(LECTURE 8)

In the last lecture we understood the relationship between group structure and Lie algebra structure (see the Dynkin theorem, for example.) In fact, there is a bijective correspondence between finite-dimensional Lie algebras and connected simply-connected fin. dim. Lie groups. A difficult implication is the construction of Lie group out of a given Lie algebra; this is based on Ado's theorem, which realizes any Lie algebra as a Lie subalgebra in $\mathfrak{gl}(n)$ for sufficiently large $n \in \mathbb{N}$.) Consequently, any Lie group is a Lie subgroup of $GL(n)$ for $n \in \mathbb{N}$ sufficiently large.

Theorem 0: A linear map $\varphi' : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism tangent to (locally defined) homomorphism of corresponding Lie groups G, H if and only if there is (locally defined) homomorphism $\varphi : G \to H$ of Lie groups fulfilling

$$\exp_H \circ \varphi' = \varphi \circ \exp_G .$$

If G is simply-connected, the homom. φ is globally defined on connected component of identity.

~~Základní strukturální algebraické vlastnosti Lieových algeber: (I/Za~~

Basic structural algebraic properties of Lie algebras (over $\mathbb{R}, \mathbb{C}$):
$\mathfrak{g}$ ... a Lie algebra, $\tilde{\mathfrak{g}} \subseteq \mathfrak{g}$ is a Lie subalgebra if $\tilde{\mathfrak{g}} \subseteq \mathfrak{g}$ is a vector subspace and $\tilde{\mathfrak{g}}$ is stable under Lie bracket (ad-action)
An ideal $I \subseteq \mathfrak{g}$ is a vector subspace such that $[x, y] \in I$ for all $x \in \mathfrak{g}$ and $y \in I$. Consequently, I is a subalgebra of $\mathfrak{g}$.
The center of Lie algebra $\mathfrak{g}$ is $Z(\mathfrak{g}) := \{ x \in \mathfrak{g} \mid [x, y] = 0 \ \forall y \in \mathfrak{g} \}$.
The Lie algebra $\mathfrak{g}$ is abelian if $Z(\mathfrak{g}) = \mathfrak{g}$.

Lemma 1: $\mathfrak{g}$ ... a Lie algebra, $I, J \subseteq \mathfrak{g}$ ideals. Define $[I, J]$ as the linear span of all $[x, y]$, $x \in I$, $y \in J$. Then the subspace $[I, J] \subseteq \mathfrak{g}$ is an ideal in $\mathfrak{g}$.

Pf: $x \in \mathfrak{g}$, $y \in I$, $z \in J$; we have $[x, [y, z]] = -[y, [z, x]] - [z, [x, y]]$ by the Jacobi identity. Because $[z, x] \in J$, $[x, y] \in I$ we have $[x, [y, z]] \in [I, J]$. ▨

A special case $I = \mathfrak{g} = J$ gives the notion $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ of derived (sub)algebra.

For $I \subseteq \mathfrak{g}$ an ideal, $\mathfrak{g}/I$ is the quotient Lie algebra (the elements of $\mathfrak{g}/I$ are $X + I$, $X \in \mathfrak{g}$, the Lie bracket is defined by $[X + I, Y + I] = [X, Y] + I$)

Consider the following descending sequence of ideals in $\mathfrak{g}$:

$$\mathfrak{g} \supseteq \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \supseteq (\mathfrak{g}')' = [\mathfrak{g}', \mathfrak{g}'] \supseteq ((\mathfrak{g}')')' = [(\mathfrak{g}')', (\mathfrak{g}')'] \supseteq \dots$$

where each $((\mathfrak{g}')')^{\cdots} = \mathfrak{g}^{(k)}$ is an ideal in $\mathfrak{g}$; $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = \mathfrak{g}'$, ...

$\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \dots$ is derived series of $\mathfrak{g}$; $\mathfrak{g}$ is <u>solvable</u> Lie algebra if $\mathfrak{g}^{(k)} = 0$ for some $k \in \mathbb{N}$.

Lemma 2: $\mathfrak{g}$ ... a Lie algebra. Then $\mathfrak{g}$ is solvable $\Leftrightarrow$ $\exists$ a sequence of ideals

$$\mathfrak{g} = I_0 \supseteq I_1 \supseteq \dots \supseteq I_{m-1} \supseteq I_m = 0 \quad \text{with } I_{k-1}/I_k$$

abelian Lie algebra for $k \in \{1, \dots, m\}$.

Proof: $\Rightarrow$ clear

$\Leftarrow$ need to prove $\mathfrak{g}^{(k)} \subseteq I_k$ for $k \in \{0, 1, \dots, m\}$. By induction: true for $k = 0$, $\mathfrak{g}^{(0)} = \mathfrak{g} = I_0$. Fix $k \in \{1, \dots, m\}$, assume $\mathfrak{g}^{(j)} \subseteq I_j$ for all $j \in \{0, 1, \dots, k-1\}$, and prove $\mathfrak{g}^{(k)} \subseteq I_k$.
By hypothesis I_{k-1}/I_k is abelian, so $[I_{k-1}, I_{k-1}] \subseteq I_k$.
Then $\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}] \subseteq [I_{k-1}, I_{k-1}] \subseteq I_k$, which completes the proof. ▨

<u>Lemma 3</u>: $\mathfrak{g}_1, \mathfrak{g}_2$ ... Lie algebras, $\varphi: \mathfrak{g}_1 \to \mathfrak{g}_2$ surjective Lie algebra homomorphism. If $k \in \mathbb{N}$, $\varphi(\mathfrak{g}_1^{(k)}) = \mathfrak{g}_2^{(k)}$ ($\Rightarrow$ if $\mathfrak{g}_1$ is solvable, then so is $\varphi(\mathfrak{g}_1) = \mathfrak{g}_2$.)

<u>Pf</u>: By induction on $k \in \mathbb{N}$, true for $k=0$. Assume true for k, then

$$\varphi(\mathfrak{g}_1^{(k+1)}) = \varphi([\mathfrak{g}_1^{(k)}, \mathfrak{g}_1^{(k)}]) = [\varphi(\mathfrak{g}_1^{(k)}), \varphi(\mathfrak{g}_1^{(k)})] =$$
$$= [\mathfrak{g}_2^{(k)}, \mathfrak{g}_2^{(k)}] = \mathfrak{g}_2^{(k+1)}. \quad \blacksquare$$

<u>Lemma 4</u>: $\mathfrak{g}$ ... a Lie algebra. Then $\mathfrak{g}^{(k+j)} = (\mathfrak{g}^{(k)})^{(j)}$ for all $k, j \in \mathbb{N}$.

<u>Pf</u>: Fix $k \in \mathbb{N}$, prove $\mathfrak{g}^{(k+j)} = (\mathfrak{g}^{(k)})^{(j)}$ by induction on j: true for $j=0$; assume true ~~for~~ for $0, \dots, j$, prove true for $j+1$:

$$\mathfrak{g}^{(k+j+1)} = [\mathfrak{g}^{(k+j)}, \mathfrak{g}^{(k+j)}] = [(\mathfrak{g}^{(k)})^{(j)}, (\mathfrak{g}^{(k)})^{(j)}],$$

and on the other hand $(\mathfrak{g}^{(k)})^{(j+1)} = [(\mathfrak{g}^{(k)})^{(j)}, (\mathfrak{g}^{(k)})^{(j)}]$. $\blacksquare$

<u>Lemma 5</u>: $\mathfrak{g}$ ... a Lie algebra, $I \subseteq \mathfrak{g}$ an ideal. Then $\mathfrak{g}$ is solvable if and only if I and $\mathfrak{g}/I$ are solvable Lie algebras.

<u>Pf</u>: If $\mathfrak{g}$ is solvable $\Rightarrow$ I is solvable ($I^{(k)} \subseteq \mathfrak{g}^{(k)}$ $\forall k \in \mathbb{N}$), and also $\mathfrak{g}/I$ is solvable by Lemma 3.

Assume now $I, \mathfrak{g}/I$ are both solvable Lie algebras. Since $\mathfrak{g}/I$ is solvable $\Rightarrow$ $\exists k \in \mathbb{N}$ $(\mathfrak{g}/I)^{(k)} = 0$. This implies $\mathfrak{g}^{(k)} + I \subseteq I$, hence $\mathfrak{g}^{(k)} \subset I$. Since I is also solvable, $\exists j \in \mathbb{N}$ such that $I^{(j)} = 0$. It follows $(\mathfrak{g}^{(k)})^{(j)} \subseteq I^{(j)} = 0$, and since $\mathfrak{g}^{(k+j)} = (\mathfrak{g}^{(k)})^{(j)}$ by Lemma 4, $\mathfrak{g}$ is solvable. $\blacksquare$

<u>Lemma 6</u>: $\mathfrak{g}$ ... a Lie algebra, $I, J \subseteq \mathfrak{g}$ solvable ideals of $\mathfrak{g}$. Then $I+J$ is solvable ideal.

<u>Pf</u>: Consider the sequence of ideals in Lie algebra $\mathfrak{g}$ $\quad I+J \supseteq J \supseteq 0$.

We have the Lie algebra isomorphism $(I+J)/J \xrightarrow{\sim} I/(I\cap J)$, where $I/I\cap J$ is solvable Lie algebra by lemma 3. Hence $(I+J)/J$ is solvable, so by lemma 5 $I+J$ is solvable. ▨

<u>Lemma 7</u> $\mathfrak{g}$ ... a (finite-dimensional) Lie algebra. Then there $\exists$ a solvable ideal $I \subseteq \mathfrak{g}$ such that $\forall$ solvable ideals is contained in I.

<u>Pf.</u> $\mathfrak{g}$ is fin-dim $\Rightarrow \exists$ solvable ideal $I \subseteq \mathfrak{g}$ of maximal dimension. Let $J \subseteq \mathfrak{g}$ solvable ideal. Then $I+J$ is solvable by lemma 6, and by maximality of I we have $I+J \subseteq I$ $\Rightarrow J \subseteq I$. ▨

The maximal solvable ideal in (finite-dimensional) $\mathfrak{g}$ is called radical of $\mathfrak{g}$, $\mathrm{rad}(\mathfrak{g})$. A fin.-dim. Lie algebra $\mathfrak{g}_{\neq 0}$ is semi-simple if $\mathrm{rad}(\mathfrak{g}) = 0$.

<u>Proposition 8</u>: $\mathfrak{g}$ ... a finite-dim. Lie algebra. The Lie algebra $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semi-simple.

<u>Pf:</u> Assume $I \subseteq \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is solvable ideal, need to prove $I = 0$. The projection $p: \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is a Lie algebra homomorphism. Define $J := p^{-1}(I)$, so J is an ideal in $\mathfrak{g}$ containing $\mathrm{rad}(\mathfrak{g})$. Let $k \in \mathbb{N}$; by lemma 3 $p(J^{(k)}) = p(J)^{(k)} = I^{(k)}$. Because I is solvable, $I^{(k)} = 0$ for some k, so $p(J^{(k)}) = 0$ and so $J^{(k)} \subseteq \mathrm{rad}(\mathfrak{g})$. Since $\mathrm{rad}(\mathfrak{g})$ is solvable, there is $j \in \mathbb{N}$ such that $(J^{(k)})^{(j)} = 0$. By lemma 4, $(J^{(k)})^{(j)} = (J^{(k+j)}) = 0$, and so J is solvable. Then $J \subseteq \mathrm{rad}(\mathfrak{g})$, hence $I = 0$. ▨

A stronger property than solvability: $\mathfrak{g}$ ... a Lie algebra, the lower central series of $\mathfrak{g}$ is a sequence of ideals

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = \mathfrak{g}', \quad \mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}], \; k \geq 2.$$

Any of $\mathfrak{g}^0, \mathfrak{g}^1, \mathfrak{g}^2$ is an ideal in $\mathfrak{g}$, and we have

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \dots$$

and $\mathfrak{g}^{(k)} \subseteq \mathfrak{g}^k$. While both $\mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$ and $\mathfrak{g}^k/\mathfrak{g}^{k+1}$ are both abelian, the quotient $\mathfrak{g}^k/\mathfrak{g}^{k+1}$ is in the center of $\mathfrak{g}/\mathfrak{g}^{k+1}$. $\mathfrak{g}$ is nilpotent if $\mathfrak{g}^k = 0$ for some $k \in \mathbb{N}$. If $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}$ is solvable.

There is a general structural theorem (we shall not give a proof)

Theorem 9: (Levi decomposition)

Let $\mathfrak{g}/\mathbb{F}$ be a finite dimensional Lie algebra over a field of characteristic 0. Then there is a subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$ such that there is a vector space isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{l}$; there is a Lie bracket on $\mathfrak{l}$ given by canonical inclusion $\mathfrak{l} \to \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{l}$ in the exact sequence

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{l} \to \mathfrak{l} \to 0,$$

and the subalgebra $\mathfrak{l}$ is uniquely determined up to a conjugation. In other words, every Lie algebra is a semi-direct product of a solvable and a semi-simple Lie algebras.

For the notion of semi-direct product, see example session.

(LECTURE 8 – Examples and exercises)

<u>Example</u>: What is the difference between left and right ideals in Lie algebras?

<u>Example</u>: The center $Z(\mathfrak{g})$ of Lie algebra $\mathfrak{g}$ is an ideal in $\mathfrak{g}$.

The kernel of a Lie algebra homomorphism is an ideal.

<u>Example</u>: If $I, J \subseteq \mathfrak{g}$ are ideals, then $[X, Y] = [Y, X]$.

<u>Example</u>: The derived algebra of $\mathfrak{sl}(2, \overset{\mathbb{R}}{\mathbb{C}})$ is $\mathfrak{sl}(2, \overset{\mathbb{R}}{\mathbb{C}})$.

Proof: $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$[e, f] = h$, $[e, h] = -2e$, $[f, h] = 2f$

and the claim follows.

<u>Example</u>: $\mathfrak{g}$ ... Lie algebra, $\mathfrak{g}'$ ... derived Lie algebra. Then the quotient Lie-algebra $\mathfrak{g}/\mathfrak{g}'$ is abelian.

<u>Example</u>: Prove $Z(\mathfrak{g}) \subseteq \mathrm{rad}(\mathfrak{g})$, so if $\mathfrak{g}$ is semi-simple, $Z(\mathfrak{g}) = 0$.

(~~Hint~~ Prove that the center $Z(\mathfrak{g})$ is a solvable ideal of $\mathfrak{g}$.)

<u>Example</u>: The Lie algebra $\mathfrak{sl}(2, \overset{\mathbb{R}}{\mathbb{C}})$ is semi-simple. In fact, the only ideals in $\mathfrak{sl}(2, \overset{\mathbb{R}}{\mathbb{C}})$ are 0 and $\mathfrak{sl}(2, \overset{\mathbb{R}}{\mathbb{C}})$.

Assume $I \subseteq \mathfrak{sl}(2, \overset{\mathbb{R}}{\mathbb{C}})$ is an ideal. Let

$X := \alpha e + \beta h + \gamma f \in I$, where the basis e, f, h is as in the previous example above, and $\alpha, \beta, \gamma \in \overset{\mathbb{R}}{\mathbb{C}}$.

Now assume $\alpha \neq 0$. We have

$$\left.\begin{array}{l}[h,x] = 2\alpha e - 2\gamma f \\ [f,x] = -\alpha h + 2\beta f\end{array}\right\} \Rightarrow \begin{array}{l}[f,[h,x]] = -2\alpha h \\ [f,[f,x]] = -2\alpha f\end{array}$$

which implies that $h, f \in I$. Because $x = \alpha e + \beta h + \gamma f \in I$ and $\alpha \neq 0$, $e \in I$ as well. Consequently, $I = sl(2, \mathbb{C}^{\mathbb{R}})$. Analogous argument works for either $\beta \neq 0$ or $\gamma \neq 0$.

Example: Let $b(2\times 2, \mathbb{C}^{\mathbb{R}}) \subseteq gl(2, \mathbb{C}^{\mathbb{R}})$ be the $\mathbb{C}^{\mathbb{R}}$-subspace of upper-triangular matrices. Then $b(2\times 2, \mathbb{C})$ is the Lie subalgebra of $gl(2,\mathbb{C})$, and $b(2\times 2, \mathbb{C})$ is solvable.

$$x_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} \in b(2\times 2, \mathbb{C}^{\mathbb{R}})$$

$a_i, b_i, d_i \in \mathbb{C}^{\mathbb{R}}$

Then $$[x_1, x_2] = \begin{pmatrix} 0 & b_1 d_2 - b_2 d_1 + a_1 b_2 - a_2 b_1 \\ 0 & 0 \end{pmatrix} \in b(2\times 2, \mathbb{C}^{\mathbb{R}})$$

$\Rightarrow b(2\times 2, \mathbb{C}^{\mathbb{R}})$ is Lie subalgebra.

Moreover,
$$b(2\times 2, \mathbb{C}^{\mathbb{R}}) = b(2\times 2, \mathbb{C}^{\mathbb{R}})^{(0)} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix},$$
$$b(2\times 2, \mathbb{C}^{\mathbb{R}})^{(1)} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix},$$
$$b(2\times 2, \mathbb{C}^{\mathbb{R}})^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Example: (Solvable Lie algebra $\not\Rightarrow$ Nilpotent Lie algebra)

$b(2\times 2, \mathbb{C}^{\mathbb{R}}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, then $b(2\times 2, \mathbb{C}^{\mathbb{R}})^1 = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$

$b(2\times 2, \mathbb{C}^{\mathbb{R}})^2 = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \ldots, b(2\times 2, \mathbb{C}^{\mathbb{R}})^k = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \quad \forall k \in \mathbb{N}$

<u>Example</u>: The space of strictly upper triangular matrices is nilpotent:

$$n(k\times k, \overset{\mathbb{R}}{\mathbb{C}}) = \begin{pmatrix} 0 & * & * & \cdots & * \\ & 0 & * & \cdots & * \\ & & 0 & * & \vdots \\ & 0 & & \ddots & * \\ & & & & 0 \end{pmatrix}$$

<u>Example</u>: Over the field F of characteristic 2, the Lie algebra $sl(2,F)$ is nilpotent:

$$[h,e] = 2e = 0, \quad [h,f] = -2f = 0, \quad [e,f] = h,$$

and so

$$sl(2,F)^0 = sl(2,F), \quad sl(2,F)^1 = \langle h \rangle, \quad sl(2,F)^k = 0$$
$$k \geq 2.$$

<u>Example</u>: (Polopřímý) Semi-direct product of Lie algebras:
Let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ be a short exact sequence of Lie algebras A, B, C and their homomorphisms i, j ($\Rightarrow$ i is monomorphism, j is epimorphism.) Let $s: C \to B$ be a section of homomorphism j, i.e. $j\circ s = Id_C$. In particular, we have $B \xrightarrow{\sim} A \oplus C$ as vector spaces. An element $b \in B$ corresponds in this vector space isomorphism to $(i^{-1}(b - s(j(b))), j(b))$
$\overset{\cap}{A} \qquad \overset{\cap}{C}$

and vice-versa: $(a,c) \in A\oplus C$ we have $i(a) + s(c) \in B$, hence there is a bijection realizing $B \xrightarrow{\sim} A \oplus C$.

Define a map
$$\varphi: C \to Hom(A,A) = End(A) \quad \text{by}$$
$$c \mapsto \varphi(c)$$
$$\varphi(c)(a) := [s(c), i(a)].$$

Because $i(A) \subseteq B$ is ideal ($i(A) = Ker(j)$), $\varphi(c)$ is defined by it. Jacobi identity implies

$$\varphi([c,c']) = \varphi(c)\circ\varphi(c') - \varphi(c')\circ\varphi(c), \quad c,c'\in C,$$

so φ is a homom. of Lie algebras $(C, \mathrm{End}(A))$.

The Lie bracket on B is

$$[i(a)+s(c), i(a')+s(c')] = i([a,a']) + s([c,c']) - \varphi(c')a + \varphi(c)(a')$$

and say $B \xrightarrow{\sim} A\oplus C$ equipped with this Lie bracket is a semi-direct product of Lie algebras A, C.

On the other hand, given homomorphism $\varphi: C \to \mathrm{Hom}(A,A)$ for Lie algebras A, C, the last formula gives Lie algebra structure on $A\oplus C$. (the underlying vector space)