LECTURE 9

We discussed various Lie groups / Lie algebras either over $\mathbb{R}$ or $\mathbb{C}$. We can then ask about, having $\underbrace{\mathfrak{g}/\mathbb{R}}_{\text{real Lie algebra}}$, its complexification $\mathfrak{g}/\mathbb{C}$, where $\mathfrak{g}/\mathbb{C} = \mathfrak{g}_{\mathbb{C}} := \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

Def 1: $V = V_{\mathbb{R}}$ … finite dim. real vectorspace, the complexification of V, denoted $V_{\mathbb{C}}$, is the space of linear combinations $v_1 + i v_2$, $v_1, v_2 \in V$, $i \in \mathbb{C}$ the complex unit. This is a real vectorspace, and becomes $\mathbb{C}$-vectorspace by $i(v_1 + i v_2) = -v_2 + i v_1$.

(We do not write $v_1 + i v_2$ pedantically as couples of real vectors (v_1, v_2) with some properties.)
We regard $V = V_{\mathbb{R}}$ as a real subspace of $V_{\mathbb{C}}$.

Lemma 2: $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}}$ … finite-dim real Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then the bracket operation $[\ ,\ ]_{\mathfrak{g}}$ on $\mathfrak{g}$ extends uniquely to $\mathfrak{g}_{\mathbb{C}}$, and $[\ ,\ ]_{\mathfrak{g}_{\mathbb{C}}}$ on $\mathfrak{g}_{\mathbb{C}}$ defines the structure of complex Lie algebra.

Pf: The requirement of $\mathbb{C}$-bilinearity of $[\ ,\ ]_{\mathfrak{g}_{\mathbb{C}}}$ implies its form

$$(*) \quad [X_1 + i X_2, Y_1 + i Y_2]_{\mathfrak{g}_{\mathbb{C}}} = \left([X_1, Y_1]_{\mathfrak{g}_{\mathbb{R}}} - [X_2, Y_2]_{\mathfrak{g}_{\mathbb{R}}}\right) + i\left([X_1, Y_2]_{\mathfrak{g}_{\mathbb{R}}} + [X_2, Y_1]_{\mathfrak{g}_{\mathbb{R}}}\right)$$

$X_1, X_2, Y_1, Y_2 \in \mathfrak{g}_{\mathbb{R}}$

so the extension is unique. To prove its existence, we have to verify $(*)$ is $\mathbb{C}$-bilinear, skew-symmetric and satisfies the Jacobi identity.

$(*)$ is clearly real bilinear, and satisfies skew-symmetry; and skew-symmetry implies $\mathbb{C}$-linearity in one entry is equivalent to $\mathbb{C}$-linearity in the other factor. Hence it remains to show

$$[i(X_1 + i X_2), Y_1 + i Y_2]_{\mathfrak{g}_{\mathbb{C}}} = i\,[X_1 + i X_2, Y_1 + i Y_2]_{\mathfrak{g}_{\mathbb{C}}}$$

LHS: $= [-X_2 + i X_1, Y_1 + i Y_2]_{\mathfrak{g}_{\mathbb{C}}} = \left(-[X_2, Y_1] - [X_1, Y_2]\right) + i\left([X_1, Y_1] - [X_2, Y_2]\right)$

RHS:

$$i\left\{ \left([X_1,Y_1]-[X_2,Y_2]\right)+i\left([X_2,Y_1]+[X_1,Y_2]\right)\right\} =$$

$$= \left(-[X_2,Y_1]-[X_1,Y_2]\right)+i\left([X_1,Y_1]-[X_2,Y_2]\right),$$

hence are equal, LHS = RHS.

As for the Jacobi identity: true for all $X, Y, Z \in \mathfrak{g} = \mathfrak{g}_{\mathbb{R}}$. With Y, Z fixed, the expression $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]$ is complex linear in X, so Jacobi identity continues to hold for $X \in \mathfrak{g}_{\mathbb{C}}$, $Y, Z \in \mathfrak{g} = \mathfrak{g}_{\mathbb{R}}$. The same argument applies to $X, Y \in \mathfrak{g}_{\mathbb{C}}$ and $Z \in \mathfrak{g} = \mathfrak{g}_{\mathbb{R}}$, and one more iteration gives Jacobi identity for all $X, Y, Z \in \mathfrak{g}_{\mathbb{C}}$. ∎

<u>Lemma 3</u>: Let $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \subseteq M_n(\mathbb{C})$ be a real Lie algebra, and assume that $\forall X \in \mathfrak{g}$, $iX \notin \mathfrak{g}$. Then the "abstract" complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ (see Lemma 2) is isomorphic (as a complex Lie algebra) to the set of matrices in $M_n(\mathbb{C})$ of the form $X + iY$, $X, Y \in \mathfrak{g}$.

<u>Pf</u>:

$$\mathfrak{g}_{\mathbb{C}} \longrightarrow M_n(\mathbb{C})$$

$$X + iY \longmapsto \underbrace{X + iY}_{\mathbb{R}}$$ lin comb of matrices in $M_n(\mathbb{C})$

This map is complex Lie alg. homomorphism, and by assumptions this map is injective, therefore isomorphism $\mathfrak{g}_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{g}_{\mathbb{R}} + i\mathfrak{g}_{\mathbb{R}} \subseteq M_n(\mathbb{C})$ of $\mathbb{C}$-Lie algebras. ∎

There is the following list of isomorphisms, see exercises for explicit proofs:

$$\left.\begin{array}{l} \mathfrak{gl}(n,\mathbb{R})_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{gl}(n,\mathbb{C}) \\ \mathfrak{u}(n)_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{gl}(n,\mathbb{C}) \\ \mathfrak{su}(n)_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{sl}(n,\mathbb{C}) \\ \mathfrak{sl}(n,\mathbb{R})_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{sl}(n,\mathbb{C}) \\ \mathfrak{so}(n)_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{so}(n,\mathbb{C}) \\ \mathfrak{sp}(n,\mathbb{R})_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{sp}(n,\mathbb{C}) \end{array}\right\} \text{as complex Lie algebras}$$

$$\mathfrak{sp}(n)_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{sp}(n,\mathbb{C})$$

If $\mathfrak{g}_{1\mathbb{R}}, \mathfrak{g}_{2\mathbb{R}}$ fulfill $(\mathfrak{g}_{1\mathbb{R}})_{\mathbb{C}} \simeq (\mathfrak{g}_{1\mathbb{R}})_{\mathbb{C}}$ as $\mathbb{C}$-Lie algebras but

$\mathfrak{g}_{1\mathbb{R}} \not\cong \mathfrak{g}_{2\mathbb{R}}$ as real Lie algebras, $\mathfrak{g}_{1\mathbb{R}}$ and $\mathfrak{g}_{2\mathbb{R}}$ are different real Lie forms of common $\mathbb{C}$-Lie algebra $(\mathfrak{g}_{1\mathbb{R}})_{\mathbb{C}} \xrightarrow{\sim} (\mathfrak{g}_{2\mathbb{R}})_{\mathbb{C}}$

Representation theory of Lie groups/algebras

V (a finite-dim vector space over $\mathbb{R}, \mathbb{C}$), $GL(V)$, the choice of basis of V identifies $GL(V) \xrightarrow{\sim} GL(n, \overset{\mathbb{R}}{\mathbb{C}})$; $GL(V)$ is a Lie group and $\mathfrak{gl}(V)$ its Lie algebra.

Def 4: A fin.-dim complex/real representation of a Lie group G is a Lie group homomorphism $\Pi : G \to GL(V)$, where V is a fin.-dim. complex/real vector space ($\dim V \geq 1$.)

A fin.-dim complex/real represent of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, where V is a fin.-dim complex/real vector space ($\dim V \geq 1$)

If Π or π are injective homomorphisms, the representation is faithful.

Def 5: Let Π be a fin.-dim. complex/real representation of a Lie group G, acting on the vector space V. A subspace $W \subseteq V$ is called invariant if $\Pi(g)w \in W$ $\forall w \in W$ $\forall g \in G$. A invariant space W is called nontrivial (or, proper) if $W \neq \{0\}$ and $W \neq V$. A representation with no non-trivial invariant subspaces is called irreducible.

Analogously (invariant, non-trivial, irreducible) for Lie algebras.

Def 6: G ... a Lie group, V, W ... fin. dim. vector spaces,

$$\left.\begin{aligned}\Pi &: G \to GL(V)\\ \Sigma &: G \to GL(W)\end{aligned}\right\}$$ representations. A linear map

$\varphi : V \to W$ is called intertwining map for Π, Σ if

$$\varphi\left(\Pi(g)v\right) = \Sigma(g)\varphi(v) \qquad \forall g \in G, v \in V.$$

Analogously for representations of Lie algebras.
If φ is an isomorphism, then φ is said to be an isomorphism of representations V, W.

Aim: Given G or $\mathfrak{g}$, determine all irreducible representations up to isomorphism.

Lemma 7: Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then every fin.-dim. complex representation π of $\mathfrak{g}$ has a unique extension to a $\mathbb{C}$-linear representation of $\mathfrak{g}_{\mathbb{C}}$, denoted π as well. Moreover, π is irreducible as a represent. of $\mathfrak{g}_{\mathbb{C}}$ iff it is irreducible as a repr. of $\mathfrak{g}$.

Pf: The unique extension of π to $\mathfrak{g}_{\mathbb{C}}$ is given by
$\pi(X+iY) = \pi(X) + i\pi(Y)$ for all $X, Y \in \mathfrak{g}$.
(It is easy to verify that this map is a homomorphism of complex Lie algebras.)
The claim on irreducibility holds because

$$\left\{\begin{array}{l}\text{a complex subspace } W \subseteq V\\ \text{is invariant under } \pi(X+iY),\\ X, Y \in \mathfrak{g}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\text{a complex subspace}\\ W \subseteq V \text{ is invariant}\\ \text{under the operators}\\ \pi(X), \pi(Y) \quad \forall X, Y \in \mathfrak{g}\end{array}\right\}$$

So the representation of $\mathfrak{g}$ and its extension to $\mathfrak{g}_{\mathbb{C}}$ have the same invariant subspaces. $\blacksquare$

Exercises 9

Example: As $\mathbb{C}$-Lie algebras, $u(n)_{\mathbb{C}} \xrightarrow{\sim} gl(n, \mathbb{C})$.

$u(n) = \{X \in M_n(\mathbb{C}) \mid X^* = -X;\ X^* = \overline{X^T}\}$; if $X^* = -X$, then $(iX)^* = iX \Rightarrow X$ and iX can't be both in $u(n)$, unless $X = 0$.

Moreover, $\forall X \in M_n(\mathbb{C})\ (\cong gl(n, \mathbb{C}))$ can be written as

$$X = X_1 + iX_2 \text{ with } X_1 = \frac{X - X^*}{2} \text{ an } X_2 = \frac{X + X^*}{2i},$$

where $X_1, X_2 \in u(n)$ $(X_1^* = -X_1,\ X_2^* = -X_2)$.

Therefore $u(n)_{\mathbb{C}} \xrightarrow{\sim} gl(n, \mathbb{C})$.

Example: Both $su(2)_{\mathbb{C}}$ and $sl(2, \mathbb{R})_{\mathbb{C}}$ are isomorphic to $sl(2, \mathbb{C})$, the real Lie algebras $su(2)$ and $sl(2, \mathbb{R})$ are not isomorphic.

We take the bases

$su(2)$: $\sigma_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $[\sigma_i, \sigma_j] = \epsilon_{ijk}\sigma_k$ $\quad i, j, k = 1, 2, 3$

$sl(2, \mathbb{R})$: $A_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $A_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $[A_1, A_2] = A_2$ $\vdots$

¿Is there an isomorphism of these real Lie algebras?

Notice that the subspace $U = \langle A_1, A_2 \rangle \subseteq sl(2, \mathbb{R})$ generated by A_1, A_2 is subalgebra, because $[A_1, A_2] = A_2 \in U$. So $sl(2, \mathbb{R})$ has 2-dim Lie subalgebra, and we prove $su(2)$ does not. Because Lie algebra isomorphism should induce 2-dim Lie subalg. in $su(2)$, this contradicts its existence. So we shall prove there is no such Lie subalgebra.

Define $\varphi : su(2) \longrightarrow \mathbb{R}^3$ $\quad e_1, e_2, e_3$ canonical basis of $\mathbb{R}^3$.

$$\sigma_i \longmapsto \varphi(\sigma_i) = e_i$$

Then $\varphi([\sigma_i, \sigma_j]) = \varphi(\epsilon_{ijk}\sigma_k) = \epsilon_{ijk}\varphi(\sigma_k) = \epsilon_{ijk} e_k = e_i \times e_j =$

$= \varphi(\sigma_i) \times \varphi(\sigma_j)$, where "x" is the cross (or, vector) product in $\mathbb{R}^3$, ϵ_{ijk} is Levi-Civita skew symmetric tensor ($= \pm 1$ according to the sign of permutation (i,j,k).)

So $\varphi([A,B]) = \varphi(A) \times \varphi(B)$, $\forall A, B \in su(2)$, and by linearity φ is an isomorphism of $su(2)$ with $(\mathbb{R}^3, \text{"x"})$.

Let $A, B \in su(2)$ be arbitrary and linearly independent. Then $\langle \varphi(A), \varphi(B) \rangle$ is a 2-dim subspace $V \subseteq \mathbb{R}^3$, hence $\varphi(A) \times \varphi(B)$ is a non-zero vector orthogonal to V. This means $\varphi([A,B]) \notin V$, and as φ is invertible, $[A,B] \notin \varphi^{-1}(V) = \langle A, B \rangle$. This means that A, B do not generate 2-dim subalgebra of $su(2)$, and as A, B were arbitrary, there is no 2-dim subalgebra in $su(2)$.

<u>Another proof</u>: Take the basis (different from the previous one)

$su(2)$: e_1, e_2, e_3 such that
$$[e_1, e_2] = 2e_3$$
$$[e_1, e_3] = -2e_2$$
$$[e_2, e_3] = 2e_1$$
(what are the corresponding matrices?)

$sl(2,\mathbb{R})$ f_1, f_2, f_3
$$[f_1, f_2] = 2f_2$$
$$[f_1, f_3] = -2f_3$$
$$[f_2, f_3] = f_1$$
— " — ?

Assume there is a vector space isomorphism A such that $[AX, AY] = A[X,Y]$ $\forall X, Y \in$ one of the algebras

Let $A: su(2) \to sl(2,\mathbb{R})$ be an isomorphism. In the bases given above, $Ae_i = \sum_j A_{ij} f_j$, $A = \{A_{ij}\}_{i,j=1}^{3}$.

Use $[Ae_1, Ae_2] = A[e_1, e_2]$ to obtain

$$2\begin{vmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{vmatrix} f_2 + 2\begin{vmatrix} A_{11} & A_{21} \\ A_{13} & A_{23} \end{vmatrix}(-f_3) + \begin{vmatrix} A_{12} & A_{22} \\ A_{13} & A_{23} \end{vmatrix} f_1 =$$

$$= 2\left(A_{31}f_1 + A_{32}f_2 + A_{33}f_3\right)$$

f_1, f_2, f_3 is a basis $\Rightarrow$ $A_{31} = \begin{vmatrix} A_{12} & A_{22} \\ A_{13} & A_{23} \end{vmatrix}$, $A_{32} = \dots$, $A_{33} = \dots$

and the substitution into the (last column determinant) expansion

$$\begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix} = A_{31}\begin{vmatrix} A_{12} & A_{22} \\ A_{13} & A_{23} \end{vmatrix} - A_{32}\begin{vmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{vmatrix} + A_{33}\begin{vmatrix} A_{11} & A_{21} \\ A_{13} & A_{23} \end{vmatrix}$$

implies $\det A = 2A_{31}^2 - A_{32}^2 - A_{33}^2$.

The other two commutation relations give

$$\det A = -A_{31}^2 + 2A_{32}^2 - A_{33}^2,$$
$$\det A = -A_{31}^2 - A_{32}^2 + 2A_{33}^2.$$

Their sum is $3\det A = 0$, so A is singular map $\mathbb{R}^3 \xrightarrow{A} \mathbb{R}^3$, hence not an isomorphism. ∎

<u>Example</u>: $\mathfrak{g}$ ... Lie algebra, $ad: \mathfrak{g} \to gl(\mathfrak{g})$

$$X \mapsto ad(X): Y \mapsto ad(X)Y = [X, Y]$$

is a Lie algebra homomorphism $X, Y \in \mathfrak{g}$

($\mathfrak{g}$ is Lie algebra, $gl(\mathfrak{g})$ is (matrix) Lie algebra)

Its kernel is $Z(\mathfrak{g})$. This is so called adjoint homomorphism.

We have to prove that the image of Lie bracket of two elements is the commutator (in $gl(\mathfrak{g})$) of images of these elements. We have for all $X_1, X_2, Y \in \mathfrak{g}$

$$ad([X_1, X_2])(Y) = [[X_1, X_2], Y].$$

On the other hand,

$$[ad(X_1), ad(X_2)](Y) = (ad(X_1) \circ ad(X_2))(Y) - (ad(X_2) \circ ad(X_1))(Y) = ad(X_1)[X_2, Y] - ad(X_2)[X_1, Y] = [X_1, [X_2, Y]] - [X_2, [X_1, Y]].$$

It follows that

$$ad([X_1, X_2])(Y) - [ad(X_1), ad(X_2)](Y) =$$

$$= [[X_1, X_2], Y] - [X_1, [X_2, Y]] + [X_2, [X_1, Y]]$$

$$= -[Y, [X_1, X_2]] - [X_1, [X_2, Y]] - [X_2, [Y, X_1]]$$

$= 0$ by Jacobi identity.

The claim about $Z(\mathfrak{g})$ is elementary.

<u>Exercise</u>: Prove that for any $X \in \mathfrak{g}$ acts $ad(X)$ as a derivation of Lie bracket, i.e.

$$ad(X)([Y_1, Y_2]) = [Y_1, ad(X)Y_2] + [ad(X)Y_1, Y_2].$$

for all $Y_1, Y_2 \in \mathfrak{g}$.

A map $\underset{\text{a linear map } \mathfrak{g} \to \mathfrak{g}}{\varphi}$ satisfying $\varphi([Y_1, Y_2]) = [Y_1, \varphi(Y_2)] + [\varphi(Y_1), Y_2]$ is called derivation of Lie algebra $\mathfrak{g}$.