LECTURE 11

It can happen that the knowledge of all irreducible representations leads to a description of all representations.

Def 1: A fin. dim. representation of a group or Lie algebra is said to be completely reducible if it is isomorphic to a direct sum of (finite number of) irreducible representations. A group or Lie algebra has the complete reducibility property if $\forall$ fin. dim. representation is completely reducible.

Most of them do not have this property, some do have (and are quite interesting)

Ex 2: Let $\Pi : \mathbb{R} \longrightarrow GL(2,\mathbb{C})$ be given by $\Pi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $x \in \mathbb{R}$. Then Π is not completely reducible.

Pf: Π is a representation of $\mathbb{R}$. In the basis $\{e_1 = \binom{1}{0}, e_2 = \binom{0}{1}\}$ of $\mathbb{C}^2$, $\langle e_1 \rangle$ is an invariant subspace. This is the only one invar. subsp.: suppose $V \subseteq \mathbb{C}^2$ is an invariant subspace, containing a vector not in $\langle e_1 \rangle$, say $v = ae_1 + be_2$, $b \neq 0$. Then $\Pi(1)v - v = be_1 \in V$, so $e_1 \in V$ and $e_2 = (v - ae_1)b^{-1} \in V$, so $V = \mathbb{C}^2 \Rightarrow \mathbb{C}^2 \neq \langle e_1 \rangle \oplus U$ for U an invariant subspace.

Proposition 3: V .. complet. red. repr. of Lie group/algebra, then

1/ $\forall\ U \subseteq V$ inv. subspace, $\exists$ another inv. subspace $W \subseteq V$ such that $V \simeq U \oplus W$.

2/ $\forall$ invariant subspace is completely reducible.

Pf: As for 1/, assume $V = U_1 \oplus \ldots \oplus U_k$, U_j = irreducible invariant subspaces, $U \subseteq V$ an invariant subspace. If $U = V$, then take $W = \{0\}$. If $U \neq V$, there is some j_1: $U_{j_1} \not\subseteq U$. Since U_{j_1} is irreducible, the invariant subspace $U_{j_1} \cap U = \{0\}$. If $U + U_{j_1} = V$, the sum is direct ($U_{j_1} \cap U = \{0\}$), and we are done. If $U + U_{j_1} \neq V$,

there is j_2 s.t. $U+U_{j_1}$ does not contain U_{j_2}, so $(U+U_{j_2}) \cap U_{j_2} = \{0\}$.
Proceeding this way, we obtain $j_1, j_2, \ldots, j_\ell$ s.t. $U + U_{j_1} + \ldots + U_{j_\ell} = V$, and the sum is direct. Then $W := U_{j_1} + \ldots + U_{j_\ell}$ is the desired complement to U.

The proof of 2/ is analogous. ▨

Def 4: If V is a fin.-dim. inner product space and G a Lie group, a representation $\Pi : G \to GL(V)$ is unitary if $\Pi(A)$ is a unitary operator on V $\forall A \in G$.

Proposition 5: G, $\mathfrak{g}$, V a fin.-dim. inner product space, Π a repr. of G on V, π the assoc. repr. of $\mathfrak{g}$ on V. If Π is unitary, then $\pi(X)$ is skew self-adjoint $\forall X \in \mathfrak{g}$. Conversely, if G is connected and $\pi(X)$ skew self-adjoint $\forall X \in \mathfrak{g}$, then Π is unitary.

Pf: Analogous to the computation of the Lie algebra of $U(n)$. If Π is unitary, then $\forall X \in \mathfrak{g}$

$$\left(e^{t\pi(X)}\right)^* = \Pi(\exp(tX))^* = \Pi(\exp(tX))^{-1} = e^{-t\pi(X)}, \quad t \in \mathbb{R}.$$

Then $\frac{d}{dt}\Big|_{t=0}$ gives $\pi(X)^* = -\pi(X)$. The opposite implication is similar. ▨

Proposition 6: G .. Lie group, Π its finite-dim unitary representation. Then Π is completely reducible. Similarly in the case of $\mathfrak{g}$ and its Lie algebra representation π which is fin. dim. unitary (i.e., $\pi(X)^* = -\pi(X)$ $\forall X \in \mathfrak{g}$), then π is completely reducible.

Pf: V .. a Hilbert space on which Π acts, $\langle\, ,\,\rangle$ the inner product on V. If $W \leq V$ an invariant subspace, let $W^\perp$ be OG-complement, i.e. $V = W \oplus W^\perp$. Is $W^\perp$ an invariant subspace for Π or π? Yes: Π is unitary $\Rightarrow$ $\Pi(A)^* = \Pi(A)^{-1} = \Pi(A^{-1})$ $\forall A \in G$. Then $\forall w \in W$ and $\forall v \in W^\perp$, we have

$$\langle \Pi(A)v, w\rangle = \langle v, \Pi(A)^* w\rangle = \langle v, \Pi(A^{-1})w\rangle = \langle v, w'\rangle = 0.$$

We used $w' := \Pi(A^{-1})w$ is in W (W is invariant) $\Rightarrow \Pi(A)v$ is OG to $\forall$ element of W. Similar argument with $\Pi(A^{-1})$ replaced by $-\pi(X)$ shows that OG-complement of an invar. subspace for π is also invariant.

Assume V is not irred, i.e. $\exists$ invariant subspace $W \subseteq V$, $W \neq \{0\}$, $\neq V$, so ~~we~~ can decompose $V = W \oplus W\perp$, where $W, W\perp$ are both invariant subspaces and thus unitary repr. of G (or, alg.) Then W and $W\perp$ are either irreducible or split as an OG-direct sum of invariant subspaces. Since V is finite-dim., continuing this process leads after finitely many steps to the direct sum of irred. invariant subspaces. ▨

<u>Theorem 7</u>: If G is a compact Lie group, every fin. dim. represent. of G is completely reducible.

<u>Pf</u>: The proof is based on the construction of Haar measure on the Lie group (i.e., a function/form on G invariant for G, $A^*\mu = \mu \;\forall A \in G$.) Then one defines

$$\langle\, ,\, \rangle_G : V \times V \longrightarrow \mathbb{C}$$
$$(v, w) \longmapsto \int_G \langle \Pi(A)v, \Pi(A)w \rangle \mu(A)$$

for any inner product $\langle\, ,\, \rangle$ on V. It is then easy to prove that $\langle\, ,\, \rangle_G$ is G-invariant. ▨

<u>Example 8</u>: $(\mathbb{R}^n, \underset{\substack{\| \\ dx_1\wedge\dots\wedge dx_n}}{dx})$, $(\mathbb{C}^*, \frac{dz}{z})$.

<u>Theorem 9 (Schur's lemma)</u>:

1/ V, W irreducible representations over $\mathbb{R}, \mathbb{C}$ of a Lie group/algebra, and $\varphi : V \to W$ splétající (intertwining) zobrazení. Pak buď $\varphi = 0$ anebo φ je isomorfismus.

2/ V .. irred. complex representation of a Lie group/algebra, and $\varphi : V \to V$ an intertwining map. Then $\varphi = \lambda\, Id$ for some $\lambda \in \mathbb{C}$.

3/ V, W - irred. $\mathbb{C}$-representations of a Lie group/algebra, and $\varphi_1, \varphi_2 : V \to W$ non-zero intertw. maps. Then $\varphi_1 = \lambda \varphi_2$ for some $\lambda \in \mathbb{C}$.

Pf: Prove it in the group case, the Lie algebra case is analogous. As for 1/, if $v \in Ker(\varphi)$, then $\varphi(\Pi(A)v) = \Sigma(A)\varphi(v) \underset{\forall A \in G}{=} 0 \Rightarrow Ker(\varphi)$ is an invariant subspace of V. Since V is irred., $Ker(\varphi) = \{0\}$ or $Ker(\varphi) = V$, so φ is either injective or trivial. Assume φ is injective, so that $Im(\varphi)$ is a non-trivial subspace of W. Moreover, $Im(\varphi)$ is invariant: $w \in W$, $w = \varphi(v)$ for some $v \in V$, then $\Sigma(A)w = \Sigma(A)\varphi(v) = \varphi(\Pi(A)v)$. Since W is irred. and $Im(\varphi)$ is nonzero & invariant, $Im_\varphi(V) = W$. Consequently, φ is either zero or $\underbrace{\text{injective \& surjective}}_{\text{isomorphism}}$.

As for 2/, V is an irred. $\mathbb{C}$-representation, $\varphi: V \to V$ ($\varphi \in End(V)$) intertwining map over $\mathbb{C}$ $\Rightarrow$ φ has at least one eigenvalue $\lambda \in \mathbb{C}$. If $U \subseteq V$ is the eigenspace for φ, then by $\varphi\Pi(A) = \Pi(A)\varphi$ each $\Pi(A)$ maps U to itself $\Rightarrow$ U is an invariant subspace. Since λ is an eigenvalue, $U \neq 0$, we must have $U = V$, so $\varphi = \lambda Id$ on all of V.

The proof of 3/ is analogous. ▨

Corollary 10: Π a complex represent. of a Lie group G. If $A \in G$ is in its center, $A \in Z(G)$, then $\Pi(A) = \lambda Id$ for some $\lambda \in \mathbb{C}$. (The same statement for $\mathfrak{g}$ and its central elements.)

Pf: We prove it in the Lie group case. If $A \in Z(G)$, then $\forall B \in G$
$\Pi(A) \cdot \Pi(B) = \Pi(A \cdot B) = \Pi(B \cdot A) = \Pi(B) \cdot \Pi(A)$
$\Rightarrow$ $\Pi(A)$ is an intertwining map of Π to itself. By Theorem 9, 2/, $\Pi(A)$ is multiple of the identity. ▨

<u>Corollary 11</u>: An irreducible complex representation of commutative Lie group/algebra is of dimension one.

<u>Pf</u>: We shall prove it in the Lie group case. If G is commutative, the center of G is all of G, so by the previous Corollary 10 is $\Pi(A)$ is a multiple of the identity for $\forall A \in G$. This implies $\forall$ subspace of V is invariant, so the only way V does not have an invariant subspace (a non-trivial one!) is if it is 1-dimensional. ∎

Exercises 11

Example 1: We have for $su(2)_{\mathbb{C}} \simeq sl(2,\mathbb{C})$, given by

$$\underbrace{j_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ j_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ j_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}}_{su(2)} \rightarrow \underbrace{X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{sl(2,\mathbb{C})}$$

with $[j_1, j_2] = 2j_3$, $[j_2, j_3] = 2j_1$, $[j_3, j_1] = 2j_2$

$[X,Y] = H$, $[H,X] = 2H$, $[H,Y] = 2Y$

$X = \frac{1}{2}(j_2 - i j_1)$
$Y = \frac{1}{2}(-j_2 - i j_1)$
$H = i j_3$

and in the complexified representation
$\pi(X) = \frac{1}{2}(\pi(j_2) - i\pi(j_1))$,
$\pi(Y) = \frac{1}{2}(-\pi(j_2) - i\pi(j_1))$
$\pi(H) = i\,\pi(j_3)$

Check that in the case of $S^m((\mathbb{C}^2)^*)$ (the homog. pol. of degree m on $\mathbb{C}^2$ in two variables z_1, z_2) the representation is unitary. The orthogonal basis is $\{z_1^k z_2^{m-k}\}_{k=0}^m$), and in the Hermitean inner product on $\mathbb{C}$-vector space j_1, j_2, j_3 act by skew symmetric endomorphisms: $\langle \pi_m(j_i) v_1, v_2 \rangle = -\langle v_1, \pi_m(j_i) v_2 \rangle$, $i = 1,2,3$. This is equivalent to

$$\langle \pi_m(H) v_1, v_2 \rangle = \langle v_1, \pi_m(H) v_2 \rangle$$
$$\langle \pi_m(\underset{Y}{X}) v_1, v_2 \rangle = \langle v_1, \pi_m(\underset{X}{Y}) v_2 \rangle$$

$\forall\, v_1, v_2 \in V$, $\langle\ ,\ \rangle$
(note these are symmetric endomorphisms.)

which is elementary to transfer from the explicit formulas we had in the last lecture.

Example 2: Recall the finite-dim representations of $sl(2,\mathbb{C})$ $Sym^2(\mathbb{C}^2)$, $\mathbb{C}^2 \simeq Sym^1(\mathbb{C}^2)$ (both are irreducible), and compute their tensor product.

<u>Example 3</u>: Let $SO(2)$ act on $\mathbb{R}^2$ via fundamental vector representation, $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = v'$.

Show that $\mathbb{R}^2$ is a real irreducible represent., but Theorem 9, 3), fails (recall that $\mathbb{R}$ is not alg. closed field.)

<u>Example 4</u>: We would like to understand the following claim: $\forall m \in \mathbb{N}$ $\exists$ an irred. repr. (complex one) of $sl(2,\mathbb{C})$ of dimension $m+1$. Any two irred. repr. of the same dimension are isomorphic. If π is an irred. $\mathbb{C}$-repr. of $sl(2,\mathbb{C})$ with dimension $m+1$, is isomorphic to (π_m, V_m) discussed in Exercises 10.

<u>Lemma</u>: u ... eigenvector of $\pi(H)$, eigenvalue $\alpha \in \mathbb{C}$. Then
$$\pi(H)(\pi(X)u) = (\alpha+2)(\pi(X)u).$$
Thus, either $\pi(X)u = 0$ or $\pi(X)u$ is an eigenvector of $\pi(H)$. Similarly, $\pi(H)(\pi(Y)u) = (\alpha-2)\pi(Y)u$, with the same conclusions as before.

<u>Pf</u>: $[\pi(H), \pi(X)] = \pi([H,X]) = 2\pi(X)$, so
$$\pi(H)\pi(X)u = \pi(X)\pi(H)u + 2\pi(X)u = \pi(X)(\alpha u) + 2\pi(X)u = (\alpha+2)\pi(X)u. \quad \square$$

Proof of the main claim: $sl(2,\mathbb{C})$, π a fin. dim. repr. on V. Since $V = V/\mathbb{C}$, $\pi(H)$ has at least one eigenvector u: $\pi(H)u = \alpha u$. Previous Lemma $\Rightarrow$ $\pi(H)\pi(X)^k u = (\alpha + 2k)\pi(X)^k u$. Since $\pi(X)^k u$ are linearly independent and V is fin.-dim vector space $\Rightarrow$ $\pi(X)^k u = 0$ for some $k \gg 0$. Let $N \in \mathbb{N}$ such that $\pi(X)^N u \neq 0$ and $\pi(X)^{N+1} u = 0$. Set $u_0 := \pi(X)^N u$, $\lambda = \alpha + 2N$, such that $\pi(H)u_0 = \lambda u_0$, $\pi(X)u_0 = 0$. Define $u_k := \pi(Y)^k u_0$, $k \in \mathbb{N}$, $\pi(H)u_k = (\lambda - 2k)u_k$ by previous Lemma. It is easy to check by induction $\pi(X)u_k = k(\lambda - (k-1))u_{k-1}$, $k \in \mathbb{N}_{\geq 1}$. V is fin.-dim., $\pi(H)$ has finite spectrum $\Rightarrow$ $\exists m \in \mathbb{N}$: $u_k = \pi(Y)^k u_0 \neq 0$ $\forall k \leq m$

$u_{m+1} = \pi(Y)^{m+1} u_0 = 0$. Then $\pi(X) u_{m+1} = 0$; and so by previous formula
$$0 = \pi(X) u_{m+1} = (m+1)(\lambda - m) u_m .$$
Since $u_m \neq 0$ and $m+1 \in \mathbb{N}_{\geq 1}$, $\lambda - m = 0$ ($\lambda = m$). Summarizing, $\forall (\pi, V)$ (irred. fin. dim) $\exists\, m \in \mathbb{N}$ and $u_0, \dots, u_m \in V$:

$$(*)\quad \pi(H) u_k = (m-2k) u_k, \quad \pi(Y) u_k = \begin{cases} u_{k+1} & k < m \\ 0 & k = m \end{cases},$$
$$\pi(X) u_k = \begin{cases} k(m-(k-1)) u_{k-1} & k > 0 \\ 0 & k = 0 . \end{cases}$$

$u_0, \dots, u_m$ are lin. independent (eigen for $\pi(H)$ of different eigenval.) their lin. span is invariant under $\pi(H), \pi(X), \pi(Y)$, and since π is irreducible $\Rightarrow$ the space is V.

Conversely - the action of $sl(2,\mathbb{C})$ defined by $(*)$ on $(m+1)$-dim vector space gives (irreducible, as can be shown) represent. One can easily shown that $(*)$ and the repr. (π_m, V_m) discussed in the previous lecture are isomorphic.

What about the non-irreducible representations?

Theorem: (π, V) .... a fin.-dim repr. of $sl(2,\mathbb{C})$.

1/ $\forall$ eigenvalue of $\pi(H)$ is an integer; if v is an eigenvector for $\pi(H)$ with eigenvalue λ and $\pi(X) v = 0$, then $\lambda \in \mathbb{N}_+$.

2/ The operators $\pi(X), \pi(Y)$ are nilpotent.

3/ Define $S : V \to V$ by $S = e^{\pi(X)} e^{-\pi(Y)} e^{\pi(X)}$. Then $S \pi(H) S^{-1} = -\pi(H)$.

4/ If $k \in \operatorname{Spec}(\pi(H))$, then $-|k|, -|k|+2, \dots, |k|-2, |k|$ also belong to $\operatorname{Spec}(\pi(H))$.

Pf: We shall prove 3/, the other claims are easy to see. We have
$$e^{\pi(X)} \pi(H) e^{-\pi(X)} = \mathrm{Ad}_{e^{\pi(X)}}(\pi(H)) = e^{\mathrm{ad}(\pi(X))}(\pi(H)),$$
and similarly for the other products in the formula

$$S\pi(H)S^{-1} = e^{\pi(X)} e^{-\pi(Y)} e^{\pi(X)} \pi(H) e^{-\pi(X)} e^{\pi(Y)} e^{-\pi(X)}.$$

Now $\overset{[X,X]=0}{\mathrm{ad}(X)(X)=0}$, $\overset{[X,H]=-2X}{\mathrm{ad}(X)(H)=-2X}$, $\overset{[X,Y]=H}{\mathrm{ad}(X)(Y)=H}$, so

$$\underbrace{e^{\mathrm{ad}(\pi(X))}}_{\mathrm{Id} + \mathrm{ad}(\pi(X)) + \underbrace{\frac{1}{2}\mathrm{ad}(\pi(X))^2 + \dots}_{=0}} (\pi(H)) = \pi(H) - 2\pi(X),$$

while $\overset{[Y,X]=-H}{\mathrm{ad}(Y)(X)=-H}$, $\overset{[Y,H]=2Y}{\mathrm{ad}(Y)(H)=2Y}$, $\overset{[Y,Y]=0}{\mathrm{ad}(Y)(Y)=0}$, so

$$\begin{aligned} e^{-\mathrm{ad}(\pi(Y))}\big(\pi(H) - 2\pi(X)\big) &= \pi(H) - 2\pi(X) \\ &\quad - 2\pi(Y) - 2\pi(H) + \tfrac{1}{2}\,4\pi(Y) \\ &= -\pi(H) - 2\pi(X). \end{aligned}$$

Finally,

$$\begin{aligned} e^{\mathrm{ad}(\pi(X))}\big(-\pi(H) - 2\pi(X)\big) &= -\pi(H) - 2\pi(X) + 2\pi(X) \\ &= -\pi(H), \end{aligned}$$

which proves the claim. $\blacksquare$