(1) (The first fundamental theorem for general linear group)

We describe a minimal system of generators of invariant on p-copies of the vector space and q-copies of the dual vector (V and V^*, respectively.) The invariants are for the natural action of $GL(V)$. Analogously for $GL(V)$ action on $\mathrm{End}(V)$

Invariants of vectors and covectors

V, $\dim_K V < \infty$, representation of $GL(V)$ on

$$W := \underbrace{V \oplus \dots \oplus V}_{p \text{ times}} \oplus \underbrace{V^* \oplus \dots \oplus V^*}_{q \text{ times}} =: V^p \oplus V^{*q}$$

given by $g(v_1, \dots, v_p, \varphi_1, \dots, \varphi_q) := (gv_1, \dots, gv_p, g\varphi_1, \dots, g\varphi_q)$

$(g\varphi_i)(v) := \varphi_i(g^{-1}v)$ contragradient repr. dual to $GL(V) \circlearrowright V$.

$\forall (i,j)$, $\left.\begin{matrix} i = 1, \dots, p \\ j = 1, \dots, q \end{matrix}\right\}$ bilinear fcion $(i|j)$ on $V^p \oplus V^{*q}$ by

$$(i,j) : (v_1, \dots, v_p, \varphi_1, \dots, \varphi_q) \longmapsto (v_i, \varphi_j) := \varphi_j(v_i)$$

These functions are called contractions, and are $GL(V)$-invariant:

$$(i,j)(g(v,\varphi)) = (g\cdot\varphi_j)(gv_i) = \varphi_j(g^{-1}gv_i) = \varphi_j(v_i) = (i|j)(v,\varphi).$$

First fundamental theorem (FFT): functions (i,j) generate the ring of invariants. The proof will be given later.

FFT for $GL(V)$: The ring of invariants for the action of $GL(V)$ on $V^p \oplus V^{*q}$ is generated by the invariants $(i|j)$:

$$K[V^p \oplus V^{*q}]^{GL(V)} = K[(i|j) \mid i = 1, \dots, p,\ j = 1, \dots, q]$$

In coordinates: fix a basis in V and its dual basis of V^*, write $v_i \in V$ as a column vector and $\varphi_j \in V^*$ as a row vector:

$$v_i = \begin{pmatrix} x_{1i} \\ \vdots \\ x_{ni} \end{pmatrix}, \quad \varphi_j = (\varphi_{j1}, \dots, \varphi_{jn}).$$

Then $X := (v_1, \dots, v_p)$ is $n \times p$ - matrix, $Y := \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_q \end{pmatrix}$ is $q \times n$ matrix,

so we obtain canonical map $V^p \oplus V^{*q} \xrightarrow{\sim} M_{n\times k}(K) \oplus M_{q\times n}(K)$.

The action of $g \in GL(n, K)$ on matrices is given by $g \cdot (X, Y) = (g \cdot X, Yg^{-1})$.

The linear map $\Psi : M_{n\times p} \times M_{q\times n} \to M_{q\times p}$

$$(X, Y) \longmapsto YX$$

has (i,j)-component

$$\Psi_{ij}(X, Y) = \sum_{\nu=1}^{n} \varphi_{i\nu} v_{\nu j} = (v_j \mid \varphi_i), \text{ i.e. } \Psi_{ij} = (j \mid i).$$

The map Ψ is $GL(V)$-invariant, hence constant on $GL(V)$-orbits:

$$\Psi(g(X, Y)) = \Psi(gX, Yg^{-1}) = Yg^{-1}gX = YX = \Psi(X, Y)$$

$\Rightarrow \Psi_{ij} = (i \mid j)$ is an invariant.

Geometric interpretation: (geometric formulation of the FFT using the language of algebraic geometry)

The image of the map Ψ is the subset $V^n_{q\times p} \subseteq M_{q\times p}$ of matrices of rank $\leq n$ (follows from the fact that $\operatorname{rank}(A\cdot B) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$ for any pair of matrices A, B.) The set $V^n_{q\times p}$ is even a closed subvariety, i.e. it is the zero set of a family of polynomials. The FFT says that the map $\Psi : M_{n\times p} \times M_{q\times n} \to V^n_{q\times p}$ is universal in the sense that $\forall\ \Phi : M_{n\times p} \times M_{q\times n} \to Z$ into an affine variety Z which is constant on orbits factors through Ψ, i.e. $\exists$ unique morphism $\overline{\Phi} : V^n_{q\times p} \to Z$ s.t. $\Phi = \overline{\Phi} \circ \Psi$.

$$\begin{tikzcd} M_{n\times p} \times M_{q\times n} \arrow[r, "\psi"] \arrow[d, "\Phi"'] & V^n_{q\times p} \arrow[d, "\Phi"] \\ Z \arrow[r, equal] & Z \end{tikzcd}$$

so that $V^n_{p\times q}$ is an algebraic analogue to the orbitspace of the action denoted $(M_{n\times p}\times M_{q\times n})/GL_n$; we say that $\psi: M_{n\times p}\times M_{q\times n} \to V^n_{q\times p}$ is an algebraic quotient w.r. to the action of GL_n, use the notation $V^n_{q\times p} = (M_{n\times p}\times M_{q\times n})/\!/GL_n$

By construction, the quotient map induces an isomorphism

$$\psi^*: K[V^n_{q\times p}] \xrightarrow{\sim} K[M_{n\times p}\times M_{q\times n}]^{GL_n}$$

$$\nwarrow \; f|_{V^n_{q\times p}}, \; f\in K[M_{q\times p}].$$

$V^n_{q\times p}\subset M_{q\times p}$ deteminantal (sub) variety, vanishing of all $(n+1)\times(n+1)$ minors, it is normal variety (its coordinate ring is integrally closed in its field of fractions.)

<u>Invariants of conjugacy classes</u>

$GL(V)$ acts on $End(V)$ by $\underset{GL(V)}{\overset{\cap}{g}} \mapsto (\underset{End(V)}{\overset{\cap}{v}} \to g^{-1}vg)$

orbits = conjugacy classes of matrices

$A\in End(V)$, $P_A(t) = \det(tE-A) = t^n + \sum_{i=1}^n (-1)^i s_i(A)t^{n-i}$, $n=\dim V$, $E\in End(V)$ the identity. The functions $s_i(A)$, are polynomial functions on $End(V)$. It is well-known that $s_i(A)$ is the i-th elementary symmetric function of the eigenvalues of A. The choice of basis of V leads to $End(V)\xrightarrow{\sim} M_n(K)$, $s_i|_D$ for diagonal matrices $D\subseteq M_n(K)$ are exactly the elem. sym. fctions σ_i on $D = K^n$. We know these are alg. independ. and generate the algebra of symm. fctions.

<u>Proposition 1</u>: The ring of invariants for the conjugation action of $GL(V)$ on $End(V)$ is generated by $s_1, \dots, s_n$:

$$K[End(V)]^{GL(V)} = K[s_1, s_2, \dots, s_n].$$

Moreover, $\{s_1, \dots, s_n\}$ are algebr. independent.

<u>Pf</u>: Define
$$S := \left\{ \begin{pmatrix} 0 & \cdots & \cdots & & a_n \\ 1 & 0 & \cdots & & a_{n-1} \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0\ 1 & 0 & a_2 \\ 0 & \cdots & & 0\ 1 & a_1 \end{pmatrix} \;\middle|\; a_1, \dots, a_n \in K \right\} \subseteq M_n(K),$$

and let $X := \{ A \in M_n(K) \mid A \text{ is conjugated to a matrix in } S \}$.
We claim that X is Zariski-dense in $M_n(K)$. In fact, the matrix $A \in X$ iff $\exists\, v \in K^n$: $v, Av, \dots, A^{n-1}v$ are linearly independent. Consider the polynomial function h on $M_n(K) \times K^n$ given by $h(A, v) := \det(v, Av, A^2v, \dots, A^{n-1}v)$.

By lemma in the section on invariant functions is the subset

$$Y := \{ (A, v) \mid v, Av, \dots, A^{n-1}v \text{ linearly independent} \}$$
$$= \left(M_n(K) \times K^n\right)_h$$

is Zariski-dense in $M_n(K) \times K^n$. Its projection onto $M_n(K)$ is X, which is then Zariski-dense, too. This implies that $\forall$ inv. function f on $M_n(K)$ is completely determined by its restriction to S.

Elementary calculation: $A \equiv A(a_1, \dots, a_n) \in S$, its charact.

polyn. is given by $P_A(t) = t^n - \sum_{i=1}^{n} a_i t^{n-1}$. Now $f(A) = q(a_1, \dots, a_n)$ with a polynomial q in n-variables, and we have $a_j = (-1)^{j+1} s_j(A)$. Hence the function

$$f - q(s_1, -s_2, s_3, \dots, (-1)^{n+1} s_n)$$

is invariant and vanishes on S $\Rightarrow f = q(s_1, -s_2, \dots) \in K[s_1, \dots, s_n]$. ∎

Exercise 2: The set of diagonalizable matrices is Zariski-dense in $M_n(K)$, in particular an invariant function on $M_n(K)$ is completely determined by its restriction to the diagonal matrices.
(K alg. closed, this is implied by the Jordan decomposition)

Traces of powers: there is another well-known series of invariant functions on End(V), namely the traces of powers of an endomorphisms:

$$Tr_k : End(V) \to K \qquad A \mapsto Tr(A^k), \; k \in \mathbb{N}$$

There are recursive formulas $Tr_k \leftrightarrow s_i$

$$Tr_k = (-1)^{k+1} k\, s_k + f_k(s_1, \dots, s_{k-1}), \qquad k \leq n$$

for certain degree k functions f_k ($k \leq n$.) The relations between Tr_k's and s_j's as those which hold for (power sums) $p_k(x) := \sum_{i=1}^{n} x_i^k$ and the elementary symmetric functions σ_j.
Hence if $char(K) > n$, s_j can be expressed in terms of the Tr_k, $k = 1, 2, \dots, n$, so we get

<u>Corrolary 3</u>: If $char(K)=0$, then $Tr_1, Tr_2, \dots, Tr_n$ generate the invariant ring $K[End(V)]^{GL(V)}$.

(does not hold for $0 < char(K) \leq n$)

<u>Exercise 4</u>: Show there is a relation for functions Tr_k, s_i:

$$(-1)^{j+1} j s_j = Tr_j - s_1 Tr_{j-1} + s_2 Tr_{j-2} - \dots + (-1)^{j-1} s_{j-1} Tr_1,$$

for all $j = 1, \dots, n$.