(LECTURE 5)

$G \dots$ a Lie group $\qquad Ad : G \to GL(T_e G)$
$\exp : T_e G \longrightarrow G$

Lemma 1:

$x \in G, \ \forall X \in T_e G : \qquad x \exp(X) x^{-1} = \exp(Ad(x) X)$

Proof: $\mathcal{C}_x : G \to G$ is Lie group homomorphism (e.g. $y_1 \to x y_1 x^{-1}$, $y_2 \to x y_2 x^{-1}$ $\Rightarrow$ $y_1 y_2 \to x y_1 y_2 x^{-1}$ and the maps are smooth)
$\quad\ \ {}^{\shortparallel}\ L_x \circ R_x^{-1}$

The commutative diagram for $H = G$, $\varphi = \mathcal{C}_x$ and $T_e \mathcal{C}_x = Ad(x)$ gives

$$\begin{array}{ccc} G & \xrightarrow{\mathcal{C}_x} & G \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ T_e G & \xrightarrow{Ad(x)} & T_e G \end{array}$$

is commutative, which is just the equality above. ▨

Lemma 2: The map $Ad : G \to GL(T_e G)$, $x \mapsto Ad(x)$ is a Lie group homomorphism.

Proof: The map $G \times G \to G$, $(x, y) \mapsto x y x^{-1}$ is smooth, its tangent map with respect to y at $y = e$ implies $x \to Ad(x)$ is $G \to End(T_e G)$ smooth. Since $GL(V)$ is open in $End(T_e G) \Rightarrow$
$Ad : G \to GL(T_e G)$ is smooth.
$\mathcal{C}_e = Id_G \Rightarrow Ad(e) = Id_{T_e G}$, and the tangent map of $\mathcal{C}_{xy} = \mathcal{C}_x \mathcal{C}_y$ at $e \in G \Rightarrow$ (the chain rule for differentiation) $Ad(xy) = Ad(x) Ad(y)$ $\forall x, y \in G$. ▨

Remark 3: Since $Ad(e) = Id_{T_e G}$, $T_{Id} GL(T_e G) = End(T_e G)$, the tangent map of Ad at e is a linear map $T_e G \to End(T_e G)$.

Definition 4: The linear map $ad : T_e G \to End(T_e G)$ is defined by $ad := T_e Ad$. By the chain rule,

$\forall\, X \in T_e G \qquad \mathrm{ad}(X) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp tX).$

(Here note $t \to tX$ is a curve in $T_e G$ such that it passes through $0 \in T_e G$ and it is tangent to X at $0 \in T_e G$).

<u>Lemma 5</u>: $\forall\, X \in T_e G$, we have $\mathrm{Ad}(\exp X) = e^{(\mathrm{ad}\, X)}$.

(Here $e^{\cdots}$ is the exponential in the matrix Lie group $GL(T_e G)$.)

<u>Proof</u>: Apply commuting square lemma to $H = GL(T_e G)$, $\varphi = \mathrm{Ad}$, $T_e\varphi = \mathrm{ad}$. Since $T_e H = T_I\, GL(T_e G) = \mathrm{End}(T_e G)$, whereas $\exp_H : X \longmapsto e^X$, we see

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ad}} & GL(T_e G) \\ \exp \uparrow & & \uparrow e^{(-)} \\ T_e G & \xrightarrow{\mathrm{ad}} & \mathrm{End}(T_e G) \end{array}$$

commutes, hence the proof. $\blacksquare$

<u>Example 6</u>: $V/\mathbb{R}$ real vector space, $x \in GL(V)$, the linear map $\mathrm{Ad}(x) : \mathrm{End}(V) \to \mathrm{End}(V)$

$$Y \longmapsto \mathrm{Ad}(x)Y = x \cdot Y \cdot x^{-1}$$

matrix multiplication

The substitution $x = e^{tX}$, and $\frac{d}{dt}\Big|_{t=0}$ gives

$$(\mathrm{ad}\, X)Y = \frac{d}{dt}\Big|_{t=0}\left(e^{tX} Y e^{-tX}\right) = X \cdot Y - Y \cdot X,$$

i.e. for matrix Lie groups is $(\mathrm{ad}\, X)Y$ the commutator bracket.

<u>Definition 7</u>: $X, Y \in T_e G$, define the Lie bracket $[X,Y] \in T_e G$ by $[X,Y] := (\mathrm{ad}\, X)Y$.

Lemma 8: The map $T_eG \times T_eG \to T_eG$, $X, Y \mapsto [X, Y]$ is bilinear and anti-symmetric $[X, Y] = -[Y, X]$, $X, Y \in T_eG$.

Proof: $\mathrm{ad}: T_eG \to \mathrm{End}(T_eG)$ is linear $\Rightarrow$ bilinearity.

Let $Z \in T_eG$, $s, t \in \mathbb{R}$. Then

$$\exp(tZ) = \exp(sZ)\exp(tZ)\exp(-sZ) =$$
$$= \exp(t\,\mathrm{Ad}(\exp(sZ))Z)$$

by previous results (i.e., first equality by $\exp(sX)\exp(tX) = \exp((s+t)X)$, second equality by lemma 1.) The tangent map for t at $t=0$ $\Rightarrow$

$$Z = \mathrm{Ad}(\exp(sZ))Z \quad , \; s \in \mathbb{R},$$

and the ~~tangent~~ map at $s=0$ $\Rightarrow$

$$0 = \mathrm{ad}(Z)\, T_0(\exp) Z = \mathrm{ad}(Z)\, Z = [Z, Z],$$

so the substitution $Z = X + Y$ and bilinearity imply the result. ∎

Lemma 9: $\varphi: G \to H$ a Lie group homomorphism. Then

$$(T_e\varphi)([X,Y]_G) = [(T_e\varphi)X, (T_e\varphi)Y]_H, \quad X, Y \in T_eG.$$

Proof: We have $\varphi \circ c_x^G = c^H_{\varphi(x)} \circ \varphi$, because

$$(\varphi \circ c_x^G)(y) = \varphi \circ (xyx^{-1}) = \varphi(x)\varphi(y)\varphi(x)^{-1}, \quad \varphi(x^{-1}) = \varphi(x)^{-1}$$
$$(c^H_{\varphi(x)} \circ \varphi)(y) = \varphi(x)\varphi(y)\varphi(x)^{-1}.$$

The tangent map of both sides at e $\Rightarrow$

$$\begin{array}{ccc} T_eG & \xrightarrow{T_e\varphi} & T_eH \\ \mathrm{Ad}_G(x)\uparrow & & \uparrow \mathrm{Ad}_H(\varphi(x)) \\ T_eG & \xrightarrow{T_e\varphi} & T_eH \end{array}$$ commutes,

and the tangent map at $x=e$ in the direction of $X \in T_eG \Rightarrow$

$$\begin{array}{ccc} T_eG & \xrightarrow{T_e\varphi} & T_eH \\ \mathrm{ad}_G(X)\uparrow & & \uparrow \mathrm{ad}_H((T_e\varphi)X) \\ T_eG & \xrightarrow{T_e\varphi} & T_eH \end{array}$$ commutes.

We write $[X,Y] = \mathrm{ad}(X)Y$; the application of $(T_e\varphi)\circ \mathrm{ad}_G(X)$ to $Y \in T_eG$, the commutativity of this diagram yields the claim. ▨

<u>Corollary 10</u>: $\forall\, X,Y,Z \in T_eG$,
$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]].$$

<u>Proof</u>: $H = GL(T_eG)$, $\varphi = \mathrm{Ad}$, $e_H = \mathrm{Id}$, $T_{\mathrm{Id}}H = \mathrm{End}(T_eG)$, and so $[A,B]_H = A\cdot B - B\cdot A \quad \forall\, A,B \in \mathrm{End}(T_eG)$.
Application of Lemma 9, $[-,-]_G = [-,-]$ and $T_e\varphi = \mathrm{ad}$, we obtain
$$\mathrm{ad}([X,Y]) = [\mathrm{ad}X, \mathrm{ad}Y]_H = \mathrm{ad}X\,\mathrm{ad}Y - \mathrm{ad}Y\,\mathrm{ad}X.$$
This applied to $Z \in T_eG$ yields the required equality.

<u>Definition 11</u>: A real Lie algebra is a real vector space V equipped with bilinear map $[-,-]: V\times V \to V$ such that $\forall\, X,Y,Z \in V$:

a/ $[X,Y] = -[Y,X]$ (anti-symmetry)

b/ $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (Jacobi identity)

Remark 12: a/ $\iff$ $[X,X] = 0 \quad \forall X \in V$

b/ $\iff$ (in view a/)

$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]],$$

or Leibnitz type rule

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$$

($[-,-]$ acts as a derivation of Lie algebra structure.)

Corollary 13: G a Lie group, then T_eG equipped with bilinear map $T_eG \times T_eG \to T_eG$

$$X, Y \mapsto [X,Y] := (\mathrm{ad}\, X)Y$$

is a Lie algebra.

Proof: Anti-linearity is proved in Lemma 8, Jacobi identity follows from Corollary 10.

Definition 14: Let V_1, V_2 be Lie algebras. A Lie algebra homomorphism $\varphi : V_1 \to V_2$ such that

$$\varphi([X,Y]_{V_1}) = [\varphi(X), \varphi(Y)]_{V_2} \text{ for all } X, Y \in V_1.$$

Notation: $G, H, \ldots$ Lie groups; $\mathfrak{g}, \mathfrak{h}, \ldots$ Lie algebras of $G, H, \ldots$ (gothic roman letters)

Lemma 15: $\varphi : G \to H$ Lie group homomorphism. Then $T\varphi \equiv \varphi_* : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. Moreover, the diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ \mathfrak{g} \arrow[u, "\exp_G"] \arrow[r, "\varphi_*"] & \mathfrak{h} \arrow[u, "\exp_H"'] \end{tikzcd}$$

commutes.

Proof: The first part is a consequence of Lemma 9, the second claim from the lemma on commutativity of

$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ T_eG \arrow[u, "\exp_G"] \arrow[r, "T_e\varphi"] & T_eH \arrow[u, "\exp_H"'] \end{tikzcd}$$

Example 16: Consider the Lie group $G = \mathbb{R}^n$. The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G) = T_0\mathbb{R}^n$ is identified with $\mathbb{R}^n$. G is commutative $\Rightarrow C_x = Id_G \quad \forall x \in G$.
Hence $Ad(x) = Id_{\mathfrak{g}} \quad \forall x \in G \Rightarrow \underset{0}{\overset{ad(X)}{\|}} \quad \forall X \in \mathfrak{g}$.

This means $[X, Y] = 0 \quad \forall X, Y \in \mathfrak{g}$.
For $X \in \mathfrak{g} \simeq \mathbb{R}^n$, the associated 1-parameter subgroup α_X is given by $\alpha_X(t) = tX$, and so $\exp(X) = X \quad \forall X \in \mathfrak{g}$.

Consider Lie group homomorphism $\varphi \equiv (\varphi_1, \to \varphi_n)$: $\mathbb{R}^n \to \mathbb{T}^n$ given by $\varphi_j(x) = e^{2\pi i x_j}$. It is elementary to see that φ is a local diffeomorphism, $Ker(\varphi) \simeq \mathbb{Z}^n$. By the isomorphism theorem for groups, φ quotient through an isomorphism of Lie groups $\tilde{\varphi}: \mathbb{R}^n/\mathbb{Z}^n \xrightarrow{\sim} \mathbb{T}^n$. This allows transfer of manifold structure from $\mathbb{T}^n$ to $\mathbb{R}^n/\mathbb{Z}^n$, and $\pi: \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is Lie group homomorphism. Since π is a local diffeomorphism, $d\pi = \pi_*: \mathfrak{g} \to \mathfrak{h}$ is isomorphism, $\mathfrak{h} := \mathrm{Lie}(\mathbb{R}^n/\mathbb{Z}^n)$.

The exponential map $\exp_{\mathbb{R}^n/\mathbb{Z}^n} : \underset{T_0(\mathbb{R}^n/\mathbb{Z}^n)}{\overset{\shortparallel}{\mathbb{R}^n}} \longrightarrow \mathbb{R}^n/\mathbb{Z}^n$ is given

by $\exp_{\mathbb{R}^n/\mathbb{Z}^n}(X) = \pi(X) = X + \mathbb{Z}^n$; $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ a local diffeomorphism.

(Examples and exercises 5)

Example (Homework from the last exercise session):

? Invariant vector fields on the (matrix) Lie group $SU(2)$?

$$G = SU(2) = \left\{ \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\}$$

$$\mathfrak{g} = \underset{\text{Lie}(SU(2))}{\overset{\|}{su(2)}} = \left\{ \begin{bmatrix} ix & -\bar{\beta} \\ \beta & -ix \end{bmatrix} \;\middle|\; x \in \mathbb{R},\ \beta \in \mathbb{C} \right\}$$

$\circledast$ Recall that $\mathfrak{g} = \overbrace{\text{Lie algebra of } SU(2)}^{\| \; T_{Id}\, SU(2)}$ is the space of 2×2 anti-Hermitean matrices.

Let us consider the basis vector $\xi = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{g}$, and $g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in G$:

$$V_\xi(g) = (L_g)_* \xi = \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} \alpha e^{it} & -\bar{\beta} e^{-it} \\ \beta e^{it} & \bar{\alpha} e^{-it} \end{pmatrix} = \begin{pmatrix} \alpha i & \bar{\beta} i \\ \beta i & -\bar{\alpha} i \end{pmatrix},$$

where we considered a (smooth) curve in G

$$\gamma : t \longmapsto \exp \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix},$$

such that $\gamma(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G$ and

$$\frac{d}{dt}\Big|_{t=0} \gamma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in T_{Id}(SU(2)).$$

The other basis elements are $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$;

$$\underset{\xi_2}{\parallel} \qquad \underset{\xi_3}{\parallel}$$

Compute $V_{\xi_2}(g)$, $V_{\xi_3}(g)$.

<u>Example</u>: The last week we introduced a map Φ_U acting by $X \to U X U^{-1}$ on the space of Hermitean matrices, $U \in SU(2)$. The question was: find Φ_U for $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in SU(2)$; we have

$$\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} x_1 & x_2 + i x_3 \\ x_2 - i x_3 & -x_1 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} x_1' & x_2' + i x_3' \\ x_2' - i x_3' & -x_1' \end{pmatrix};$$

and so we get by elementary calculation

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underbrace{\begin{pmatrix} |\alpha|^2 - |\beta|^2 & -2\,\mathrm{Re}(\alpha\beta) & 2\,\mathrm{Im}(\alpha\beta) \\ 2\,\mathrm{Re}(\alpha\bar{\beta}) & \mathrm{Re}(\alpha^2 - \beta^2) & \mathrm{Im}(\beta^2 - \alpha^2) \\ 2\,\mathrm{Im}(\alpha\bar{\beta}) & \mathrm{Im}(\alpha^2 + \beta^2) & \mathrm{Re}(\alpha^2 + \beta^2) \end{pmatrix}}_{\underset{\Phi_U}{\parallel}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

<u>Example</u>: Adjoint action of Lie group $SU(2)$ on $T_I(SU(2))$ ($=$ Lie algebra $su(2)$)

$$T_I(su(2)) = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle_{\mathbb{R}}$$

$g = A \in SU(2)$, $X \in su(2)$; the adjoint action $Ad(A)$ is given by $X \longrightarrow A \cdot X \cdot A^{-1}$; $X \in su(2)$ implies

$$(A \cdot X \cdot A^{-1})^* = (A^{-1})^* \cdot X^* \cdot A^* = -A \cdot X \cdot A^{-1}$$

because $AA^* = Id$, $X = -X^*$ (the operation $*$ is transpose conjugate, i.e. $Y^* = \overline{Y}^T$) Hence $A \cdot X \cdot A^{-1} \in su(2)$. The explicit calculation is

$$A \cdot X \cdot A^{-1} = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} a-ib & -c-id \\ c-id & a+ib \end{pmatrix}$$

$$= (a^2+b^2-c^2-d^2) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$+ (-2ad - 2bc) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$+ (-2ac + 2bd) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Analogous computation works for the other basis elements $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

<u>Example</u> : Describe the adjoint map $ad : \mathfrak{g} \to End(\mathfrak{g})$ for $\mathfrak{g} = T_I G$ the tangent space of the identity of $G = SU(2)$.

Recall $su(2) = \{ X \in Mat(2\times 2, \mathbb{C}) \mid X = -\underset{\overline{X}^T}{\underset{\shortparallel}{X^*}},\ \underset{0}{\underset{\shortparallel}{tr(X)}} \}$.

We choose as a basis of $su(2)$ the elements

$$su(2) = \langle U, V, W \rangle_{\mathbb{R}}, \quad U = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$V = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$W = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

The Lie algebra structure on $su(2)$ (i.e., the map ad) is given by commutator ($\Leftarrow$ $SU(2)$ is a matrix Lie group):

$$[U,V] = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = W$$

$$[V,W] = \frac{1}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \frac{1}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = U$$

$$[W,U] = \quad \ldots\ldots \quad = V$$

As we know, $ad(X)Y = [X,Y]$, hence

<u>U</u> :
$$U \to [U,U] = 0$$
$$V \to [U,V] = W \qquad ,$$
$$W \to [U,W] = -V$$

<u>V</u> :
$$U \to [V,U] = -W$$
$$V \to 0$$
$$W \to [V,W] = U$$

(Do it for W by Yourself)

$$\Rightarrow \quad ad(U) = \begin{array}{cc} \begin{array}{ccc} U & V & W \end{array} & \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} & \begin{array}{c} U \\ V \\ W \end{array} \end{array}$$

$$ad(V) = \begin{array}{cc} \begin{array}{ccc} U & V & W \end{array} & \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} & \begin{array}{c} U \\ V \\ W \end{array} \end{array}$$

$$ad(W) = \begin{pmatrix} \text{complete} \\ \text{it !} \end{pmatrix}$$

Notice that $ad(U)$, $ad(V)$ and $ad(W)$ are real orthogonal matrices (i.e., equal to minus transpose) for $SO(3, \mathbb{R})$:

$$ad(U),\ ad(V),\ ad(W) \in T_I(SO(3,\mathbb{R})).$$

This is not a chance: there is quite often a bilinear form on $T_I(G)$, preserved by the action of ad-mapping.