline{1})$

a gen. of $\pi_1(\ \ _2, x_0)$

$f(\gamma(t)) = e^{2\pi i t \frac{\Omega_{12}}{K}} v \xrightarrow{t \to 1} e^{2\pi i \frac{\Omega_{12}}{K}} v$

$\xrightarrow{t \to 0} v$

$\Rightarrow \rho^{KZ_2}(\underset{PB_2}{(\gamma)}) = e^{2\pi i \frac{\Omega_{12}}{K}} \in GL(V^{\otimes 2})$

is the monodr. representation

$\gamma(t) = \left(\frac{1}{2}(3 - e^{\pi i t}), \frac{1}{2}(3 + e^{\pi i t})\right) \qquad \gamma(0) = (\underline{1}, \underline{2}) \qquad \gamma(1) = (\underline{2}, \underline{1})$

a gen. of $\pi_1(Y_2, x_0)$

$f(\gamma(t)) = e^{\pi i t \frac{\Omega_{12}}{K}} v \xrightarrow{t \to 0} v$

$\xrightarrow{t \to 1} e^{\pi i \frac{\Omega_{12}}{K}} v$

$((\underline{1}, \underline{2}), v_1 \otimes v_2) = ((\underline{2}, \underline{1}), v_2 \otimes v_1)$

$\cap$

$Y_2 \times_{S_2} V^{\otimes 2}$

$P : V^{\otimes 2} \to V^{\otimes 2}$

$v_1 \otimes v_2 \mapsto v_2 \otimes v_1$

$\rho^{KZ_2}([\gamma]) = P(e^{\pi i \frac{\Omega_{12}}{K}})$

$\cap$

B_2