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<u>Example 1:</u> (Invariants under simultaneous conjugation) Pairs of 2×2 matrices, $\operatorname{char}(K) \neq 2$. The invariants of pairs of 2×2 matrices $(A, B) \in M_2(K) \times M_2(K)$, under simultaneous conjugation, are generated by functions

$$\begin{aligned}
Tr_A &: (A, B) \longmapsto Tr(A) \\
Tr_{A^2} &: (A, B) \longmapsto Tr(A^2) \\
Tr_B &: (A, B) \longmapsto Tr(B) \\
Tr_{B^2} &: (A, B) \longmapsto Tr(B^2) \\
Tr_{A\cdot B} &: (A, B) \longmapsto Tr(A\cdot B).
\end{aligned}$$

Moreover, the five invariants are alg. independent.

Pf: The Jacobi criterion can be used to verify the last statement. It is also sufficient to consider the traceless matrices $M_2'(K)$, since $M_2(K) = K \oplus M_2'(K)$. There is a Zariski dense set $U \subseteq M_2'(K) \times M_2'(K)$, where $\forall$ pair $(A, B) \in U$ is equivalent to one of the form

$$\left(\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \begin{pmatrix} a & 1 \\ c & -a \end{pmatrix} \right), \quad t \neq 0, c \neq 0$$

and such a pair is equivalent to the pair with $-t$ and $-a$. Thus, an invariant function f restricted to these pairs depends only on t^2, a^2, at and c. But

$$t^2 = \tfrac{1}{2} Tr A^2, \quad a^2 + c = \tfrac{1}{2} Tr B^2, \quad at = \tfrac{1}{2} Tr A\cdot B$$

and so $$c = \tfrac{1}{2} Tr(B^2) - \frac{(Tr(AB))^2}{Tr(A^2)}$$

which implies that f can be written as a rational fun in $Tr(A^2)$, $Tr(B^2)$ and $Tr(AB)$. Since f is a polynom. fun on $M_2' \times M_2'$ and given invariants alg. independ.

$\Rightarrow$ f is a pol. f-ion in such invariants. $\boxtimes$

Invariants under simultaneous conjugation

Linear action of $GL(V)$ on $\mathrm{End}(V)^m := \underbrace{\mathrm{End}(V)\oplus\ldots\oplus\mathrm{End}(V)}_{m\times}$

by $g\,(A_1,\ldots,A_m) := (gA_1g^{-1},\ldots,gA_mg^{-1})$.

What are the invariants under the action?

$\forall$ finite sequence $i_1, i_2,\ldots,i_k$ $1\le i_\nu\le m$ we define a function

$$\mathrm{Tr}_{i_1\ldots i_k} : \mathrm{End}(V)^m \longrightarrow K,\ (A_1,\ldots,A_m)\longmapsto \mathrm{Tr}(A_{i_1}A_{i_2}\ldots A_{i_k})$$

These generalized traces are invariant f-ions on $\mathrm{End}(V)^m$:

(First fundamental theorem for matrices) If $\mathrm{char}(K)=0$, the ring of functions on $\mathrm{End}(V)^m$ invariant under simultaneous conjugation is generated by invariants $\mathrm{Tr}_{i_1,\ldots,i_k}$:

$$K[\mathrm{End}(V)^m]^{GL(V)} = K[\mathrm{Tr}_{i_1,\ldots,i_k} \mid k\in\mathbb{N},\ 1\le i_1,\ldots,i_k\le m].$$

The proof will be given later: for $\mathrm{char}(K)=0$, it is invariant to the FFT for $GL(V)$. Also, the traces $\mathrm{Tr}_{i_1,\ldots,i_k}$ of degree $k\le n^2$ already generate the ring of $GL(V)$-invariants.

Endomorphisms of tensors

We study m-fold tensor product $V^{\otimes m}$ as an $S_m\times GL(V)$-module. There is a close relationship between FFT, and obtain a relation between irrep. of GL_n and S_m, due to Schur.

V, $GL(V)\circlearrowright V$, $V^{\otimes m} := \underbrace{V\otimes\ldots\otimes V}_{m}$

$\dim_K V<\infty$ $\quad g(v_1\otimes\ldots\otimes v_m) := gv_1\otimes\ldots\otimes gv_m \quad g\in GL(V)$

The symmetric group S_m acts on $V^{\otimes m}$ by

$$\sigma(v_1 \otimes \cdots \otimes v_m) := v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}, \quad \sigma \in S_m.$$

It is clear that the actions of $GL(V)$ and S_m on $V^{\otimes m}$ do commute. We denote by $\langle GL(V) \rangle$ the linear subspace of $\mathrm{End}(V^{\otimes m}) = V^{\otimes m} \otimes V^{\otimes m *}$, spanned by elements of $GL(V)$, i.e. $\rho_{V^{\otimes m}}(GL(V)) \subseteq \mathrm{End}(V^{\otimes m})$. Similarly, we introduce the subspace $\langle S_m \rangle \subseteq \mathrm{End}(V^{\otimes m})$. Both are subalgebras and they centralize each other:

$$ab = ba \text{ for all } a \in \langle GL(V) \rangle \text{ and all } b \in \langle S_m \rangle$$

Def 2: W/K, $\dim_K W < \infty$, $A \subset \mathrm{End}(W)$ a subalgebra, the centralizer of A in $\mathrm{End}(W)$ (or, commutant) is the subalgebra of elements in $\mathrm{End}(W)$ commuting with A:

$$Z_A(\mathrm{End}\, W) \equiv A' := \{ b \in \mathrm{End}(W) \mid ab = ba \ \forall a \in A \}.$$

Equivalently, A' is the algebra of A-linear endomorph. of W (regarded as an A-module): $A' = \mathrm{End}_A(W)$.

Theorem 3: Consider the standard actions of $GL(V)$ and S_m on $V^{\otimes m}$, and $\langle GL(V) \rangle$, $\langle S_m \rangle$ the subalgebras of $\mathrm{End}(V^{\otimes m})$ spanned by linear operators from $GL(V)$ and S_m, respectively. Then

a/ $\mathrm{End}_{S_m}(V^{\otimes m}) = \langle GL(V) \rangle$,

b/ If $\mathrm{char}(K) = 0$, then $\mathrm{End}_{GL(V)}(V^{\otimes m}) = \langle S_m \rangle$.

We shall prove a/, and then the Double centralizer theorem from which follows b/. We shall also prove the result on

$V^{\otimes m}$ as an $S_m \times GL(V)$ - module.

Pf: a/ It is well known that there is an isomorphism

$\gamma : \mathrm{End}(V)^{\otimes m} \xrightarrow{\sim} \mathrm{End}(V^{\otimes m})$ given by

$\gamma(A_1 \otimes \dots \otimes A_m)(v_1 \otimes \dots \otimes v_m) = A_1 v_1 \otimes \dots \otimes A_m v_m.$

The representation $GL(V) \longrightarrow \mathrm{End}(V)^{\otimes m}$ is given by

$$g \longmapsto g \otimes \dots \otimes g,$$

and we claim that the action of S_m on $\mathrm{End}(V)^{\otimes m}$ is

$\sigma \in S_m$: $\quad \sigma(A_1 \otimes \dots \otimes A_m) = A_{\sigma^{-1}(1)} \otimes \dots \otimes A_{\sigma^{-1}(m)}.$

In fact, we have for any $v_1 \otimes \dots \otimes v_m \in V^{\otimes m}$

$\sigma(\gamma(A_1 \otimes \dots \otimes A_m)(\sigma^{-1}(v_1 \otimes \dots \otimes v_m))) =$

$\sigma(A_1 v_{\sigma(1)} \otimes \dots \otimes A_m v_{\sigma(m)}) =$

$A_{\sigma^{-1}(1)} v_1 \otimes \dots \otimes A_{\sigma^{-1}(m)} v_m =$

$\gamma(A_{\sigma^{-1}(1)} \otimes \dots \otimes A_{\sigma^{-1}(m)})(v_1 \otimes \dots \otimes v_m).$

This implies that γ induces an isomorphism

$$\begin{array}{ccc} \left(\mathrm{End}(V)^{\otimes m}\right)^{S_m} & \xrightarrow{\sim} & \mathrm{End}_{S_m}(V^{\otimes m}) \\ \cap \text{ symmetric tensors} & & \cap \\ \mathrm{End}(V)^{\otimes m} & & \mathrm{End}(V^{\otimes m}) \end{array}$$

The claim of Theorem follows from Lemma 4 applied to $X := GL(V) \subseteq W := \mathrm{End}(V)$ ▨

<u>Lemma 4</u>: $W/_K$, $\dim_K W < \infty$, $X \subseteq W$ Zariski dense subset.
Then the linear span of the tensors $x \otimes \dots \otimes x$, $x \in X$, is the linear span of all symmetric tensors, $\Sigma_m \subseteq W^{\otimes m}$ (subspace).

⑤ [Recall: $X \subseteq W$ is Zariski dense if $\forall$ fcion $f \in K[W]$ vanishing on X is the zero function]

Pf: $w_1, \ldots, w_n$ a basis of W, $\dim_K W = n$. Then $B := \{ w_{i_1} \otimes \ldots \otimes w_{i_m} \}_{i_1, \ldots, i_m}$ is a basis of $W^{\otimes m}$, stable for the action of S_m. The elements $w_{i_1} \otimes \ldots \otimes w_{i_m}$ and $w_{j_1} \otimes \ldots \otimes w_{j_m}$ belong to the same orbit of S_m iff $\forall\, w_i$ appears the same number of times in both expressions. In particular, $\forall$ orbit of S_m has a unique representative basis element of the form $w_1^{\otimes h_1} \otimes \ldots \otimes w_n^{\otimes h_n}$ with $\sum_{j=1}^{n} h_j = m$. Denote by $r_{h_1, \ldots, h_n} \in W^{\otimes m}$ the elements of linear span (subspace) of elements in this S_m-orbit, $\langle r_{h_1, \ldots, h_n} \mid \sum_{j=1}^{n} h_j = m \rangle$ is a basis of symmetric tensors $\Sigma_m \subseteq W^{\otimes m}$. Now the claim of lemma follows from: $\forall$ linear fcion $\lambda : \Sigma_m \to K$ vanishing on all $\overbrace{x \otimes \ldots \otimes x}^{m}$, $x \in X$, is the zero function. For $x = \sum_{j=1}^{n} x_j w_j$, $\overbrace{x \otimes \ldots \otimes x}^{m} = \sum_{\substack{h_1, \ldots, h_n \\ \sum h_i = m}} x_1^{h_1} \ldots x_n^{h_n} r_{h_1, \ldots, h_n}$,

and so $\lambda(x^{\otimes m}) = \sum_{\substack{h_1, \ldots, h_n \\ \sum h_i = m}} a_{h_1, \ldots, h_n} x_1^{h_1} \ldots x_n^{h_n}$ with $a_{h_1, \ldots, h_n} = \lambda(r_{h_1, \ldots, h_n}) \in K$.

This is a polynom. in $x_1, \ldots, x_n$, vanishing on X by our assumptions. Hence it is zero polynomial in $K[W]$, so all $a_{h_1, \ldots, h_n}$ are zero $\Rightarrow \lambda$ is zero function. ▨

Exercise 5: 1/ Let $F_d := K[x_1, \ldots, x_n]_d$ denote the vector space of homogeneous elements of degree d ($\deg(x_i) = 1$ $\forall\, i = 1, \ldots, n$). Assume $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) > d$. Show that F_d is linearly spanned by d-th degree of linear forms (elements, i.e. $\sum_{i=1}^{n} \alpha_i x_i$, $\alpha_i \in K$)

(6) 2/ Let $\rho : G \to GL(V)$ be an irreducible repr., assume $\mathrm{End}_G(V) = K$.

Denote $V^n := \underbrace{V \oplus \dots \oplus V}_{n}$ the G-module identified with $V \otimes K^n$.

Prove that

a/ $\mathrm{End}_G(V^n) = M_n(K)$ in a canonical way,

b/ $\forall$ G-submodule of V^n is of the form $V \otimes U$ for some $U \subseteq K^n$.

c/ If $\mu : H \to GL(W)$ is an irred. H-module/representation, $V \otimes W$ is a simple $G \times H$-module.

d/ $\langle G \rangle = \mathrm{End}(V)$, because $\langle G \rangle$ is a $G \times G$-submodule of $\mathrm{End}(V) = V^* \otimes V$.

e/ $V = V/K$, $V_L := V \otimes_K L$, L/K a field extension.
Then the representation of G on V_L is irreducible.
(use $\langle G \rangle_L = \mathrm{End}_L(V_L)$.)