Linear algebraic groups

$V = V/\mathbb{C}$ vector space, $\dim_{\mathbb{C}} V = n$ (fin. dimen.), $GL(V)$ = the (general) linear group of automorphisms of $(V, +, \mathbb{C})$.

Choice of basis $\Rightarrow$ $GL(V) \xrightarrow{\sim} GL(n, \mathbb{C})$ $n \times n$ invertible $\mathbb{C}$-valued matrices

The $\mathbb{C}$-algebra $End(V)$, $\mathbb{C}$-linear maps $V \to V$, is $\mathbb{C}$-vector space

Choice of basis $\Rightarrow$ $End(V) \xrightarrow{\sim} M_n(\mathbb{C})$ $n \times n$ ~~invertible~~ $\mathbb{C}$-valued matrices, $\mathbb{C}$-vector space

For $g \in M_n(\mathbb{C})$, $1 \le i, j \le n$, $x_{ij}(g)$ is the (i,j)-th entry of g

Def 1: a/ A subgroup $G \le GL(n, \mathbb{C})$ is a lin. alg. group if there exists a set A of polynomial functions on $M_n(\mathbb{C})$ s.t. $G = \{ g \in GL(n, \mathbb{C}) \mid f(g) = 0 \ \ \forall f \in A \}$.

A generates the defining ideal of G.

b/ $\mu : V \xrightarrow{\sim} \mathbb{C}^n$ for $V = V/\mathbb{C}$ given by the choice of basis in V, induces $\mu' : End(V) \xrightarrow{\sim} \underset{\substack{\parallel \\ M_n(\mathbb{C})}}{Mat(n, \mathbb{C})}$. A subgroup $G < GL(V)$ is linear alg. group if $\mu'(G)$ is a lin. alg. subgroup of $GL(n, \mathbb{C})$. This definition is basis-independent.

Remark: The Hilbert basis theorem $\Rightarrow$ $\forall$ linear. alg. group can be defined by a finite number of polyn. equations.

Example 2: a/ (the general linear group)

$$G = GL(n, \mathbb{C}), \ A = \emptyset, \text{ in general } GL(V).$$

b/ $G = SL(n, \mathbb{C})$, pol. equation $\det(g) - 1 = 0$, the special linear group. We have $SL(V) = GL(V)|_{\det(g) - 1}$, and $\det(g) - 1$ is independent of the choice of basis.

c/ $B(n, \mathbb{C}) < GL(n, \mathbb{C})$ the subgroup of upper-triangular matrices (Borel subgroup) defined by

$$A = \{ x_{ij}(g) \mid g \in GL(n, \mathbb{C}) \text{ and } \underset{i > j}{i, j \in \{1, \dots, n\}} \}$$

d/ The subgroup of diagonal matrices $D(n,\mathbb{C}) < GL(n,\mathbb{C})$ (example of Cartan subgroup), where

$$A = \{ x_{ij}(g) \mid g \in GL(n,\mathbb{C}) \text{ and } \underset{i \neq j}{i,j \in \{1,..,n\}} \}$$

e/ The subgroup of upper triangular matrices with diagonal entries equal to 1 (example of unipotent subgroup) $N^+(n,\mathbb{C}) < GL(n,\mathbb{C})$

$$A = \{ x_{ij}(g) \mid g \in GL(n,\mathbb{C}) \text{ and } \begin{array}{l} x_{ii}(g) - 1 \text{ for } i = 1,..,n \\ x_{ij}(g) \text{ for } j < i \end{array} \}.$$

In particular, for $n=2$ get $N^+(2,\mathbb{C}) \xrightarrow{\sim} \mathbb{C}$ as abelian lin. alg. subgroup.

f/ $\Gamma \in GL(n,\mathbb{C})$ defines a non-degenerate bilinear form B_Γ by

$$B_\Gamma(x,y) = x^T \Gamma y \quad \text{for } x,y \in \mathbb{C}^n.$$

Then the linear alg. group G_Γ is defined by

$$G_\Gamma = \{ g \in GL(n,\mathbb{C}) \mid B_\Gamma(gx,gy) = B_\Gamma(x,y) \text{ for all } x,y \in \mathbb{C}^n \}$$

or, $\quad G_\Gamma = \{ g \in GL(n,\mathbb{C}) \mid g^T \Gamma g = \Gamma \}.$

If B_Γ is symmetric, G_Γ is orthogonal lin. alg. grp.

— " — skew-symmetric, — " — symplectic — " — .

<u>Remark 3</u>: $g^T \Gamma g = \Gamma \Leftrightarrow (g^T)^{-1}\Gamma = \Gamma g \Leftrightarrow \Gamma^{-1}(g^T)^{-1}\Gamma = g$.

Denoting $\sigma_\Gamma(g) := \Gamma^{-1}(g^T)^{-1}\Gamma$, we get $g \in G_\Gamma$ iff $\sigma_\Gamma(g) = g$. This map has some interesting properties:

a/ σ_Γ is an automorphism of G_Γ: $\sigma_\Gamma(e) = e$,
$\sigma_\Gamma(g_1 g_2) = \sigma_\Gamma(g_1)\sigma_\Gamma(g_2)$

b/ If $\Gamma^T = \pm\Gamma$, then σ_Γ is involution: $(\sigma_\Gamma)^2 = Id$.

The ring of regular functions $\mathcal{O}(G)$ of lin. alg. group G

(3) **Def 4**: The ring of regular functions $\mathcal{O}(GL(n,\mathbb{C}))$ is an algebra over $\mathbb{C}$ generated by (matrix entries) $x_{ij}(g)$ and $\det^{-1}(g)$, $i,j \in \{1,\dots,n\}$:

$$\mathcal{O}(GL(n,\mathbb{C})) := \mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}, \det^{-1}]$$

$V/\mathbb{C}$, $\dim_{\mathbb{C}} V = n$, choose a basis, this induces isomorphism $\mu: GL(V) \xrightarrow{\sim} GL(n,\mathbb{C})$.
We define regular functions on $GL(V)$ to be of the form $f \circ \mu$, $f \in \mathcal{O}(GL(n,\mathbb{C}))$

$$\mathcal{O}(GL(V)) = \{ f \circ \mu \mid f \in \mathcal{O}(GL(n,\mathbb{C})) \}$$

(does not depend on the choice of basis, acting by automorphisms of coordinate ring.)

$G \leq GL(V)$, $f \in \mathcal{O}(G)$ is regular if it is a restriction of a regular function on $GL(V)$.

Remarks 5:

(a) The affine alg. set $GL(n,\mathbb{C}) = GL_n(\mathbb{C})$ is a principal open set (in $\mathbb{C}^{n^2}$) For X an affine alg. set, $f \in \mathcal{O}(X)$ (regular function on X), the principal open set $X_f := \{x \in X \mid f(x) \neq 0\}$.

X_f is isomorphic to an affine alg. set in higher dimension: for $G = GL_n(\mathbb{C})$, $f(g) = \det(g)$ and

$$G \simeq \{ (x,t) \in \underset{\substack{\parallel \\ Mat(n,\mathbb{C}) = M_n(\mathbb{C})}}{End(\mathbb{C}^n)} \times \mathbb{C} \mid t\det(x) - 1 = 0 \}.$$

(b) A way to define regular functions on $GL(V)$ without a choice of basis: for $B \in End(V)$, let $f_B : End(V) \to \mathbb{C}$ be defined by $f_B(Y) = tr(Y \cdot B)$. If $V = \mathbb{C}^n$, $B = E_{ij}$ elementary matrix, then $f_{E_{ij}}(Y) = x_{ji}(Y)$. Because $\underset{End(V) \to End(V)^*}{B \to f_B}$ is $\mathbb{C}$-linear, $f_B|_{GL(n,\mathbb{C})}$ is a linear combination of matrix-entry functions, and so regular. The algebra of regular functions on $GL(n,\mathbb{C})$ is generated by $\{f_B \mid B \in M_n(\mathbb{C})\}$ and $\det^{-1}$, which can reformulated as $\mathcal{O}(GL(V))$ is generated by $\{f_B, B \in \mathrm{End}(V)\}$ and $\det^{-1}$ (obviously, basis independent.)

(c) Analogously to step (b), there is basis-free definition of a linear algebraic group. For $V/_{\mathbb{C}}$, $G < GL(V)$ is linear alg. group if G is a closed subset of $GL(V)$ (in the Zariski topology.) This agrees with Def 1: the Zariski closed subset of $GL(V)$ are defined by $f(x_{11}(g), \dots, x_{nn}(g), \det^{-1}) = 0$, with f a polynomial of n^2+1 variables. Since $\det(g) \neq 0$, we can multiply f by $\det(g)^k$ for $k \gg 0$ to obtain polynomial equation in matrix entries of g.

The set of regular fuions $\mathcal{O}(G)$ on G is commutative $\mathbb{C}$-algebra. It has finite set of generators, f_B for $B \in \mathrm{End}(V)$ and $\det^{-1}|_G$.
"basis elements

Define the ideal

$$I_G := \{f \in \mathcal{O}(GL(V)) \mid f(G) = 0\}$$

in $\mathcal{O}(GL(V))$. The quotient map by I_G induces an isomorph.

$$\mathcal{O}(GL(V))/I_G \xrightarrow{\sim} \mathcal{O}(G).$$

<u>Examples 6:</u>

(a) $D(n, \mathbb{C}) \leq GL(n, \mathbb{C})$ the subgroup of diagonal matrices. The coordinate fuions x_{ij} are in $I_{D(n,\mathbb{C})}$ if $i \neq j$. The functions x_{ii} are algebr. independent and $\det^{-1}|_{D(n,\mathbb{C})} = \left(\prod_{i=1}^{n} x_{ii}\right)^{-1} \Rightarrow$
$\mathcal{O}(D(n,\mathbb{C})) = \mathbb{C}[x_{11}, x_{11}^{-1}, \dots, x_{nn}, x_{nn}^{-1}]$ and it is called algebr. torus of rank n.

(b) $\left.\begin{array}{l} G \leq GL(n,\mathbb{C}) \\ H \leq GL(m,\mathbb{C}) \end{array}\right\}$ lin. alg. groups $\Rightarrow G \times H \leq GL(n,\mathbb{C}) \times GL(m,\mathbb{C}) \leq GL(m+n, \mathbb{C})$
with $\mathcal{O}(G \times H) \simeq \mathcal{O}(G) \otimes \mathcal{O}(H)$

$$\mathcal{O}(G\times H) \ni \text{"} f' \times f'' \text{"} \longleftarrow f' \otimes f''$$

$$(g, h) \mapsto f'(g) f''(h)$$

Morphisms of linear alg. groups:

Def 7: $\varphi : G \to H$ morphism of linear alg. groups G, H fulfills (is given by):

1/ φ is a homomorphism of groups,

2/ φ is regular, i.e. $\varphi^*(f) := f \circ \varphi \in \mathcal{O}(G) \quad \forall f \in \mathcal{O}(H)$.

G, H lin. alg. groups are isomorphic if $\exists$ morphism $\varphi : G \to H$ s.t. φ^{-1} exists and is a morphism.

Def 8: G... lin. alg. group, $g \in G$. The left (right) G-translation L_g (R_g) on $\mathcal{O}(G)$ is defined by

$$\left.\begin{aligned}(L_g f)(x) &:= f(g^{-1}x)\\ (R_g f)(x) &:= f(xg)\end{aligned}\right\} f \in \mathcal{O}(G).$$

Lemma 9: G... lin. alg. group. Then

1/ The group mult. $\mu : G \times G \to G$ is a regular map,

2/ The inversion $\eta : G \to G$ — " —,

3/ If $f \in \mathcal{O}(G)$ $\exists p \in \mathbb{N}$ and $f_i', f_i'' \in \mathcal{O}(G)$, $i \in \{1, \dots, p\}$ s.t.

$$f(gh) = \sum_{i=1}^{p} f_i'(g) \otimes f_i''(h) \quad \text{for } \forall g, h \in G$$

$$\mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G) \underset{\text{Example 6, (b)}}{\xrightarrow{\simeq}} \mathcal{O}(G \times G) \xrightarrow{\mu : G \times G \to G} \mathcal{O}(G)$$

4/ For $g \in G$, the maps L_g, R_g are regular.

Pf: 1/ Follows from multipl. of matrices $x_{ik}(gh) = \sum_j x_{ij}(g) x_{jk}(h)$

and from the relation $\det(gh) = \det(g)\det(h)$

$$\begin{array}{c}\mathcal{O}(G \times G)\\ \downarrow \simeq\\ \mathcal{O}(G) \otimes \mathcal{O}(G)\end{array}$$

2/ A consequence of the Cramer's rule for the computation of $x_{ij}(g^{-1})$ in terms of $x_{kl}(g)$, $k, l \in \{1, \dots, n\}$.

3/ The formula in 3/ holds for $f = x_{ij}|_G$, and by the multiplicative property of determinant it holds for $f = \det^{-1}|_G$. Since these generate the algebra $\mathcal{O}(G)$, the identity holds for $\forall$ regular fcions.

4/ Follows from 3/, when g is fixed. ▨

<u>Lemma 10</u>: $H \leq G$ closed subgroup of a linear alg. group G.
I_H ... the ideal of function in $\mathcal{O}(G)$ vanishing on H.
Then $H = \{g \in G \mid R_g(I_H) \subseteq I_H\}$

<u>Pf</u>:
a/ If $g \in H$ and $f \in I_H$, then $R_g(f)(g') = f(g'g) = 0 \ \forall g' \in H$
$\Rightarrow R_g(f) \in I_H$, one inclusion is proved.

b/ If $R_g(I_H) \subset I_H$, then for $f \in I_H$ we get
$0 = R_g(f)(e) = f(g) \Rightarrow g \in H$. ▨

Representations of lin. algebraic groups

(we work with $\mathbb{C}$-representations)

<u>Def 11</u>: G .. lin. alg. gp. A representation of G is a pair (ρ, V), $V = V/\mathbb{C}$ and $\rho : G \to GL(V)$ is a group homomorphism.
V is <u>regular (rational)</u>, if $\dim V < \infty$ and the functions

$$G \longrightarrow \mathbb{C}$$
$$g \longmapsto \langle \rho(g)v, v^* \rangle \quad , \quad v \in V, \ v^* \in V^*$$
$$\quad \quad \quad \quad \quad \quad \quad \quad \ \ \parallel \ v^*(\rho(g)v)$$

(called matrix coefficients of ρ) are regular for all $v \in V$, $v^* \in V^*$ (equivalently, ρ is a morphism of linear alg. group.)

$$\underbrace{\{g \mapsto \langle \rho(g)v, v^* \rangle\}}_{\text{reg. fcion on } G} \longleftarrow \underbrace{\{\rho(g) \to \langle \rho(g)v, v^* \rangle\}}_{\text{reg. fcion on } GL(V)}$$

Def 12: V, $\dim_{\mathbb{C}} V = \infty$. The representation (ρ, V) of G is locally regular if for any fin. dim. $F \subseteq V$, $\dim_{\mathbb{C}} F < \infty$, there is a G-invariant fin. dim.

$$F \subseteq E \quad (\rho(g)E \subseteq E)_{\forall g \in G} \text{ s.t. } \rho|_E \text{ is a}$$

regular representation (see Def 11.)

Def 13: Let (ρ, V) and (τ, W) be two regular representations of G. A $\mathbb{C}$-linear map $T : V \to W$ is called an intertwining map if $T \circ \rho(g) = \tau(g) \circ T$ for all $g \in G$.

The map T is (called) isomorphism of (ρ, V) and (τ, W) if T is a bijection and an intertwining map.

The representations (ρ, V) and (τ, W) are isomorphic (equivalent) if there $\exists$ an isomorphism $(\rho, V) \xrightarrow{\sim} (\tau, W)$.

The representation (ρ, V) with $V \neq \{0\}$ is reducible, if there exists a G-invariant subspace $W \subseteq V$ s.t. $W \neq \{0\}$ and $W \neq V$. A representation, which is not reducible, is (called) irreducible.

Examples 14:

a/ Sub/factor representations: if (ρ, V) is a regular representation and $W \subset V$ is G-invariant, then $\tau := \rho|_W$ is a regular representation of G on W. We also obtain a representation τ of G on the quotient space V/W by $\tau(g)(v + W) = \rho(g)v + W$. A direct proof of the regularity of such represent. is clear.

(b) Defining representation: Let $G < GL(V)$ lin. alg. groups.
The representation $\rho(g) = g$ on V is regular, since the matrix coefficients $\langle g \cdot v, v^* \rangle = f_{v \otimes v^*}(g)$ are regular functions on G.

(c) Dual representation: (ρ, V) a regular representation. The dual (contragradient) represent. (ρ^*, V^*) is defined by

$$\langle \rho^*(g)(v^*), v \rangle = \langle v^*, \rho(g^{-1})(v) \rangle \qquad \begin{array}{l} v \in V \\ v \in V^* \end{array}$$

and by its very definition, (ρ^*, V^*) is regular repr.

$M^T \in \mathrm{End}(V^*)$ dual to $M \in \mathrm{End}(V) \Rightarrow \rho^*(g) = \rho(g^{-1})^T$.

If $W \subseteq V$ G-invariant subspace, then
$$V^* \supseteq W^\perp = \{ v^* \in V^* \mid \langle v^*, w \rangle = 0 \quad \forall\, w \in W \}$$
is a G-invariant subspace of V^*. In particular, (ρ, V) is irreducible $\Rightarrow$ (ρ^*, V^*) is also irred.
The canonical isom. $(V^*)^* \simeq V$ implies $(\rho^*)^* \simeq \rho$.

(d) Direct sum: (ρ, V), (σ, W) regular repr. of G. Define the direct sum representation $\rho \oplus \sigma$ on $V \oplus W$ by

$$(\rho \oplus \sigma)(g)\,(v, w) := (\rho(g)v, \sigma(g)w), \qquad \begin{array}{l} g \in G \\ v \in V \\ w \in W. \end{array}$$

Then $(\rho \oplus \sigma, V \oplus W)$ is a regular representation of G, since
$$\langle (\rho \oplus \sigma)(g)\,(v, w), (v^*, w^*) \rangle = \langle \rho(g)v, v^* \rangle + \langle \sigma(g)w, w^* \rangle$$
for $v \in V$, $v^* \in V^*$, $w \in W$, $w^* \in W^*$.

(e) Tensor product: (ρ, V), (σ, W) regular repr. of G, define the tensor product repr. $\rho\otimes\sigma$ on $V\otimes W$ by

$$(\rho\otimes\sigma)(g)[v\otimes w] = \rho(g)v \otimes \sigma(g)w \qquad \begin{matrix} g\in G \\ v\in V \\ w\in W \end{matrix}$$

which is regular since

$$\langle(\rho\otimes\sigma)(g)[v\otimes w], v^*\otimes w^*\rangle = \langle\rho(g)v, v^*\rangle\langle\sigma(g)w, w^*\rangle,$$

$v\in V, v^*\in V^*, w\in W, w^*\in W^*$.

(f) Representation induced on $\mathrm{End}(V)$:
$G < GL(V)$, (ρ, V) the defining representation. Consider the representation $\rho\otimes\rho^*$ on $V\otimes V^* \simeq \mathrm{End}(V)$, given by $g \mapsto (A \to g.A := gAg^{-1})$ for $A\in\mathrm{End}(V)$. The previous examples imply this is a regular represent.

<u>Lemma 15</u>: The representations $(L, \mathcal{O}(G))$ and $(R, \mathcal{O}(G))$ are locally regular.

Pf: As we already proved, $\forall f\in\mathcal{O}(G)$ there are $f_i', f_i''\in\mathcal{O}(G)$ s.t. $f(gh) = \sum_i f_i'(g) f_i''(h)$. Hence
$L(x)f = \sum_i f_i'(x^{-1}) f_i''$, $R(x)f = \sum_i f_i''(x) f_i'$.
It follows that the subspaces
$V_L(f) := \langle L(x)f \mid x\in G\rangle$, $V_R(f) := \langle R(x)f \mid x\in G\rangle$
are finite-dimensional and left/right invariant for G-translation.
This is easily generalized to a fin.-dim. $E\subseteq\mathcal{O}(G)$ with a basis $\{f_1, \dots, f_p\}$ $\Rightarrow$ $V_L = \sum_i V_L(f_i)$ resp. $V_R = \sum_i V_R(f_i)$ are regular representations. ∎

(Lie algebra of a linear algebraic group)

The situation for Lie groups: $\mathrm{Lie}(G) := T_e G$, $e \in G$ unit element

$C^\infty(G, TG)^G \subseteq C^\infty(G, TG) = \mathfrak{X}(G)$,

left-invariant vector field

$C^\infty(G, TG)^G \xrightarrow{\sim} T_e G$ as Lie algebras

$G < GL(m, \mathbb{R}) \Rightarrow \mathrm{Lie}(G) \leq \mathfrak{gl}(m, \mathbb{R})$

$X \in \mathfrak{gl}(m, \mathbb{R})$ is in $\mathrm{Lie}(G)$ iff $\exp(tX) \in G$ $\forall t \in \mathbb{R}$

On a smooth manifold: tangent vectors are defined by equivalence classes of smooth curves in it (passing through a point at which the tangent vector is considered.)

$X \in \mathfrak{X}(M)$: Leibnitz property $X(fg) = X(f)g + fX(g)$,

$\Rightarrow X \in \underbrace{\mathrm{Der}}_{\text{derivations}}(\mathcal{F}(M))$ algebra of smooth functions $f, g \in C^\infty(M)$,

We distinguish

Lie grps	lin. alg. grps
$G < GL(V)$	$\mathbb{G} < \mathbb{G}L(V)$
$\mathrm{Lie}(G)$	$\mathfrak{g}$

(it can be shown that $\mathrm{Lie}(G)$ and $\mathfrak{g}$ coincide a (Lie) subalgebras of $\mathrm{End}(V)$.)

<u>Def 16</u>: A derivation of an algebra A is a linear map $D: A \to A$ s.t. $D(a \cdot b) = D(a)b + aD(b)$. In the cas $\mathcal{O}(G) = A$, with G a linear algebr. group, a derivation of A is called a vector field on G. The space of all derivations on $\mathcal{O}(G)$ is denoted by $\mathfrak{X}(G)$.

<u>Def 17</u>: A derivation in $\mathfrak{X}(G)$ is left-invariant, if it commutes with left translations L_g, $g \in G$. The space of left-invariant derivations on $\mathcal{O}(G)$ is (called) the Lie algebra of G, denoted $\mathfrak{g}$, or $\mathfrak{X}_L(G)$.

$\mathfrak{X}(G)$ ... infty-dim. Lie algebra: D_1, D_2 vector fields $\Rightarrow [D_1, D_2]$ is a vector field

$$D_1(ab) = D_1(a)b + aD_1(b)$$
$$D_2(ab) = D_2(a)b + aD_2(b) \Rightarrow [D_1,D_2](ab) = [D_1,D_2](a)b + a[D_1,D_2](b)$$
for $[D_1,D_2] = D_1 \circ D_2 - D_2 \circ D_1$

Moreover, the bracket of invariant vector fields is invariant vector field $\Rightarrow$ the Lie algebra $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{X}(G)$.

<u>Def 18</u>: G .. a lin. alg. group. If $x \in G$, then a tangent vector to G at x is a $\mathbb{C}$-lin map $v: \mathcal{O}(G) \to \mathcal{O}(G)$ fulfilling
$$v(fg) = v(f)g(x) + f(x)v(g) \qquad \forall f, g \in \mathcal{O}(G)$$
The set of all tangent vectors at x is (called) tangent space $T_x G$ of G at x.

Questions: Is $\mathfrak{g}$ finite dimensional? If yes, what is its dimension?
If $G < H$ lin. alg. groups, subgroups of $\mathrm{End}(V)$, how $\mathfrak{h}$ and $\mathfrak{g}$ are related?

We show that $\mathfrak{g} \xrightarrow{\sim} T_e G$; there is $\theta: \mathfrak{g} \to T_e G$ given by evaluation at $e \in G$, and we prove θ is an isomorphism.

<u>Th 19</u>: G .. lin. alg. groups. The evaluation map $\theta: \mathfrak{g} \to T_e G$ defined by $\theta(X) = X(e)$, $X \in \mathfrak{g}$, is an isomorphism of $\mathbb{C}$-vector spaces.

<u>Pf</u>: We introduce the extension map $\eta: T_e G \to \mathfrak{g}$ by
$$(\eta(X)f)(g) = X(L_{g^{-1}}(f)), \quad X \in T_e G, f \in \mathcal{O}(G), g \in G$$

(12) We prove the series of claims:

a/ $\eta(X)$ is a derivation, i.e. $\eta(X) \in \mathcal{X}(G)$ ≡ vector fields on G

We have for $f, f' \in \mathcal{O}(G)$, $X \in T_eG$, $y \in G$

$$\eta(X)(ff')(y) = X(L_{y^{-1}}(ff')) = X(L_{y^{-1}}(f))f'(y) + X(L_{y^{-1}}(f'))f(y)$$
$$= [\eta(X)f]f'(y) + [\eta(X)f']f(y).$$

b/ $\eta(X) \in \mathfrak{g}$, i.e. $\eta(X)$ commutes with the left translations by G

We have for $f \in \mathcal{O}(G)$, $X \in T_eG$, $y \in G$, $x \in G$

$$[L_y(\eta(X)f)](x) = [\eta(X)f](y^{-1}x) = X(L_{x^{-1}y}f) = X(L_{x^{-1}}(L_yf))$$
$$= \eta(X)(L_y(f))(x).$$

c/ $\eta \circ \theta = Id_{\mathfrak{g}}$, in particular η is surjective

We have for $f \in \mathcal{O}(G)$, $X \in \mathfrak{g}$, $y \in G$

$$\underset{[(\eta\circ\theta)(X)]f}{\underset{\parallel}{\eta(\theta(X)f)(y)}} = \theta(X)(L_{y^{-1}}f) = X(L_{y^{-1}}f)(e) = L_{y^{-1}}(X(f))(e) =$$
$$= X(f)(y).$$

d/ $\theta \circ \eta = Id_{T_eG}$, in particular η is injective

We have for $X \in T_eG$, $f \in \mathcal{O}(G)$

$$\underset{(\theta\circ\eta)(X)(f)}{\underset{\parallel}{\theta(\eta(X))(f)}} = \eta(X)(f)(e) = X(L_{e^{-1}}(f)) = X(f).$$

▨

Particular description in the case $G = GL(n, \mathbb{C})$:

<u>Th 20</u>: Let $G = GL(n, \mathbb{C})$, $A \in M_n(\mathbb{C})$ and X_A a derivation of $\mathcal{O}(G)$ defined by $X_A(f)(g) = \frac{d}{dt}\Big|_{t=0} f(g(Id + tA))$, $g \in G$, $f \in \mathcal{O}(G)$, $t \in \mathbb{R}$.

a/ the vector field X_A is left-invariant,

b/ if $E_{ij} := X_{E_{ij}}$, $E_{ij} \in M_n(\mathbb{C})$ elementary matrix, then

$$X_A = \sum_{i,j} a_{ij} E_{ij} \quad \text{with } E_{ij} = \sum_{k=1}^{n} x_{ki} \frac{\partial}{\partial x_{kj}},$$

c/ $[X_A, X_B] = X_{[A,B]}$ for $A, B \in M_n(\mathbb{C})$, so the map

d/ $\begin{array}{ccc} M_n(\mathbb{C}) & \to & \mathfrak{g} \\ A & \mapsto & X_A \end{array}$ is complex Lie algebra homomorphism

Recall that given $B \in \mathrm{End}(V)$, we define a function $\mathrm{End}(V) \xrightarrow{f_B} \mathbb{C}$

$Y \mapsto f_B(Y) = tr_V(YB)$.

Then $(X_A f_B)(g) = f_{AB}(u)$

$\mathrm{End}(V) \xrightarrow{f_{AB}} \mathbb{C}$

$Y \mapsto f_{AB}(Y) = tr_V(YAB)$

e/ The map $\begin{array}{ccc} M_n(\mathbb{C}) & \to & \mathfrak{g} \\ A & \mapsto & X_A \end{array}$ is an isomorphism.

Pf: a/ Both sides are equal to

$$L_g(X_A(f))(u) = X_A(f)(g^{-1}u) = \frac{d}{dt}\Big|_{t=0} f(g^{-1}u(Id + tA)),$$

$$X_A(L_g(f))(u) = \frac{d}{dt}\Big|_{t=0} F(u(Id + tA)) = \frac{d}{dt}\Big|_{t=0} f(g^{-1}u(Id+tA)).$$

$$F(x) = f(g^{-1}x)$$

b/ The first formula follows from linearity in A, the second from the chain rule for differentiation and the relation

$$u e_{ij} = \sum_{k=1}^{n} x_{ki}(u) e_{kj}.$$

c/ The linearity of the map $A \mapsto X_A$, it is sufficient to consider the case $A = e_{ij}$, $B = e_{kl}$: $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}$ (true by the matrix multiplication.) Hence it is sufficient to show $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$:

$$\sum_{p,q} x_{pi} \frac{\partial}{\partial x_{pj}}(x_{qk}) \frac{\partial}{\partial x_{ql}} - \sum_{p,q} x_{qk} \frac{\partial}{\partial x_{ql}}(x_{pi}) \frac{\partial}{\partial x_{pj}} =$$

$$= \sum_{p,q} \delta_{jk} \delta_{pq} x_{pi} \frac{\partial}{\partial x_{ql}} - \sum_{p,q} \delta_{il} \delta_{pq} x_{qk} \frac{\partial}{\partial x_{pj}} = \delta_{jk} E_{il} - \delta_{il} E_{kj}.$$

d/ Let $A, B \in \mathrm{End}(V)$. Then

$$(X_A(f_B))(u) = \frac{d}{dt}\Big|_{t=0} f_B(u(Id+tA)) = \frac{d}{dt}\Big|_{t=0} tr(uB + tuAB) =$$
$$= tr(uAB) = f_{AB}(u).$$

e/ injectivity of $A \mapsto X_A$: if $X_A(f)=0$ $\forall f \in \mathcal{O}(G)$, then
$X_A(f_B)(\underset{\overset{\shortparallel}{e}}{I}) = f_{AB}(I) = tr(AB) = 0$ $\forall B \in End(V)$.
Because the bilinear form $\begin{matrix} End\,V \times End\,V \to \mathbb{C} \\ A \;,\; B \mapsto t(AB) \end{matrix}$ is non-degenerate,
$A=0$.
surjectivity of $A \mapsto X_A$: first, if $v \in T_e(GL(V))$ and $f \in \mathcal{O}(GL(V))$, then $v(f) = \sum_{i,j} \frac{\partial f}{\partial x_{ij}}(e)\, v(x_{ij})$, which follows from the fact that $v(1) = v(1\cdot 1) = 2v(1) \Rightarrow v(1)=0$, and v annihilates all terms in the Taylor expansion of f at e of order at least two (if $g, g' \in \mathcal{O}(G)$ both vanish at e, then $v(g\cdot g')=0$.)
Now assume $X \in \mathcal{X}_L(GL(V))$, we want to find $A \in End(V)$ s.t. $X = X_A$. Let us define the matrix $A = \{a_{ij}\}$ by $a_{ij} = X(e)(x_{ij})$. By definition of X_A,
$$X_A(e)(x_{ij}) = \frac{d}{dt}\Big|_{t=0} x_{ij}(I+tA) = a_{ij} = X(e)(x_{ij})$$
$\underset{\text{(by the initial claim)}}{\Rightarrow}$ $X(e) = X_A(e)$ $\underset{\text{TR.19}}{\Rightarrow}$ $X = X_A$. ∎

<u>Def 21</u>: G - lin. alg. grp. The subgroup $H < G$ is a closed subgroup if there is a system $\{f_\alpha\}_{\alpha \in A}$ of functions in $\mathcal{O}(G)$ such that
$$H = \{g \in G \mid f_\alpha(g) = 0 \quad \forall \alpha \in A\},$$
i.e. H is closed in G in the Zariski topology.

Recall that G is lin. alg. group iff there is $V/\mathbb{C}$ s.t. G is a closed subgroup of $GL(V)$. For G lin. alg. group, $H<G$ a closed subgroup $\Rightarrow$ H is lin. alg. group: $\mathcal{O}(H) = \{f = \tilde{f}|_H \mid \tilde{f} \in \mathcal{O}(G)\}$.

Th 22: $H < G$ closed subgroup of lin. alg. group G. Let us define the map $\alpha : T_eH \to T_eG$ by $\alpha(v)(\tilde{f}) = v(\tilde{f}|_H)$, $v \in T_eH$, $\tilde{f} \in \mathcal{O}(G)$.

Then a/ the map α is injective and $\alpha(T_eH) = \{w \in T_e(G) \mid \underset{\overset{\shortparallel}{0}}{w(I_H)}\}$

($\Rightarrow$ T_eH is identified with a vector subspace of T_eG)

b/ in the light of the isomorphism $\eta : T_eG \to \mathfrak{X}_L(G) \equiv \mathfrak{g}$,

$$\mathfrak{h} = T_eH = \{X \in T_eG \mid \eta(X)(I_H) \subset I_H\}.$$

Pf: a/ $i : H \to G$ embedding, $i^* : \mathcal{O}(G) \to \mathcal{O}(H)$

$$\tilde{f} \mapsto i^*(\tilde{f}) = \tilde{f} \circ i = \tilde{f}|_H, \quad \tilde{f} \in \mathcal{O}(G)$$

We have $I_H = \{\tilde{f} \in \mathcal{O}(G) \mid \tilde{f}|_H = 0\}$, $\mathcal{O}(H) = \mathcal{O}(G)/I_H$.
The map i^* is surjective, therefore α is injective. All elements $\alpha(v)$, $v \in T_eH$, annihilates by definition I_H, and vice versa: if $w \in T_eG$ annihilates I_H, it quotients to a functional on $\mathcal{O}(H)$ satisfying the Leibniz property ($\Rightarrow$ defines an element $v \in T_eH$ with $\alpha(v) = w$.)

b/ there are two inclusions:

- if $X \in T_eG$, $\eta(X)(I_H) \subseteq I_H$, then $\forall f \in I_H$ and $e \in H$:
$$0 = \eta(X)(f) = X(L_{e^{-1}} f) = X(f), \text{ so } X \in T_eH.$$
- we have $L_{x^{-1}}(f) \in I_H$ $\forall f \in I_H$, $x \in H$. Then if $X \in \mathfrak{h}$, we get for all $x \in H$
$$\eta(X)(f)(x) = X(L_{x^{-1}}(f)) = 0 \quad \Rightarrow \eta(X)(f) \in I_H. \ \blacksquare$$

In the case $H < G = GL(V)$, $End(V) \equiv$ Lie algebra of $GL(V)$, $X_A = \eta(A)$, $A \in End(V)$, we have:

Corollary 23: $G < GL(V)$ lin. alg. group. Then $\mathfrak{g} = \{A \in M_n(\mathbb{C}) \mid X_A(I_G) \subseteq I_G\}$.

 Description of tangent vectors at $e \in G$:

<u>Lemma 24</u>: G... lin. alg. grp, $\varphi: \mathbb{C} \to G$ a rational map given by $\mathbb{C} \to M_n(\mathbb{C})$ s.t. $\varphi(0) = Id$ and $\varphi(z) \in G$ $\forall z \in \mathbb{C}$ except possibly for a finite set of non-zero complex numbers. Then the matrix $A := \frac{d}{dz}\big|_{z=0} \varphi(z)$ belongs to $\mathfrak{g}$.

<u>The tangent map</u>: the transition from lin. alg. groups G to their Lie algebras $\mathfrak{g}$ reduces many problems to linear algebras.

<u>Def 25</u>: G, G' lin. alg. grps, $\varphi: G \to G'$ morphism, $\varphi^*: \mathcal{O}(G') \to \mathcal{O}(G)$ the map of their regular (coordinate) rings, $\varphi^*(f) = f \circ \varphi$.
Then the tangent map (differential of φ) $d\varphi: \mathfrak{g} \to \mathfrak{g}'$ is defined by

$$[(d\varphi)(X)](f') = X(\varphi^*(f')) \qquad \forall f' \in \mathcal{O}(G'),\ X \in \mathfrak{g}$$

(the range of $d\varphi$ is in $\mathfrak{g}'$.) $\qquad d\varphi(X) \in \mathfrak{g}'$

<u>Lemma 26</u>: The map $\varphi^*: \mathcal{O}(G') \to \mathcal{O}(G)$, $\varphi^*(f') = f' \circ \varphi$, commutes with the left translation:

$$L_{g^{-1}} \circ \varphi^* = \varphi^* \circ L_{\varphi^{-1}(g)} \qquad \forall g \in G.$$

<u>Pf</u>: $f' \in \mathcal{O}(G')$, evaluate both sides applied to f' at $g' \in G$:

LHS: $\quad L_{g^{-1}}(f' \circ \varphi)(g') = f'(\varphi(g \cdot g'))$,

RHS: $\quad L_{\varphi^{-1}(g)}(f')(\varphi(g')) = f'(\varphi(g)\varphi(g'))$. $\blacksquare$

<u>Th 27</u>: Let $\varphi: G \to H$ be a morphism of alg. grps. Then $d\varphi: \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of $\mathbb{C}$-Lie algebras. If K is a linear alg. grp., $\rho: H \to K$ another morphism, then $d(\rho \circ \varphi) = (d\rho) \circ (d\varphi)$. In particular, if $K = G$ and $\rho \circ \varphi$ is the identity map, then

then $d(\varphi \circ \psi) = Id_{\mathfrak{g}}$ $\Rightarrow$ isomorphic lin. alg. groups do have isomorphic Lie algebras.

Pf: $X, Y \in \mathfrak{g}$, $X' := d\varphi(X)$, $Y' := d\varphi(Y)$. For $f' \in \mathcal{O}(G')$,

$$[X', Y'](f') = [d\varphi(X), d\varphi(Y)](f') = d\varphi(X)\, d\varphi(Y)(f') -$$

$$- d\varphi(Y)\, d\varphi(X)(f') = d\varphi(X)(d\varphi(Y)f') - d\varphi(Y)(d\varphi(X)f')$$

$$= X(\varphi^*(d\varphi(Y)f')) - Y(\varphi^*(d\varphi(X)f')),$$

and also

$$(d\varphi[X,Y])(f') = [X,Y](f' \circ \varphi) = XY(f' \circ \varphi) - YX(f' \circ \varphi),$$

and the equality follows if

$$\varphi^*(d\varphi(Y)f') = Y(f' \circ \varphi) = Y(\varphi^*(f'))$$ (and similarly for $Y \Leftrightarrow X$.)

The evaluation at $g \in G$ gives

RHS: $Y(\varphi^*(f'))(g) = Y(L_{g^{-1}} \varphi^*(f'))$,

LHS: $\varphi^*(d\varphi(Y)f')(g) = d\varphi(Y)(f')(\varphi(g)) = d\varphi(Y)(L_{\varphi^{-1}(g)}(f'))$

$$= Y(\varphi^*(L_{\varphi^{-1}(g)}(f'))).$$

Now Lemma 26 implies equality of RHS and LHS. ∎

<u>Differential of a representation</u>: special example of a differential of a morphism is differential of representation.

$\pi : G \to GL(V)$ a repr. of lin. alg. grp. G
($\Rightarrow \pi$ is a morphism of lin. alg. grps.) Then $d\pi : \mathfrak{g} \to End(V)$ is a morphism of Lie algebras. A consequence of Th 27 is

<u>Th 28</u>: Let $\pi : G \to GL(V)$ a representation of G on V. Then for all $f' \in \mathcal{O}(GL(V))$, and all $A \in \mathfrak{g}$,

$$X_A(f'\circ\pi) = X_{d\pi(A)}(f')\circ\pi,$$

and the value $d\pi(A)$ is characterized by this relation uniquely.

Pf: The formula follows from $X(\varphi^*(f')) = \varphi^*(d\varphi(X))(f')$, $f'\in\mathcal{O}(GL(V$ discussed in the proof of Th 27.

Assume $\exists\, D\in End(V)$ such that $X_A(f'\circ\pi) = X_D(f')\circ\pi$ holds $\forall f'\in\mathcal{O}(GL(V))$. Then $\forall\, C\in End(V)$

$$X_A(f_C\circ\pi)(I) = (X_D f_C)\circ\pi(I) = f_{DC}(I) = tr_V(DC),$$

so the map $End(V)\to\mathbb{C}$ is a linear fuional

$$C\longmapsto X_A(f_C\circ\pi)(I)$$

on $End(V)$ represented by $tr_V(D\cdot ?)$. Because this is non-degenerate bilinear form on $End(V)$, D is determined uniquely. ■

Examples:

a/ sub-, quotient (factor)-representation: (ρ, V) regular repr. of G, $W\subset V$ G-invariant subspace, $\sigma = \rho|_W$ and τ is the factor-represent. on V/W. Then $d\rho$ restricts to $d\sigma$ on W, and the diff. $d\tau$ is the factor-repr. of $d\rho$ on V/W.

b/ Defining repr: $G\overset{\iota}{<}GL(n,\mathbb{C})$ lin. alg. gp., $(\iota,\mathbb{C}^n)$ def. repr. of G given by inclusion $\iota: G\to GL(n,\mathbb{C})$. We claim $d\iota(A) = A$ $\forall A\in End(\mathbb{C}^n)$. It is sufficient to verify the formula in Th 28 with $d\iota(A) = A$, $A\in\mathfrak{g}$. The formula is $X_A(f'\circ\iota) = X_A(f')\circ\iota$, and both sides evaluated at $g\in G$ are equal to $A(L_{g^{-1}}(f'))$.

(1) Dual representation: (π, V) regular repr., for $C \in End(V)$

c/ $C^T \in End(V^*)$ denotes dual map. The dual represent. (π^*, V^*) on V^* is given by $\pi^*(g) := \pi(g^{-1})^T$. As for $d\pi^*$, we have for $\forall C \in End(V^*)$

$$(f_C \circ \pi^*)(g) = tr_{V^*}(\pi(g^{-1})^T C) = tr_V(C^T \pi(g^{-1})) = (f_{C^T} \circ \pi)(g^{-1}).$$

Then for $A \in \mathfrak{g}$, we have

$$X_A(f_C \circ \pi^*)(I) = \frac{d}{dz}\Big|_{z=0} (f_{C^T} \circ \pi)((I+zA)^{-1}) = -X_A(f_{C^T} \circ \pi)(I).$$

Therefore, we get

$$tr_{V^*}(d\pi^*(A)C) = -tr_V(d\pi(A)C^T) = -tr_{V^*}(d\pi(A)^T C)$$

$\Rightarrow$ because it holds $\forall C \in End(V^*)$, we conclude
$d\pi^*(A) = -(d\pi(A))^T$ for $A \in \mathfrak{g}$.

d/ Direct sum: $(\rho, V), (\sigma, W)$ regular represent. of G, their direct sum $\pi := \rho \oplus \sigma$ on $U := V \oplus W$ is defined by

$$(\rho \oplus \sigma)(g)(v, w) = (\rho(g)v, \sigma(g)w), \quad g \in G, v \in V, w \in W.$$

Then $d\pi(X) = d\rho(X) \oplus d\sigma(X)$, $X \in \mathfrak{g}$.

e/ Tensor product: $(\pi_1, V_1), (\pi_2, V_2)$ regular repr. of G, $\pi := \pi_1 \otimes \pi_2$ be their tensor product on $V := V_1 \otimes V_2$. Then

$$d(\pi_1 \otimes \pi_2)(X) = d\pi(X) = d\pi_1(X) \otimes I + I \otimes d\pi_2(X).$$

To prove this: $End(V_1 \otimes V_2) \cong End(V_1) \otimes End(V_2)$, so $d\pi(A)$, $A \in \mathfrak{g}$, is determined by the action of the vector field X_A on functions

$$(f_{C_1 \otimes C_2} \circ \pi)(g) = tr_{V_1 \otimes V_2}(\pi_1(g)C_1 \otimes \pi_2(g)C_2) =$$
$$= (f_{C_1} \circ \pi_1)(g)(f_{C_2} \circ \pi_2)(g), \quad C_i \in End(V_i)$$

for $i = 1, 2$.

Since X_A is a derivation,

$$X_A(f_{C_1\otimes C_2}\circ \pi)(g) = X_A\big((f_{C_1}\circ\pi_1)(g)\cdot(f_{C_2}\circ\pi_2)(g)\big) =$$

$$\big(X_A(f_{C_1}\circ\pi_1)\big)(g)(f_{C_2}\circ\pi_2)(g) + (f_{C_1}\circ\pi_1)(g)\big(X_A(f_{C_2}\circ\pi_2)\big)(g),$$

and its evaluation at $g=I$ gives the claim.

f/ Representation induced on $\mathrm{End}(V)$: $G<GL(V)$ lin. alg. grp., (ρ, V) its defining repr., consider the repr. $\sigma := \rho\otimes\rho^*$ on $V\otimes V^* \simeq \mathrm{End}(V)$. The previous examples give

$$d\sigma(A) = d\rho(A)\otimes I - I\otimes d\rho(A)^T,$$

and $T: V\otimes V^* \xrightarrow{\sim} \mathrm{End}(V)$, $T(v\otimes v^*)(u) = v^*(u)v$ $\forall u\in V$, translates σ to $(\tau, \mathrm{End}(V))$ by $\tau(g)(Y) = \rho(g)Y\rho(g^{-1})$, $Y\in\mathrm{End}(V)$, and $(d\tau, \mathrm{End}(V))$ is given by

$$d\tau(A)(Y) = d\rho(A)Y - Yd\rho(A), \quad A\in\mathfrak{g}.$$

g/ The adjoint repr.: $G<GL(n,\mathbb{C})$ lin. alg. grp., the repr. of $GL(n,\mathbb{C})$ on $\mathrm{End}(\mathbb{C}^n)$ by $A\mapsto gAg^{-1}$, is regular. The restriction to G is regular repr. of G.

<u>Lemma 29</u>: $A\in\mathfrak{g}$, $g\in G$. Then $gAg^{-1}\in\mathfrak{g}$.

<u>Pf</u>: The right regular repr. of $GL(n,\mathbb{C})$ is $(R_g f)(g') = f(g'g)$ for $g, g'\in GL(n,\mathbb{C})$, $f\in\mathcal{O}(GL(n,\mathbb{C}))$. For $A\in M_n(\mathbb{C})$,

$$\big((R_g X_A R_{g^{-1}})f\big)(g') = \big(X_A R_{g^{-1}} f\big)(g'g) = \frac{d}{dt}\Big|_{t=0}(R_{g^{-1}}f)(g'g(I+tA))$$

$$= \frac{d}{dt}\Big|_{t=0} f(g'(I+tgAg^{-1})) = (X_{gAg^{-1}}f)(g').$$

Assuming $A\in\mathfrak{g}$, $g\in G$, $f\in I_G$, we have $R_{g^{-1}}f\in I_G$, so $X_A R_{g^{-1}}f = 0$. By previous calculation, $X_{gAg^{-1}}f = 0$, which proves $gAg^{-1} \in \mathfrak{g}$. ▨

We define $Ad(g)A = gAg^{-1}$, $g \in G$ and $A \in \mathfrak{g}$.
Then $(Ad, \mathfrak{g})$, $Ad(g) : \mathfrak{g} \to \mathfrak{g}$, is called the adjoint representation of G. For $A, B \in \mathfrak{g}$,
$Ad(g)([A, B]) = [Ad(g)(A), Ad(g)(B)]$, so that $Ad(g)$ acts by Lie algebra automorphisms, $Ad : G \to Aut(\mathfrak{g})$.

<u>Lemma 30</u>: The differential $(Ad, \mathfrak{g})$ is the representation

$$ad : \mathfrak{g} \longrightarrow End(\mathfrak{g}), \text{ given by}$$

$$ad(A)(B) = [A, B], \quad A, B \in \mathfrak{g},$$

and $ad(A)$ is a derivation of $\mathfrak{g}$, $ad(\mathfrak{g}) \subset Der(\mathfrak{g})$.

<u>Pf</u>: This is special case of the point f/, Examples. ▨

The additive Lie group $\mathbb{R}$ and the connected multiplicative group $\mathbb{R}_{>0} = \mathbb{R}_+$ of pos. real numbers are isomorphic under the map $x \to \exp(x)$. In the theory of alg. groups, the additive group $\mathbb{C}$ and the multiplicative group $\mathbb{C}^*$ are not isomorphic.

Rational representations of $\mathbb{C}$, exponential, nilpotent and unipotent matrices

We can extend the domain of exp to $\mathbb{C}$, to get a bijection we have to restrict to a domain in $\mathbb{C}$, e.g. $\{z \in \mathbb{C} \mid |z-1| < 1\}$. In general, the exp map on $M_n(\mathbb{C})$ maps the space of nilpotent matrices biholomorphically to the space of unipotent matrices.

$\mathbb{C}$ ... additive alg. group, this structure comes from the embedding $\mathbb{C} \xhookrightarrow{\varphi} SL(2,\mathbb{C})$, $z \longmapsto \varphi(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = I + zE_{12}$. The regular functions on $\mathbb{C}$ are the polynomials in z, the Lie algebra of $\mathbb{C}$ is given by E_{12}, $(E_{12})^2 = 0$. Thus $\varphi(z) = \exp(zE_{12})$, and we determine all regular represent. of $\mathbb{C}$.

Recall: Lie grp vs Lie alg.
G Lie(G)

$\varphi : \mathbb{R} \to G$ 1-param. sbgp., for $G \leq GL(n,\mathbb{R})$ as the expon. map $\varphi_A(t) = \exp(tA)$, $A \in \mathrm{Lie}(G) \subset \mathrm{Mat}_n(\mathbb{R})$
$t \in \mathbb{R}$

The same works over $\mathbb{C}$, if we restrict the domain of holom. map $z \xrightarrow{\varphi} \varphi(z) = \exp(zA)$, $z \in \mathbb{C}$, to the subspace of nilpotent complex matrices in $M_n(\mathbb{C})$.

<u>Def 31</u>: A matrix $A \in M_n(\mathbb{C})$ is nilpotent if $A^k = 0$ for some $k \in \mathbb{N}$. A linear map $U \in M_n(\mathbb{C})$ is (called) unipotent if $U - \mathrm{Id}$ is nilpotent.

For $A \in M_n(\mathbb{C})$ nilpotent, i.e. $A^k = 0$ for some $k \in \mathbb{N}$, we define

$$\exp(A) = \sum_{j=0}^{k-1} \frac{1}{j!} A^j = \mathrm{Id} + B, \quad \text{with}$$

$B = A + \frac{1}{2!}A^2 + \dots + \frac{1}{(k-1)!}A^{k-1}$ a nilpotent matrix. This follows from $B = A\underbrace{\left(\mathrm{Id} + \frac{1}{2!}A + \dots + \frac{1}{(k-1)!}A^{k-2}\right)}_{B'}$, such that $AB' = B'A$ and A being nilpotent. Hence $\exp(A)$ is unipotent.

Conversely, if $U = \mathrm{Id} + B$ is unipotent, B is nilpotent. We define $\log U := \sum_{j=1}^{k-1} \frac{(-1)^{j+1}}{j!} B^j$ for $B^k = 0$.

For A nilpotent and $z \in \mathbb{C}$, zA is nilpotent. The function $z \to \varphi(z) = \log(\exp(zA))$ is a polynomial in z s.t. $\varphi(0) = 0$ and $\left.\frac{d\varphi}{dz}\right|_{z=0} = A$. Hence we have, by the substitution principle, that exponential function is a bijective polynomial map from the nilpotent elements in $M_n(\mathbb{C})$ onto unipotent elements in $GL(n, \mathbb{C})$ (with inverse $U \mapsto \log U$), i.e. $\log(\exp(zA)) = zA$ or $\exp(\log(\mathrm{Id} + A)) = \mathrm{Id} + A$.

Subst. principle: $\forall$ equation involving power series in a complex var. z that holds as an identity of absol. conv. series when $|z| < r$ also holds as an identity of matrix power series in a matrix variable X, and the series converge absolutely in the matrix norm when $\|X\| < r$.

<u>Lemma 32</u>: (Taylor's formula) Assume $A \in M_n(\mathbb{C})$ is nilpotent, and $f \in \mathcal{O}(GL(n,\mathbb{C}))$. Then there is $k \in \mathbb{N}$ s.t. $(X_A)^k f = 0$, and
$$f(\exp(A)) = \sum_{j=0}^{k-1} \frac{1}{j!} \left[(X_A)^j f\right](I).$$

Pf: We know $\det(\exp(zA)) = 1$, hence $\forall f \in \mathcal{O}(GL(n,\mathbb{C}))$, the function $z \xrightarrow{\varphi} \varphi(z) = f(\exp(zA))$ is a polynomial. Hence $\exists k \in \mathbb{N}$ s.t. $\left(\frac{d}{dz}\right)^k \varphi = 0$, so it is sufficient to evaluate $\varphi(1)$ by means of the Taylor expansion at point 0: $\left[\left(\frac{d}{dz}\right)^j \varphi\right](0) = (X_A^j f)(I)$. ∎

<u>Theorem 33</u>: $G < GL(n,\mathbb{C})$ lin. alg. group, $\mathfrak{g}$ its Lie algebra.

a/ If $A \in M_n(\mathbb{C})$ is nilpotent, then $A \in \mathfrak{g} \Leftrightarrow \exp(A) \in G$.

b/ Assume $A \in \mathfrak{g}$ is nilpotent matrix, (ρ, V) a regular representation of G. Then $d\rho(A)$ is a nilpotent transformation on V, and $\rho(\exp(A)) = \exp(d\rho(A))$.

Pf: a/ For $f \in I_G$ and $A \in \mathfrak{g}$, $X_A^m f \in I_G$ $\forall m \in \mathbb{N}_0$. By Taylor's formula $f(\exp(A)) = 0$, thus $\exp(A) \in G$.

Conversely, if $\exp A \in G$, then the function $z \xrightarrow{\varphi} \varphi(z) = f(\exp(zA))$ on $\mathbb{C}$ vanishes when $z \in \mathbb{Z}$, so it must vanish for all $z \in \mathbb{C}$ since it is a polynomial. Hence $(X_A f)(I) = 0$ $\forall f \in I_G$, and by the left G-invariance of $X_A \Rightarrow X_A f(g) = 0$ $\forall g \in G$ $\Rightarrow A \in \mathfrak{g}$.

b/ Apply Taylor's formula to the fin. dim. space of regular functions f^ρ_B, defined for the regular repr. (ρ, V) by by $f^\rho_B(g) = tr_V(\rho(g)B)$, for $B \in End(V)$. There is a positive integer k s.t.

$$0 = X_A^k f^\rho_B(I) = tr_V(d\rho(A)^k B) \quad \forall B \in End(V).$$

By non-degeneracy of tr_V, $d\rho(A)^k = 0$. The Taylor's formula applied to f^ρ_B results in

$$tr_V(B\rho(exp(A))) = \sum_{m=0}^{k-1} \frac{1}{m!} X_A^m f^\rho_B(I) = \sum_{m=0}^{k-1} \frac{1}{m!} tr_V(d\rho(A)^m B)$$

$$= tr_V(B\, exp(d\rho(A))) \qquad \forall B \in End(V). \quad ▨$$

<u>Corollary 34</u>: If (π, V) is a regular representation of the additive group $\mathbb{C}$. Then there exists a unique nilpotent $A \in End(V)$ s.t. $\pi(z) = exp(zA)$ $\forall z \in \mathbb{C}$.

<u>Def 35</u>: A lin. alg. group is connected if the ring of regular functions $\mathcal{O}(G)$ has no zero divisors.

(Recall: $a \in R$ is (left) zero divisor if the map $a: R \to R$, $r \mapsto a \cdot r$ is not injective.)

<u>Examples</u>:

a/ $\mathbb{C}$ and $\mathbb{C}^*$ are connected alg. groups, $\mathcal{O}(\mathbb{C}) = \mathbb{C}[t]$, $\mathcal{O}(\mathbb{C}^*) = \mathbb{C}[t, t^{-1}]$.

b/ G, H connected alg. grps, then $G \times H$ is connected as well.

c/ Assume G is connected lin. alg. group, $\rho: G \to H$ is a surjective homomorph. of alg. groups. Then $\rho^*: \mathcal{O}(H) \to \mathcal{O}(G)$ is injective and H is connected.

(26) <u>Theorem 35</u>: Assume that lin. alg. group G is generated by unipotent elements. Then G is connected both as a lin. alg. group and as a Lie group. In particular, the classical Lie groups $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $SO(n, \mathbb{C})$ and $Sp(n, \mathbb{C})$ are connected for all $n \geq 1$ both as algebraic groups and Lie groups.