Representation theory of GL_n and S_m, characters

We already learned the notion of rational representation for linear algebraic group, their complete reducibility and many other properties. We shall start by an exercise:

Exercise 1 : Let us consider the embedding $GL_n(K) \times GL_m(K) \hookrightarrow GL_{m+n}(K)$,

and prove $\qquad A, B \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

a/ $GL_n(K) \times GL_m(K)$ is Zariski closed in $GL_{m+n}(K)$,

b/ There is canonical isomorphism $K[GL_n] \otimes_K K[GL_m] \xrightarrow{\sim} K[GL_n \times GL_m]$

$$f \otimes g \to \big((A,B) \mapsto f(A) g(B)\big)$$

(start from the subspace $M_n \times M_m \subset M_{m+n}$)

c/ For $G \subset GL_n(K)$, $H \subset GL_m(K)$ subgroups, $G \times H \subset GL_{m+n}(K)$ via $G \times H \hookrightarrow GL_n(K) \times GL_m(K) \hookrightarrow GL_{m+n}(K)$. Then b/ induces an isomorphism $K[G] \otimes_K K[H] \xrightarrow{\sim} K[G \times H]$

(b/ $\Rightarrow$ the map exists and is unique; choose a basis (over K) $\{f_i\}_{i \in I}$ of $K[G]$ and assume the function $\sum_i f_i \otimes h_i$ is identically $= 0$ on $G \times H$. Then it follows $h_i(B) = 0$ $\forall B \in H$.)

For $G < GL(V)$ lin. alg. group, if the represent. of G on $V^{\otimes m}$ is completely reducible for $\forall m \in \mathbb{N}$, then $\forall$ rational representation of G is completely reducible. E.g., $\forall$ rational represent. of $T_n < GL_n(K)$ (diagonal matrices = Cartan subgroup) is completely reducible.

Frobenius reciprocity (Motivated by the structure of induced modules for the pair of finite groups (group and its subgroup).)

Def 2 : Let $H \leq G \leq GL_n(K)$ linear algebr. groups, W a rational H-module. Then the induced G-module

$\mathrm{Ind}_H^G(W) := \{\eta : G \to W \text{ regular} \mid \eta(gh^{-1}) = h\,\eta(g)\ \forall h \in H\}$

$G \subseteq \mathrm{End}(V)$ closed, $W \subseteq w$ closed $\Rightarrow$ regular map is a restr. of polyn. map $\mathrm{End}(V) \to W$

$\mathrm{Ind}_H^G(W) = \mathrm{Mor}_H(G, W)$

or, $\mathrm{Ind}_H^G(W) = [K[G] \otimes W]^H$, $h \cdot (f \otimes w) := f^h \otimes hw$, $h \in H, g \in G$ $f^h(g) := f(g \cdot h)$

($\Rightarrow$ $\mathrm{Ind}_H^G(W)$ is a loc. finite, rational G-mod.)

__Theorem 3__: Let $H < G < GL_n(K)$ linear alg. groups, V a rational G-module and W a rational H-module. There is canonical isomorphism

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) \xrightarrow{\sim} \mathrm{Hom}_H(V|_H, W)$$

given by $\varphi \longmapsto e_W \circ \varphi$ with $e_W : \mathrm{Ind}_H^G(W) \to W$, $\alpha \mapsto \alpha(e)$, $e \in G$.

__Pf__: The map $\varphi \mapsto e_W \circ \varphi$ is well-defined and linear. Its inverse is: for $\psi : V|_H \to W$ a H-linear map and $v \in V$, we define $\varphi_v : G \to W$ by $\varphi_v(g) := \psi(g^{-1}v)$. The map φ_v is clearly H-equivariant, and so $\varphi_v \in \mathrm{Ind}_H^G(W)$. Another elementary calculation shows that $\varphi : V \to \mathrm{Ind}_H^G(W)$, $v \mapsto \varphi_v$ is G-equivariant linear map, and $\psi \mapsto \varphi$ is the inverse map to $\varphi \longmapsto e_W \circ \varphi$. ∎

__Unipotent subgroup of $GL_n(K)$, fixed vectors and highest weights__

For $i \neq j$, $s \in K$, define $u_{ij}(s) := E + sE_{ij} \in GL_n(K)$

(E: unit matrix; E_{ij}: elementary matrix)

We have $u_{ij}(s) \cdot u_{ij}(s') = u_{ij}(s+s')$, hence $U_{ij} := \{u_{ij}(s) \mid s \in K\}$ is 1-param. subgroup of $GL_n(K)$,

isomorphic to additive group $(K, +, 0)$: $U_{ij} \xrightarrow{\sim} K^+$.
$\underset{K^+}{\overset{\parallel}{}}$

Moreover, U_{ij} is normalized by T_n (Cartan subgroup of $GL(V)$):

$$t\, u_{ij}(s)\, t^{-1} = u_{ij}(t_i t_j^{-1} s) \text{ for all } t = \begin{pmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_n \end{pmatrix} \in T_n,\ s \in K.$$

It is well-known that the elements $u_{ij}(s)$, $i<j$ and $s \in K$, generate the subgroup U_n of upper triangular matrices which are unipotent

$$U_n := \left\{ \begin{pmatrix} 1 & * & \cdots & * \\ & 1 & & \vdots \\ & & \ddots & * \\ 0 & & & 1 \end{pmatrix} \right\} = \langle u_{ij}(s) \mid i<j,\ s \in K \rangle.$$

$\parallel$
standard unipotent subgroup of $GL_n(K)$.

<u>Lemma 4</u>: Let λ be a weight of W, $w \in W_\lambda$ a weight vector. Then there are elements $w_k \in W_{\lambda + k(\epsilon_i - \epsilon_j)}$, $k \in \mathbb{N}$, with $w_0 = w$ such that

$$u_{ij}(s)\, w = \sum_{k \geq 0} s^k w_k, \quad s \in K.$$

<u>Pf</u>: The map $\varphi: K \to W$, $s \mapsto u_{ij}(s) w$ is a polynomial map, because W is a rational GL_n-module and $\det u_{ij}(s) = 1$. Therefore, $\varphi(s) = \sum_{k \geq 0} s^k . w_k$ for suitable $w_k \in W$. For $t \in T_n$, we get

$$t \cdot \varphi(s) = t u_{ij}(s) w = (t u_{ij}(s) t^{-1})(t w) = u_{ij}(t_i t_j^{-1} s)(t w)$$
$$= u_{ij}(t_i t_j^{-1} s)(\lambda(t) . w) = \sum_{k \geq 0} \lambda(t) (t_i t_j^{-1} s)^k . w_k \quad \forall s \in K.$$

Thus $t \cdot w_k = \lambda(t) (t_i t_j^{-1})^k . w_k \ \forall k \geq 0$, so $w_k \in W_{\lambda + k(\epsilon_i - \epsilon_j)}$. ∎

Analogously, let W be a non-trivial $GL_n(K)$-module. Then $W^{U_n} \neq 0$, and for $w \in W^{U_n}$ a weight vector of weight λ and $W' := \langle GL_n w \rangle < W$ ~~the~~ $GL_n(K)$-submodule generated by w, the weight space W'_λ is equal to Kw and the other weights of W' are all $\prec \lambda$ (in the order on $\mathfrak{h}^*$.)

① **Exercise 5:** $A = \{a_{ij}\}_{i,j=1}^{n} \in M_n(K)$ the determinants of submatrices $A_r = \{a_{ij}\}_{i,j=1}^{r}$, $\det A_r$, are called the principal minors. Show that $U_n^- T_n U_n$ ($U_n^- \equiv U_n^T$) is the set of matrices in $M_n(K)$ whose principal minors are all $\neq 0$. In particular, $U_n^- T_n U_n$ is Zariski dense in $M_n(K)$. $U_n^- T_n U_n \equiv$ open cell of GL_n.

For $\mathrm{char}(K) = 0$, $GL_n(K)$-module W, then W is simple iff $\dim W^{U_n} = 1$. In this case W^{U_n} is a 1-dimensional weight space W_λ and all other weights of W are $< \lambda$.

Exercise 6: Assume $\mathrm{char}(K) = 2$, $V = K^2$ and $G = GL_2(K)$ acting on $W = S^2V = \mathrm{Sym}^2 V$ (second symmetric power of V.)

a/ $W^{U_2} = Ke_1^2$, $\langle GL_2(K)e_1^2 \rangle = Ke_1^2 \oplus Ke_2^2$ is isomorphic to V,

b/ $(W^*)^{U_2} = K x_1x_2 \oplus K x_2^2$, $K x_1 x_2$ is the determinant represent. and $\langle GL_2(K) x_2^2 \rangle = W^*$.

For $\mathrm{char}(K) = 0$, if W is simple $GL_n(K)$-module then $\mathrm{End}_{GL_n}(W) = K$.

Example 7: $\mathrm{char}(K) = 0$, $V = K^n$

a/ $GL_n(K)$-modules $\Lambda^j V$, $j = 1, \ldots, n$ ($V = K^n$) are simple with hw $\epsilon_1 + \cdots + \epsilon_j$. The $GL_n(K)$-modules $S^k V$, $k \in \mathbb{N}$, are simple with highest weight $k\epsilon_1$. ($(\Lambda^j V)^{U_n} = K(e_1 \wedge e_2 \wedge \cdots \wedge e_j)$, $(S^k V)^{U_n} = Ke_1^k$.)

b/ If W is a simple $GL_n(K)$-module of hw $\lambda = p_1\epsilon_1 + \cdots + p_n\epsilon_n$, then the dual module W^* is simple of hw $\lambda^* := -p_n\epsilon_1 - p_{n-1}\epsilon_2 - \cdots - p_1\epsilon_n$ ($p_n\epsilon_1 + \cdots + p_1\epsilon_n = \sigma_0 \cdot \lambda$ is the lowest weight of W, $\sigma_0 \in S_n$: $i \xrightarrow{\sigma_0} n+1-i$, and the weights of W^* are $\{-\mu \mid \mu \text{ a weight of } W\}$.

Exercise 8: Show that the vectorspace $M_n' := \{A \in M_n(K) \mid Tr(A) = 0\}$ form a simple module over $GL_n(K)$ w.r.to $(g, A) \mapsto g \cdot A \cdot g^{-1}$, of highest weight $\epsilon_1 - \epsilon_n$.

Two simple GL_n-modules are isomorphic if and only if they have the same highest weight. For example, this allows explicit characterization of $S_m \times GL_n$-module structure of $V^{\otimes m}$, $m \in \mathbb{N}$:

Proposition 9: Let $V = K^n$. The $S_m \times GL_n$-module $V^{\otimes m}$ admits an isotypic decomposition of the form

$$V^{\otimes m} = \bigoplus_{\lambda} V_\lambda(n) = \bigoplus_{\lambda} M_\lambda \otimes L_\lambda(n), \quad \text{where}$$

$L_\lambda(n)$ is a GL_n-module of hw λ, M_λ is a simple S_m-module and λ runs through the set $\{\sum p_i \epsilon_i \mid p_1 \geq p_2 \geq \dots \geq p_n \geq 0, \sum p_i = m\}$.

Irreducible characters of $GL(V)$ and S_m: $\underset{char(K)=0}{}$ $\left\{\begin{array}{l}\text{Characters of irred. polyn. repr. of } GL_n \\ \qquad\qquad \text{Schur polynomials}\end{array}\right\}$ $\Updownarrow$ $\{$Characters of symmetric groups

T_n .. Cartan subgroup of $GL_n(K)$ (subgroup of diagon. matrices)

$\rho : GL_n(K) \to GL(W)$ rational repr.

$$\chi_\rho : (x_1, \dots, x_n) \longmapsto Tr\left(\rho\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}\right), \quad \text{character of } \rho \;\; \chi_W$$

Lemma 10: Let ρ, ρ' be rational repr. of GL_n.

a/ $\chi_\rho \in \mathbb{Z}[x_1, x_1^{-1}, \dots]$, and $\chi_\rho \in \mathbb{Z}[x_1, \dots, x_n]$ for ρ polynomial.

b/ χ_ρ is a symmetric function.

c/ If ρ, ρ' are equivalent repr. of GL_n, then $\chi_\rho = \chi_{\rho'}$.

Pf: a/ and c/ follow from definition. As for b/: this follows from the action of S_n on T_n by conjugation. In fact, the group S_n of permutation matrices in GL_n normalizes Cartan

⊂ subgroup (torus T_n):

$$\sigma^{-1}\begin{pmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_n \end{pmatrix}\sigma = \begin{pmatrix} t_{\sigma(1)} & & & 0 \\ & t_{\sigma(2)} & & \\ & & \ddots & \\ 0 & & & t_{\sigma(n)} \end{pmatrix}$$

$\Rightarrow$ S_n acts on the character group $\mathcal{X}(T_n)$ defined by $\sigma(\chi(t)) := \chi(\sigma^{-1}t\sigma)$, $t \in T_n$, $\chi \in \mathcal{X}(T_n)$ ($\Leftrightarrow \sigma(\epsilon_i) = \epsilon_{\sigma(i)}$ $\forall i = 1, \dots, n$)

For a rational represent. $\rho: GL_n \to GL(W)$ the linear map $\rho(\sigma)$ induces an isomorphism $W_\chi \xrightarrow{\sim} W_{\sigma(\chi)}$ ($\Rightarrow$ the weights of W are invariant under the action of S_n.) ∎

As a consequence of character theory, χ_W determines the weights and their multiplicities of (rational repr. W), so that two representations with the same character are equivalent.

<u>Proposition 11</u>: a/ W_1, W_2 rational repr. of GL_n. Then $\chi_{W_1 \oplus W_2} = \chi_{W_1} + \chi_{W_2}$ and $\chi_{W_1 \otimes W_2} = \chi_{W_1} \cdot \chi_{W_2}$.

b/ If W is an irred. repr., polynomial of degree m. Then χ_W is a homogen. pol. of degree m.

c/ The character of W^* is $\chi_{W^*}(x_1, \dots, x_n) = \chi_W(x_1^{-1}, \dots, x_n^{-1})$.

<u>Examples 12</u>: $V = K^n$, $\chi_{V^{\otimes m}} = (x_1 + \dots + x_n)^m$,

$$\chi_{S^2V} = \sum_{i \le j} x_i x_j, \quad \chi_{\Lambda^2 V} = \sum_{i<j} x_i x_j, \quad \chi_{det} = x_1 x_2 \dots x_n,$$

$$\chi_{V^*} = x_1^{-1} + \dots + x_n^{-1}, \quad \chi_{\Lambda^{n-1}V} = \sum_{i=1}^{n} x_1 \dots \hat{x}_i \dots x_n = (x_1 \dots x_n)(x_1^{-1} + \dots + x_n^{-1}).$$

S^jV: χ_{S^jV} are complete symm. pol.: $h_j(x_1, \dots, x_n) = \sum_{i_1 \le i_2 \le \dots} x_{i_1} x_{i_2} \dots x_{i_j}$ $= \chi_{S^jV}$,

generating function of $\{\chi_{S^jV}\}_{j \in \mathbb{N}}$ is

$$\prod_{i=1}^{n} \frac{1}{1 - x_i t} = \sum_{j=0}^{\infty} h_j t^j,$$ $h_j \equiv$ special examples of Schur polynom.