KHOVANOV COHOMOLOGY

(Turner – 5 lectures on Khovanov homol.
Mezger – An introd. to Kh. hom.) (1)

J : links $\longrightarrow \mathbb{C}[q, q^{-1}]$

reminder on Jones polynomial

Kauffman bracket : $\langle O \rangle = 1$

$\langle L \sqcup O \rangle = (q+q^{-1}) \langle L \rangle$

circle in disc which doesn't intersect L

$\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$

0-sm. 1-sm.

oriented link

$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$ the Jones pol. normalized

n_+ = number of positive crossings in L :

negative

$J(L) := (q+q^{-1}) \hat{J}(L)$ unnormalized Jones pol. → will be used in the sequel.

– relation to previous lectures : $a = iq^{\frac{1}{2}}$

$\boxed{\text{Thm}}$ J is an isotopy invariant of links

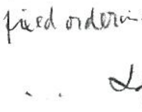
another viewpoint :

cube of complete smoothings of L : $\{0,1\}^n$, $n :=$ # of crossings of L

- i-th coordinate α_i of $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0,1\}^n$ is 0 $\equiv$ i-th crossing $\times$ is replaced by 0-smoothing : $\asymp$ (fixed ordering of cr.)
- ... $\alpha_i = 1$ ... $\times$ is replaced by 1-sm. : $)($
- α is called a (complete) smoothing, Γ_α refers to the smoothened link
- draw an edge in $\{0,1\}^n$ between α and α' iff α, α' differ at exactly one component

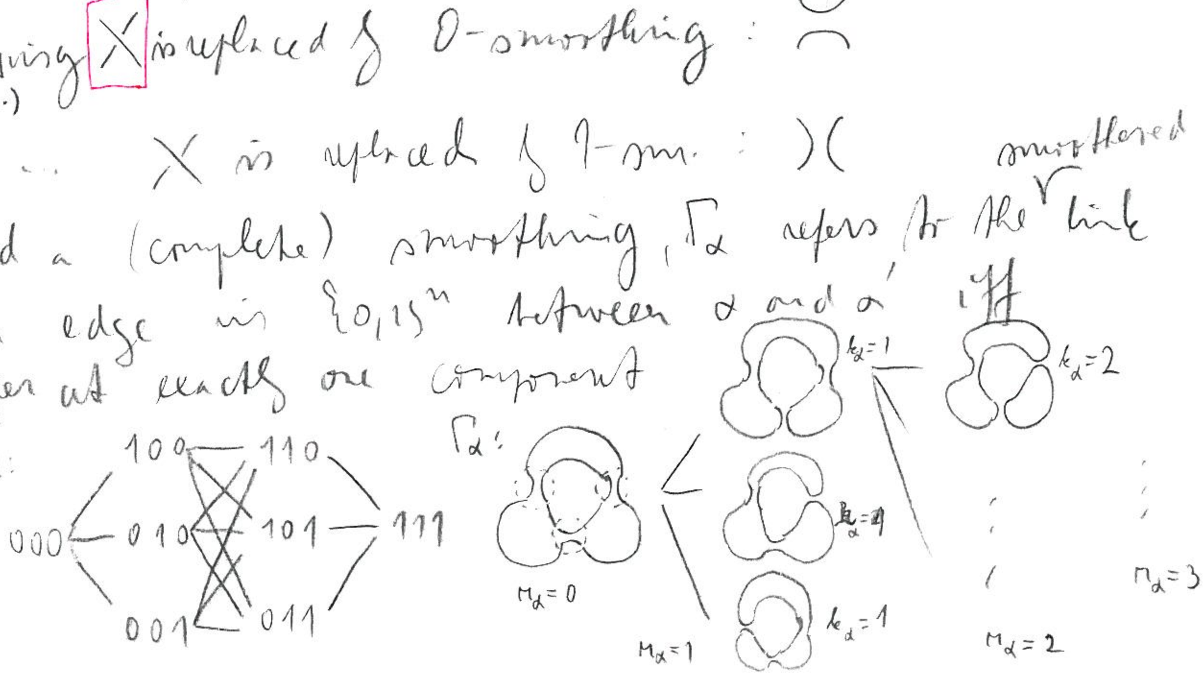
HOPF LINK BETTER

Ex.



trefoil knot

α :



α a comp. sm. (1-smoothings)

$r_\alpha :=$ number of 1's in α

$k_\alpha :=$ number of circles in Γ_α

observe: $$\langle L \rangle = \sum_{\alpha \in \{0,1\}^n} (-1)^{r_\alpha} q^{r_\alpha} (q+q^{-1})^{k_\alpha}$$

$$J(L) = (-1)^{n_-} q^{n_+ - 2n_-} \sum_\alpha (-1)^{r_\alpha} q^{r_\alpha} (q+q^{-1})^{k_\alpha}$$

Construction of the Khovanov complex

$\boxed{\text{def}}$ q-graded (just a name) vector space $V = \bigoplus_{l \in \mathbb{Z}} V^l$, $v \in V^l \equiv |v| = l$ q-degree

$(V[k])^l := V^{l-k}$ (another notation $\uparrow^k V$)

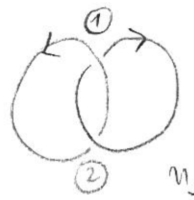
$$q\dim V := \sum_l q^l \dim V^n$$

observe $$q\dim(V \oplus V') = q\dim V + q\dim V'$$
$$q\dim(V \otimes V') = q\dim V \cdot q\dim V'$$
$$q\dim(V[k]) = q^k \cdot q\dim V$$

Fix $V := \Bbbk\{1, x\}$, $\left.\begin{array}{l} |1| := 1 \\ |x| := -1 \end{array}\right\}$ q-degrees

$$C^{i,*}(L) := \bigoplus_{\substack{\alpha \in \{0,1\}^n \\ r_\alpha = i + n_-}} V^{\otimes k_\alpha}[r_\alpha + n_+ - 2n_-]$$

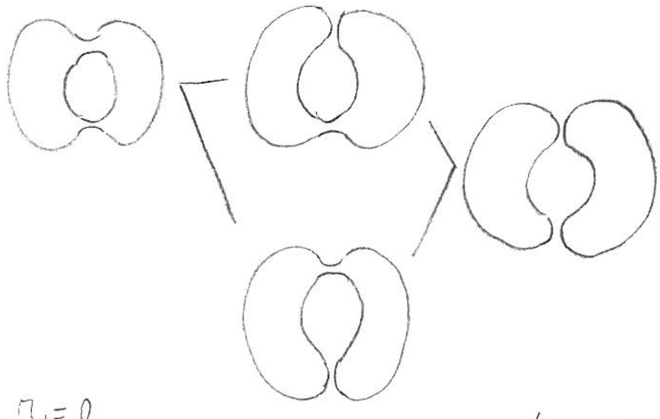
$\boxed{\text{Ex.}}$



$n_+ = 0$

$n_- = 2$

$$\begin{array}{ccccc} & & 10 & & \\ & \diagup & & \diagdown & \\ 00 & & & & 11 \\ & \diagdown & & \diagup & \\ & & 01 & & \end{array}$$



$r_\alpha = 0$ $\qquad$ $r_\alpha = 1$ $\qquad$ $r_\alpha = 2$

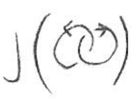
$$\begin{array}{ccc} & V_{10} & \\ V_{00} & \oplus & V_{11} \\ & V_{01} & \\ \| & \| & \| \\ C^{-2,*} & C^{-1,*} & C^{0,*} \end{array}$$

$V_{00} = V^{\otimes 2}[-4]$ $\qquad$ $V_{11} = V^{\otimes 2}[-2]$

$V_{10} = V_{01} = V[-3]$

$$q\dim C^{-2,*} - q\dim C^{-1,*} + q\dim C^{0,*} =$$ (q-Euler characteristic of the cpx $C^{*,(*)}$ (differ. not yet explicit!))

$$= q^{-4}(q+q^{-1})^2 - 2q^{-3}(q+q^{-1}) + q^{-2}(q+q^{-1})^2 =$$

$$= q^{-2} + \cancel{2q^{-4}} + \underline{q^{-6}} - \cancel{2q^{-2}} - \cancel{2q^{-4}} + 1 + \cancel{2q^{-2}} + \underline{q^{-4}} = q^{-6} + q^{-4} + q^{-2} + 1 = J($$  $)$

We will later define a differential(s)

$$0 \to C^{-n_-,*}(L) \to \cdots \to C^{i,*}(L) \xrightarrow{d^i} C^{i+1,*}(L) \to \cdots \to C^{n_+,*}(L) \to 0$$

of q-degree 0. Then $H^{i,*}_{KH}(L) := \operatorname{Ker} d^i / \operatorname{Im} d^{i-1}$ is the <u>i-th Khovanov homology</u> of L.

Even without knowledge of d^*, we

$\boxed{\text{def}}$ <u>q-Euler characteristics</u> of $C^{*,*}$ is

$$\chi_q(C^{*,*}) := \sum_{k\in\mathbb{Z}} (-1)^k \, q\dim H^{k,*}(C^{*,*}, d^*)$$

analogous to the standard ("non-q") result:

$\boxed{\text{lemma}}$ $\chi_q(C^{*,*}) = \sum_{k\in\mathbb{Z}} (-1)^k \, q\dim C^{k,*}$

$\boxed{\text{observe}}$ $\chi_q(C^{*,*}(L)) = J(L)$

<u>PF</u>: $\chi_q(C^{**}(L)) = \sum_{k\in\mathbb{Z}} (-1)^k \sum_{\substack{\alpha \text{ s.t.} \\ n_\alpha = k+n_-}} q^{n_\alpha + n_+ - 2n_-}(q+q^{-1})^{k_\alpha} =$

$$= \sum_\alpha (-1)^{n_\alpha - n_-} q^{n_\alpha + n_+ - 2n_-}(q+q^{-1})^{k_\alpha} = (-1)^{n_-} q^{n_+ - 2n_-} \underbrace{\sum_\alpha (-1)^{n_\alpha} q^{n_\alpha}(q+q^{-1})^{k_\alpha}}_{=J(L)}$$

$\square$

Construction of the Khovanov differential

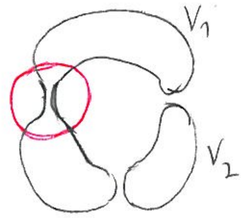
FROB. ALG. ON V!

Let $\alpha \xrightarrow{(\cdots * \cdots)} \alpha'$ in $\{0,1\}^n$

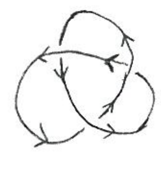
$\alpha = (\cdots 0 \cdots)$, $\alpha' = (\cdots 1 \cdots)$

$$\Gamma_\alpha \xrightarrow{(*)} \Gamma_{\alpha'} \quad \overset{\text{want}}{\Longrightarrow} \quad V_\alpha \subset C^{i,*}(L) \xrightarrow[\text{a lin map}]{d_{\alpha\alpha'}} V_{\alpha'} \subset C^{i+1,*}(L)$$

(the arrow points to α' iff $n_{\alpha'} > n_\alpha$ $(\Leftrightarrow n_{\alpha'} = n_\alpha + 1)$)

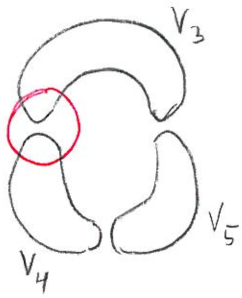
$\Gamma_\alpha =$ 

$V_\alpha = (V_1 \otimes V_2)[2-3]$



$n_+ = 3$
$n_- = 0$

$\downarrow d_{\alpha\alpha'}$

$\Gamma_{\alpha'}$ 

$V_{\alpha'} = (V_3 \otimes V_4 \otimes V_5)[3-3]$

- each V_i corresponds to a circle
- all V_i's are copies of V

Replacing the red 0-sm. in Γ_α by a 1-sm. either

1) disconnects a circle, thus $\Gamma_{\alpha'}$ has one more circle than Γ_α

or 2) connects two circles, thus $\Gamma_{\alpha'}$ less Γ_α.

Other circles are left intact.

1) Let V_a correspond to the circle which gets disconnected by passing to $\Gamma_{\alpha'}$, let V_b, V_c correspond to the two circles in $\Gamma_{\alpha'}$ created by the disconnection.

Then define $V_\alpha \xrightarrow{d_{\alpha\alpha'}} V_{\alpha'}$ by requiring it to be 1_V on each V_i-factor other than V_a, and let it be $V_a \xrightarrow{\Delta} V_b \otimes V_c$

in the example

$$\begin{array}{ccc} (V_1 \otimes V_2)[-1] & & (v_1 \otimes v_2)[-1] \\ \downarrow d_{\alpha\alpha'} := (\Delta \otimes 1_V)[1] & & \downarrow \\ V_3 \otimes V_4 \otimes V_5 & & \Delta(v_1) \otimes v_2 \\ & & \in V_3 \otimes V_4 \end{array}$$

2) V_a, V_b get connected, V_c is the circle merged

$$\begin{array}{c} V_a \otimes V_b \otimes \cdots \; [k] \\ \downarrow m \otimes 1_V \otimes 1_V \otimes \cdots \; [1] \\ V_c \otimes \cdots \; [k+1] \end{array}$$

$\boxed{\text{def}}$ on V, there is a Frobenius alg. str. given by

$m : V \otimes V \to V \qquad 1 \cdot 1 = 1, \quad 1 \cdot x = x = x \cdot 1, \quad x \cdot x = 0$

$\Delta : V \to V \otimes V \qquad \Delta(1) = 1 \otimes x + x \otimes 1, \quad \Delta(x) = x \otimes x$

notice that $|m| = -1 = |\Delta|$ (q-degrees).

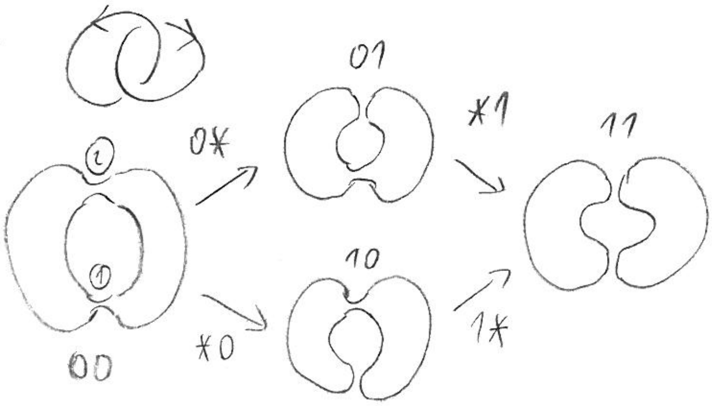
$d^i : C^{i,*}(L) \to C^{i+1,*}(L)$

Then define $\underbrace{d^i(v) := \sum_{\substack{\alpha' \\ r_{\alpha'} = r_\alpha + 1}} d_{\alpha\alpha'}(v) \cdot (-1)^{\{= \# \text{ of 1's to the left of } * \text{ in } \alpha \xrightarrow{\}} \alpha'}}_{\text{for } v \in V_\alpha \subset C^{i,*}(L)}$

$$\left[\begin{array}{l} \alpha = (\dots \alpha_{p-1}, 0, \alpha_{p+1}, \dots) \\ \qquad \downarrow (\dots \alpha_{p-1}, *, \alpha_{p+1}, \dots) \\ \alpha' = (\dots \alpha'_{p-1}, 1, \alpha'_{p+1}, \dots) \\ \\ \qquad \alpha_m = \alpha'_m \Leftarrow m \neq p \\ \text{then } (-1)^{\# \dots} = (-1)^{|\{\alpha_m \mid \alpha_m = 1 \text{ and } m < p\}|} \\ \qquad = (-1)^{\alpha_1 + \dots + \alpha_{p-1}} \end{array}\right.$$

$n_+ = 0, \; n_- = 2$

$\boxed{\text{Ex.}}$



$r_\alpha = 0 \qquad r_\alpha = 1 \qquad r_\alpha = 2$

$$\begin{array}{ccccc} & & V[-3] & & \\ & \nearrow^{m[1]} & & \searrow^{\Delta(1)} & \\ (V \otimes V)[-4] & & \oplus & & V \otimes V[-2] \\ & \searrow_{m[1]} & & \nearrow_{-\Delta[1]} & \\ & & V[-3] & & \end{array}$$

$0 \to C^{-2,*} \longrightarrow C^{-1,*} \longrightarrow C^{0,*} \to 0$

d^{-2}		d^{-1}	
$1 \otimes 1 \longmapsto$	$\begin{matrix} 1 \\ \oplus \\ 1 \end{matrix}$	$\begin{matrix} 1 \\ \oplus \\ 0 \end{matrix} \longmapsto$	$1 \otimes x + x \otimes 1$
$1 \otimes x \longmapsto$	$\begin{matrix} x \\ \oplus \\ x \end{matrix}$	$\begin{matrix} x \\ \oplus \\ 0 \end{matrix} \longmapsto$	$x \otimes x$
$x \otimes 1 \longmapsto$	$\begin{matrix} x \\ \oplus \\ x \end{matrix}$	$\begin{matrix} 0 \\ \oplus \\ 1 \end{matrix} \longmapsto$	$-1 \otimes x - x \otimes 1$
$x \otimes x \longmapsto$	$\begin{matrix} 0 \\ \oplus \\ 0 \end{matrix}$	$\begin{matrix} 0 \\ \oplus \\ x \end{matrix} \longmapsto$	$-x \otimes x$

sign [?]

$H^{i,j}$

$i \backslash j$	-6	-5	-4	-3	-2	-1	0
0					$\mathbb{k}$		$\mathbb{k}$
-1							
-2	$\mathbb{k}$		$\mathbb{k}$				

$\chi_q(C^{*,*}(L)) = +(1 + q^{-2}) + (q^{-4} + q^{-6})$ ok

$H^{-2,*}_{KH}(L) \cong \mathbb{k}\{1 \otimes x - x \otimes 1, x \otimes x\}[-4]$

$H^{-1,*}_{KH}(L) \cong \dfrac{\mathbb{k}\{1 \oplus 1, \cancel{\dots}\, x \oplus x\}}{\mathbb{k}\{1 \oplus 1, x \oplus x\}[-3]}$

$\cong 0$

$H^{0,*}_{KH}(L) \cong \dfrac{\mathbb{k}\{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\}[-2]}{\mathbb{k}\{x \otimes x, 1 \otimes x + x \otimes 1\}[-2]}$

$\cong \mathbb{k}\{1 \otimes 1, 1 \otimes x\}[-2]$

$\boxed{\text{Thm}}$ $d^{i+1} d^i = 0$

PF: $v \in V_\alpha \subset C^{i,*}(L)$

$$d^{i+1} d^i (v_\alpha) = d^{i+1} \sum_{\substack{\alpha' \\ \alpha \xrightarrow{\xi} \alpha'}} d_{\alpha\alpha'}(v) \, (-1)^{\xi} =$$

$$= \sum_{\alpha \xrightarrow[\xi]{} \alpha' \xrightarrow[\xi']{} \alpha''} d_{\alpha'\alpha''} \, d_{\alpha\alpha'}(v) \, (-1)^{\xi + \xi'} = \sum_{\alpha \rightrightarrows \alpha''} \left(d_{\alpha_1'\alpha''} d_{\alpha\alpha_1'}(v) (-1)^{k+(l+1)} + d_{\alpha_2'\alpha''} d_{\alpha_2'\alpha}(v) (-1)^{k+l} \right),$$

$$\alpha = (\underbrace{\underbrace{\ldots}_{k \text{ 1's}} 0 \ldots}_{l \text{ 1's}} 0 \ldots) \qquad \alpha' = \begin{cases} (\ldots 1 \ldots 0 \ldots) = \alpha_1' \\ \text{or} \\ (\ldots 0 \ldots 1 \ldots) = \alpha_2' \end{cases} \qquad \alpha'' = (\ldots 1 \ldots 1$$

$$\begin{tikzcd} & \alpha_1' \arrow[dr] & \\ \alpha \arrow[ur] \arrow[dr] & & \alpha'' \\ & \alpha_2' \arrow[ur] & \end{tikzcd}$$ face of the cube $\{0,1\}^n$

suffices: $\forall$ face, $d_{\alpha_1'\alpha''} d_{\alpha\alpha_1'}(v) = d_{\alpha_2'\alpha''} d_{\alpha\alpha_2'}(v)$.

This can be computed directly, but here is a more enlightening way:

<u>Correspondence between c. Frobenius algebras and 2D TQFT's.</u>

$\boxed{\text{def}}$ Bord_2 is the foll. symmetric monoidal cat:

objects = disjoint union of oriented circles (homeo class of)

morph. = $A \to B$ is an oriented 2-manifold C

w. boundary $\partial C = A \sqcup \overline{B}$ ← reversed or.

i.e. a bordism between A and B.

$A \xrightarrow{1_A} A$ is a tube; composition is gluing of bordisms

$\otimes$ = disjoint union

$\emptyset$ = unit w.r.t. $\otimes$

$\boxed{\text{def}}$ 2D TQFT is a monoidal ftr $F : \text{Bord}_2 \to \text{Vec}$ $(\text{std.} \otimes, \Bbbk)$

i.e. $F(A \sqcup B) \cong F(A) \otimes F(B)$ (nat. iso) ← part of the data

$F(\emptyset) \cong \Bbbk$