Covariants

V, W - fin. dim K-vector spaces, $\varphi: W \to V$ is morphism (polynomial map) if the coordinate functions of φ w.r. to some basis of V are pol. fcions on W ($f \mapsto \varphi^* f$; basis independent.)

E.g. $W \to \underbrace{W \otimes \dots \otimes W}_{r \text{ times}}$, $w \mapsto w \otimes \dots \otimes w$, or

$W \to S^r(W)$, r-th symmetric power of W, $w \mapsto w^r$.

$\varphi: W \to V$, $\varphi^*(f) = f \circ \varphi \in \underset{\| K[W]}{\mathcal{O}[W]}$ for $f \in \underset{\| K[V]}{\mathcal{O}[V]}$, and $\varphi^*: K[V] \to K[W]$ algebra homomorphism, which determines φ: in the basis $\{e_1, \dots, e_n\}$ we have $V \xrightarrow{\sim} K^n \Rightarrow$ $K[V] \xrightarrow{\sim} K[y_1, \dots, y_n]$, $\varphi \equiv (\varphi_1, \dots, \varphi_i \dots)$ of $\varphi: W \to K^n$ is $\varphi_i = \varphi^*(y_i)$.

<u>Exercises 1</u>: a/ $\varphi \mapsto \varphi^*$ defines a bijection between set of morphisms $W \to V$ and K-alg. homom. $K[V] \to K[W]$. If φ^* is surjective, then φ is injective; φ^* is injective iff $\varphi(W)$ is Zariski dense in V.

b/ Every morphism φ can be written uniquely as a sum of homog. components of φ (i.e., $\varphi^*(V^*) \subseteq K[W]_d$; $\varphi: W \to V$ is d-homogeneous if $\varphi(\lambda w) = \lambda^d \varphi(w)$ $\forall w \in W$, $\lambda \in K$.)

c/ $K[SL(2,K)]$ the algebra of functions $f|_{SL(2,K)}$, $f \in K[M_2(K)]$. Show that the kernel of restriction map $K[M_2] \to K[SL(2)]$ is the ideal generated by det:

$$K[SL(2)] = K[\bar{a}, \bar{b}, \bar{c}, \bar{d}] \xleftarrow{\sim} K[a,b,c,d]/(ad - bc - 1).$$

(2) W, V - G-modules, $\varphi: W \to V$ morphism satisfying $\varphi(g\cdot w) = g\cdot\varphi(w)$ $\forall w\in W, g\in G$ is equivariant morphism.

Def 2: W, V - G-modules, a covariant of W of type V is a G-equivariant morphism (polyn. map) $\varphi: W \to V$.

Example 3: $\left.\begin{array}{l} W \to W\otimes W\otimes\dots\otimes W \\ W \to S^r(W) \end{array}\right\}$ covariants w.r. to $GL(W)$

Number of constructions with covariants:

- $\varphi: W\to V$ covariant, $f\in K[V]^G$ an invariant $\Rightarrow f\circ\varphi\in K[W]^G$
- V_1, V_2 G-modules, $p: V_1\otimes V_2 \to U$ projection on some G-module U
  $\left.\begin{array}{l}\varphi_1 \text{ covariant of type } V_1 \\ \varphi_2 \quad \text{———}\shortparallel\text{———} \quad V_2\end{array}\right. \Rightarrow (\varphi_1,\varphi_2)_U : W \xrightarrow{(\varphi_1,\varphi_2)} V_1\times V_2 \to V_1\otimes V_2 \xrightarrow{p} U$
  is covariant of type U

  E.g. $\mathrm{char}(K)\neq 2$, standard repr. of $SL(2)$ on $V=K^2$. Then $V\otimes V \cong S^2V\oplus K$, the projection $p: V\otimes V\to K$ is $(x_1,x_2)\otimes(y_1,y_2)\mapsto x_1y_2 - y_1x_2$, so for φ,ψ covariants of type V we get $(\varphi,\psi)_K = \varphi_1\psi_2 - \varphi_2\psi_1$ $(\varphi\equiv(\varphi_1,\varphi_2),\ \psi\equiv(\psi_1,\psi_2))$ covariant of type K.

In general, covariants of a fixed type form a module over the ring of invariants.

- For any $GL(n)$-module W two covariants of type M_n can be multiplied via (linear projection = matrix multiplication) $p: M_n\otimes M_n \to M_n$, so the covariants of type M_n form a (non-commutative) algebra over the ring of invariants.

$\varphi: W\to V$ covariant of type V, $\varphi^*|_{V^*}: V^* \to K[W]$, $\lambda\mapsto\lambda\circ\varphi$ is G-homomorphism, and φ^* (hence, φ) is completely determined by this linear map $\varphi^*|_{V^*}$.

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Prop. 4: Let W, V be G-modules. The covariants of W of type V are in bijection with the G-hom. $V^* \to K[W]$.

Pf: Alg. hom. $K[V] \to K[W]$ are in bijective correspondence with K-linear maps $V^* \to K[W]$. Thus morphisms $W \to V$ and K-linear maps $V^* \to K[W]$ are bijective, which induces a bijection between the subset of G-equivariant morphisms and the subset of G-homomorphisms. ▨

Exercise 5: a/ For $G = GL(W)$ or $G = SL(W)$. The only covariants of W of type W are the scalar multiples of Id, $t \cdot Id : W \to W$, $t \in K$.

b/ W a representation of $SL(2)$, $U \leq SL(2)$ the subgroup of upper triangular unipotent matrices. There is an isomorphism

$$K[W \oplus K^2]^{SL(2)} \xrightarrow{\sim} K[W]^U$$
$$f \longmapsto \bar{f}, \quad \bar{f}(w) = f(w, (1,0)).$$

Classical invariant theory

Problem: Describe generators + relations for the ring of invariants $K[W]^G$.

First fundamental th. (FFT) specifies generators,
Second —— " —— (SFT) —"— relations.

Exercise 6: $\mathbb{Z}_2 = \{Id, \sigma\}$ acts on fin. dim. v. space V by $\sigma(v) = -v$ ($char(K) \neq 2$). Determine generators for the ring of invariants $K[V]^\sigma = K[V]^{\mathbb{Z}_2}$.

(Hint: Invariants are even degree polynomials.)

<u>Remark 7</u>: Sometimes one has to replace K by alg. extension L or alg. closure $\bar{K}$ of K, and to use geometric arguments. For $\rho: G \to GL(W)$ on $W = W/K$, we have for $W_{\bar{K}} := \bar{K} \otimes_K W$ that $\bar{K} \otimes_K K[W]^G = \bar{K}[W_{\bar{K}}]^G$.

Here on $\bar{K}[W_{\bar{K}}]^G$ we can replace G by $\bar{G}$, $G < \bar{G}$, with a representation $\bar{\rho}: \bar{G} \to GL(W_{\bar{K}})$ provided that $\rho(G)$ is Zariski dense in $\bar{\rho}(\bar{G}) < GL(W_{\bar{K}})$. A typical examples are $G = GL_n(K)$, $SL_n(K)$ with $\bar{G} = GL_n(\bar{K})$, $SL_n(\bar{K})$.

<u>Theorem 8</u> (Hilbert) W a G-module, assume the G-mod $K[W]$ is completely reducible (i.e., direct sum of irreducibles.) Then $K[W]^G$ is finitely generated.

<u>Pf</u> (Outline): G-mod $K[W]$ is completely reducible $\Rightarrow$ there is G-equivariant projection $R: K[W] \to K[W]^G$ (it is identity on $K[W]^G$) R is called Reynolds operator, and we have $R(hf) = hR(f)$ for $h \in K[W]^G$ and $f \in K[W]$. It follows that $R(K[W]I) = K[W]I \cap K[W]^G = I$, for any ideal $I \subseteq K[W]^G$. Denote by I_h the maximal homogeneous ideal $I_h := \bigoplus_{d>0} K[W]_d^G$ of $K[W]^G$.

By Hilbert basis theorem, $K[W]I_h \subseteq K[W]$ is finitely gener. ideal, i.e. $\exists\, f_1, .., f_s \in I_h$ homog. polyn. s.t. $K[W]I_h = (f_1, .., f_s)$. But then $I_h = R(K[W]I_h)$, as an ideal of $K[W]^G$, is also generated by $f_1, .., f_s$. By Exercise 9, the homog. system of generators of I_h is also a system of generators for $K[W]^G$. ∎

⑤ In the same way as we did it for finite groups, the proof shows $K[W]^G$ is Noetherian (equivalently, every ideal of $K[W]^G$ is finit. generated: it is a consequence of $K[W]I \cap K[W]^G = I$ and Noetherian property of $K[W]$.)

<u>Exercise 9</u>: $A = \bigoplus_{i \geq 0} A_i$ graded K-algebra, $A_i \cdot A_j \subseteq A_{i+j}$; assume the ideal $A^+ = \bigoplus_{i>0} A_i$ is fin. gener. Then A is finitely generated A_0-algebra: if A^+ is generated by homog. elements $a_1, \dots, a_n$, then $A = A_0[a_1, \dots, a_n]$.

In char $K = 0$ case, we have explicit bound for the degree of gener.

<u>Th. 10</u>: (Noether) Assume char$(K) = 0$. Then $\forall$ repr. W of finite group G is $K[W]^G$ generated by the invariants of degree $\leq$ to the order of G.

(The proof was already discussed.) In general, there are various bounds on the degree of generators, most optimal for cyclic ~~of~~ or abelian groups.