(1)

Riemannian man. as metric spaces

M... smooth (differentiable) manifold, $\dim(M) = n$.

Riemm. metric : a family of (positive definite) inner products: $g_p : T_pM \times T_pM \to \mathbb{R}$, $p \in M$

such that $\forall$ $X, Y \in \mathcal{X}(M)$ smooth is

$p \longmapsto g_p(X(p), Y(p))$ a smooth fusion $M \to \mathbb{R}$.

In the local coordinates $(x_1, \dots, x_n)$, $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ is a basis of T_pM $\forall p \in \mathcal{U} \equiv (x_1, \dots, x_n)$, $\{dx_1, \dots, dx_n\}$ the dual basis of $T_p{}^*M$ and $g = \sum_{i,j} g_{ij}\, dx^i \otimes dx^j$.

$\forall \gamma : \langle a, b \rangle \to M$ smooth curve, the length $L(\gamma) := \int_a^b \|\gamma'(t)\|\, dt$.

Then $\forall$ connected Riemannian man. (M, g) becomes metric space:

$d(x, y) := \inf\{L(\gamma) \mid \gamma(a) = x, \gamma(b) = y, \gamma \text{ smooth curve}\}$,

$d : M \times M \to \mathbb{R}_{\geq 0}$.

For (M, g) compact (top. space), $\forall$ $x, y \in M$ $\exists$ a geodesic whose length is $d(x, y)$. Without compactness, this need not be true (e.g., $M = \mathbb{R}^2 \setminus \{0\}$, $x = (0, 1)$, $y = (-1, 0)$ and induced flat metric from $\mathbb{R}^2$, $\ell(x, y) = 2$ but there is no realizing geodesic.

A Riem. man. (M, g) is (geodesically) complete if $\forall p \in M$, the exponential map $\exp_p$ is defined $\forall$ $v \in T_pM$, i.e. any geodesic $\gamma(t)$ starting at $p \in M$ is defined $\forall t \in \mathbb{R}$. The Hopf-Rinow theorem $\Longrightarrow$ M is (geodesically) complete

$\Longleftrightarrow$

M is metrically complete (for d)

$\Longleftrightarrow$

M is topologically complete

M is complete $\Longrightarrow$ (M is non-extendable) M is not isometric to an open proper submanifold of any other Riem. man.

Locally symmetric spaces – Motivation, Examples and Applications

(M, g) – complete Riem. man. $\mathrm{Isom}(M, g)$ – the isometry group (the group of diff. $f: M \to M$, preserving metric : $f^*(g) = g$.)

<u>Def</u> (Locally sym. space) and sym. space

(M, g) complete Riem. manifold is called locally symmetric space if for any $x \in M$, the local geodesic symmetry s_x is a local isometry.
(M, g) — , — is called a symmetric space if it is locally symmetric and every local isometry s_x extends to a global isometry of (M, g).

Some explanation: (M, g), $\forall x \in M$ $\exists$ (a normal) neighborhood U such that

1/ $\forall y \in U$ is connected to x by a unique geodesic,

2/ $\exists$ star-shaped domain (i.e. $\exists x_0$ in domain such that $\forall y$ in it the line segment is also in it) in T_xM, $V \subseteq T_xM$ containing the origin $0 \in T_xM$ such that $\exp: V \to U$ is a diffeomorphism.

On U, there is a geodesic symmetry s_x, defined by reversing geodesic passing through x, i.e., for any geodesic $\gamma(t)$, $t \in \mathbb{R}$, $\gamma(0) = x$ ($\gamma: I \to M$, $\mathrm{Im}\,\gamma \subseteq U$) holds

$$(s_x \gamma)(t) = \gamma(-t), \quad \underbrace{\gamma(t) \in U,}_{\text{local condition}} \ t \in \mathbb{R}.$$

In terms of exponential map, we have the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{s_x} & U \\ \uparrow \exp & & \uparrow \exp \\ V & \xrightarrow{-Id} & V \end{array}$$

From this comm diagram follows that s_x is a diff. of U. Since $s_x \neq Id$ and $s_x^2 = Id$, s_x is involutive local geodesic symmetry at x.

If M is symmetric, then for all $t \in \mathbb{R}$ holds

$$s_x(\gamma(t)) = \gamma(-t).$$

Symmetric spaces are locally symmetric spaces (they are usually considered of finite volume.)

Natural questions:

1/ Why are locally symmetric spaces important?

2/ How to construct them, are there many?

3/ What are their geometric and analytic properties?

~~4/~~

Locally symmetric vs symmetric spaces:

__Prop__: If (M,g) is a complete locally symmetric space, then its universal covering space $X = \tilde{M}$ equipped with the pull-backed Riemannian metric $\tilde{g}$ is (globally) symmetric space.

Let $\pi_1(M) =: \Gamma$ be the fundamental group of M. Then Γ acts isometrically and properly on $X = \tilde{M}$, and

$$M = {}_{\Gamma}\backslash X.$$

Hence locally symmetric spaces are quotients of symmetric spaces.

Proper action: the map $G \times X \to X \times X$, $(g, x) \mapsto (x, g \cdot x)$ is proper (the inverse images of compact subsets are compact)

__Prop__: If X is a symmetric space, then $G = Isom^\circ(X)$ (the identity component of the isometry group $Isom(X)$ of X) is a Lie group and acts transitively on X.

Fix $x_0 \in X$, its stabilizer in G is denoted $K = \{g \in G \mid g x_0 = x_0\}$.

Then K is compact subgroup of G, $G/K \simeq X$, $gK \mapsto g x_0$, X is homogeneous space.

Not all homogeneous spaces are symmetric spaces, e.g., $SL(2,\mathbb{R})$ with its left invariant metric (similarly, any non-compact semi-simple Lie group) is not a symmetric space (the metric is not right invariant)

Γ acts isometrically on X and is a discrete subgroup of G, hence
Γ '' the fundamental group of X

any locally sym. space is of the form $M = \Gamma \backslash G / K$,

$(G, K, \Gamma) \longrightarrow$
$\Updownarrow$
locally symmetric spaces

$\left\{ \begin{array}{l} G \text{... connected Lie group,} \\ \Gamma \text{... discrete subgroup of } G, \\ K \text{... a compact subgroup of } G. \end{array} \right.$

Reversing the process: $\underbrace{\text{compact, non-compact, flat types}}_{\text{according to the sectional curvatures of}}$ of symmetric spaces

focus on non-compact type symmetric spaces, constructed by
- G - connected non-compact semisimple Lie group, G-invariant metric on G
- K - a maximal compact subgroup
- G/K with G-invariant metric is symmetric space of non-compact type.

- Any discrete, torsion-free subgroup $\Gamma \subseteq G$ acts isometrically and fixed-point free on X, $\Gamma \backslash X$ is a locally symmetr. space, (Γ are often arithmetic subgroups of algebraic groups G.)

In many cases Γ is not torsion-free (e.g., $SL(2,\mathbb{Z}) \subseteq SL(2,\mathbb{R})$) then $\Gamma \backslash X$ is not necessarily smooth, but even in these situations, is the quotient locally symm. space (it is an orbifold or V-manifold)

Basic example:

$G = SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{R},\ ad - bc = 1 \right\}$,

$K = SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\} \simeq S^1$,

$\Gamma = SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \right\}$.

The modular group $\Gamma = SL(2,\mathbb{Z})$ is not torsion-free. In fact,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq Id \quad , \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^4 = Id .$$

The examples of torsion-free subgroups for any $N \geq 1$ are the principal congruence subgroups of $SL(2,\mathbb{Z})$:

$$\Gamma_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = Id \bmod N \right\}.$$

(Γ_N is torsion free for $N \geq 3$.)

The realization of $X = G/K = SL(2,\mathbb{R})/SO(2)$: the hyperbolic plane $\mathbb{H} = \{ z = x+iy \in \mathbb{C} \mid x \in \mathbb{R},\ y \in \mathbb{R}_+ \}$ is equipped with metric $ds^2 = \dfrac{dx^2+dy^2}{y^2}$ of constant scalar curvature $R_{\mathbb{H}} = -1$

($\mathbb{H}$ is a non-compact Riemann surface, $\dim_{\mathbb{R}} \mathbb{H} = 2$,) and $SL(2,\mathbb{R})$ acts isometrically by fractional linear transformations:

$$G \times \mathbb{H} \to \mathbb{H}$$

$$\left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \mapsto \frac{az+b}{cz+d} \quad , \qquad cz+d \neq 0.$$

This action is transitive on $\mathbb{H}$ and $\mathbb{H}$ is a symmetric space: the map

$$s_i : z \mapsto -\frac{1}{z} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot z =: s_i(z) \quad , \qquad z \in \mathbb{H}$$

is the geodesic symmetry at $i \in \mathbb{H}$. The conjugation by elements $SL(2,\mathbb{R})$ implies that for any point $x \in \mathbb{H}$ the geodesic symmetry i_x is a global isometry as well. Notice $Isom^\circ(\mathbb{H}) = PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/_{\pm I}$, since $-Id$ acts trivially on $\mathbb{H}$. The stabilizer of $i \in \mathbb{H}$ is $K = SO(2) \subseteq G$, hence

$$X = SL(2,\mathbb{R})/SO(2) \simeq \mathbb{H},$$
$$g \cdot SO(2) \mapsto g \cdot i.$$

Locally symmetric space for (G, K, Γ) is $\Gamma \backslash \mathbb{H}$.

The structure of $\Gamma \backslash X$ is encoded in the notion of fundamental domain.

<u>Def.</u> A fundamental domain for the discrete group Γ acting on $\mathbb{H}$ is an open subset $\Omega \subseteq \mathbb{H}$ such that

1/ $\forall$ coset $\Gamma . x$ contains at least one point in the closure $\overline{\Omega}$, i.e. $\mathbb{H} = \Gamma \overline{\Omega}$.

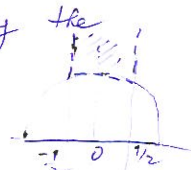
2/ No two different interior points of Ω lie in one Γ-orbit, i.e. $\gamma \cdot \Omega$ for $\gamma \in \Gamma$ are disjoint open subsets of $\mathbb{H}$.

We can find a set F: $\Omega \subseteq F \subseteq \overline{\Omega}$, such that $\forall$ orbit contains exactly one point in F: $\underset{\text{set isomorphism}}{\Gamma \backslash \mathbb{H} \simeq F}$.

F fundamental set for Γ (usually not open in $\mathbb{H}$.) The topology of $\Gamma \backslash \mathbb{H}$ is given by homeomorphism $\Gamma \backslash \mathbb{H} \simeq \overline{\Omega}/\sim$, where $\sim$ is the equivalence relation and $\overline{\Omega}/\sim$ has the quotient topology.

We shall find a good fundamental domain for $SL(2,\mathbb{Z})$, motivated by problems in representations of integers by quadratic forms.

<u>Prop</u>: A fundamental domain Ω for $\Gamma = SL(2,\mathbb{Z})$ on $\mathbb{H}$ is given by the region $\Omega = \{ z = x+iy \in \mathbb{H} \mid |z| > 1, \ -\frac{1}{2} < x < \frac{1}{2} \}$.



<u>Pf</u>: We show

1/ $\forall$ Γ-orbit contains at least one point in $\overline{\Omega}$,

2/ show that no two points of Ω lie in one orbit.

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, $z \mapsto z+1$ $\forall z \in \mathbb{H}$, every orbit of Γ contains a point z with $Re(z) \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$. Consider the Γ-orbit $\Gamma . z$ of such a point. To control the imaginary part of points in $\Gamma . z$, we choose $\gamma \in \Gamma$ such that $Im(\gamma . z)$ is maximal: if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \text{ then } Im(\gamma . z) = \frac{y}{|cz+d|^2}.$$

Since $a,b,c,d \in \mathbb{Z}$, $cz+d \in \mathbb{Z} + \mathbb{Z}z$ (a lattice in $\mathbb{C} \simeq \mathbb{R}^2$) and hence $Im(\gamma . z)$ is uniformly bounded and the maximum value is achieved. The translation $z \mapsto z+1$ does not change the imaginary part, so we can (and we shall) assume that for γz with maximal imaginary part, $Re(\gamma z) \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$.

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Then for all $\gamma \in \Gamma$, $\mathrm{Im}(z) \geq \mathrm{Im}(\gamma\cdot z)$, $z \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$.

We claim that $|z| \geq 1$. Otherwise $|z| < 1$, and taking $S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma$

$$S\cdot z = -\frac{1}{z} = -\frac{x - iy}{|z|^2} = -\frac{x}{|z|^2} + i\frac{y}{|z|^2}, \text{ hence}$$

$$\mathrm{Im}(S\cdot z) = \frac{y}{|z|^2} > \mathrm{Im}(z) \quad (\text{assuming } |z|<1,)$$ and this contradicts $\mathrm{Im}(z) \geq \mathrm{Im}(\gamma\cdot z)$ $\forall \gamma \in \Gamma$. The point 1/ is completed.

To prove the second claim, suppose $z, \gamma z \in \Omega$ for some $\gamma \in \Gamma$. Then we have shown $\frac{y}{|cz+d|^2} \geq y$, hence $|cz+d| \leq 1$. Since $|cz+d| \geq |c|\,\mathrm{Im}(z)$ and $\underbrace{(|z| \geq 1,\ \mathrm{Re}(z) \in \langle -\frac{1}{2}, \frac{1}{2} \rangle)} \Rightarrow \mathrm{Im}(z) \geq \frac{\sqrt{3}}{2} > \frac{1}{2}$,

$|cz+d| \geq |c|\,\mathrm{Im}(z) \Rightarrow |c| < 2$, i.e. $c = 0, \pm 1$. Case by case discussion:

A/ $c=0 \Rightarrow |cz+d| \leq 1$ implies $|d| \leq 1$. Then $ad - bc = 1$ implies $ad = 1$ and so $|d| = 1$. For $d=1$ we have $a=1$, so $\gamma z = z + b$. Since $z, \gamma z \in \Omega$, $b = 0 \Rightarrow \gamma = Id$. When $d = -1$, we have $a = -1$, $\gamma z = -z - b$, hence $z, \gamma z \in \Omega$ implies $b = 0$, so $\gamma = -Id$.

B/ $c = +1 \Rightarrow |cz+d| \leq 1$ implies $|z+d| \leq 1$. Since $|\mathrm{Re}(z)| < \frac{1}{2}$ and d is integral, $|d| \leq 1$, hence $d = 0, \pm 1$. The case $d=0$ can not happen, otherwise $|z+d| = |z| > 1$ by the definition of Ω and this contradicts $|cz+d| \leq 1$. The case $d=1$ can not happen as well, otherwise $|z+d| = |z+1| > |z| > 1$ since $|\mathrm{Re}(z+1)| > |\mathrm{Re}(z)|$ and this again contradicts $|z+d| \leq 1$. The case $d=-1$ is excluded by similar reasoning.

C/ $c = -1$ The proof is analogous to B/.

A/ & B/ & C/ $\Rightarrow \gamma = \pm Id$, and this completes the proof of 2/. $\square$

A consequence of the identification of fundamental domain Ω is

<u>Corollary:</u> The group $SL(2,\mathbb{Z})$ is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

<u>Pf:</u> $\mathbb{H}$ is covered by translates $\gamma\cdot\overline{\Omega}$, $\gamma\in\Gamma$, of $\overline{\Omega}$, which are reflections with respect to the side of such domains. The domains sharing common sides with $\overline{\Omega}$ are $T^{-1}(\overline{\Omega})$, $T(\overline{\Omega})$ and $S(\overline{\Omega})$. Hence any domain is of the form $f(S,T)\,\overline{\Omega}$, where $f(S,T)$ is an element in the subgroup $\langle S,T\rangle$ generated by S,T. Then for $\forall z\in\mathbb{H}\ \exists f(S,T)$ such that $f(S,T).z\in\overline{\Omega}$.

Take $z_0\in\Omega$; for any $\gamma\in\Gamma\ \exists\ f(S,T)\in\langle S,T\rangle$ such that $f(S,T)\cdot\gamma\cdot z_0\in\overline{\Omega}$. Since $z_0\in\Omega$,

$$f(S,T)\cdot\gamma\cdot z_0 = z_0 \underset{\substack{\text{Prop.}\\ \text{above}}}{\Longrightarrow} f(S,T)\cdot\gamma = \pm Id.$$

Since $\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}_{-Id} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = S^2$, $\gamma\in\langle S,T\rangle$.

Hence $\Gamma = \langle S,T\rangle$, i.e. Γ is generated by S,T. ▨

<u>Remark:</u> The fundamental domain Ω can be regarded as the Dirichlet domain for $\Gamma = SL(2,\mathbb{Z})$ with center iy_0, $y_0>1$:

$$\Omega = \{z\in\mathbb{H} \mid d(z, iy_0) < d(\gamma z, iy_0),\ \gamma\in\Gamma\}.$$

(d is the distance (length) function induced by hyperbolic metric.)

<u>Corollary:</u> The quotient $\Gamma\backslash\mathbb{H}$ is non-compact but has finite area.

<u>Pf:</u> The fund. domain Ω is contained in the subset

$$S = \{z = x+iy\in\mathbb{H} \mid y>\tfrac{\sqrt{3}}{2},\ -\tfrac{1}{2}<x<\tfrac{1}{2}\}.$$

Then

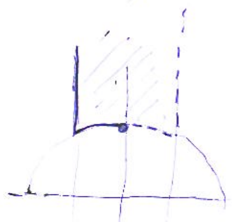
$$\text{Area}(\Gamma\backslash\mathbb{H}) = \int_\Omega \frac{dx\,dy}{y^2} \leq \int_S \frac{dx\,dy}{y^2} = \int_{\frac{\sqrt{3}}{2}}^{+\infty} \frac{dy}{y^2} < +\infty.$$

(8) $\Gamma\backslash\mathbb{H}$ is non-compact: let $t_j \in \mathbb{R}$, $j \in \mathbb{N}$: $t_j \xrightarrow{j\to\infty} \infty$. Identify it_j with its image in $\Gamma\backslash\mathbb{H}$, and it can not converge to any point in $\Gamma\backslash\mathbb{H}$.

$\blacksquare$

The exact form of fundamental set for $SL(2,\mathbb{Z})$ is described as follows:

<u>Prop</u>: Let F be the union of Ω and $\partial\overline{\Omega} \cap \{z \in \mathbb{H} \mid \mathrm{Re}(z) \leq 0\}$, i.e. the left "half" of the boundary $\partial\overline{\Omega}$.
Then F is an exact fundamental form of Γ, i.e. it intersect $\forall$ Γ-orbit at one point.



<u>Pf</u>. The proof follows from the action of T, S on $\partial\overline{\Omega}$.

$\blacksquare$

Fundamental domain Ω and quadratic forms

$$\left\{\begin{matrix}\text{fund. domain} \\ \Omega\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{reductions of} \\ \text{binary quadratic} \\ \text{forms}\end{matrix}\right\}$$

Lagrange, Legendre, Gauss, ...

Let $f(u,v) = au^2 + buv + cv^2$, $a,b,c \in \mathbb{Z}$
an integral binary quadratic forms

Two basic questions in arithmetic of quadratic forms:
1/ Find $n \in \mathbb{Z}$ represented by f, $n = f(u,v)$ for some $u,v \in \mathbb{Z}$
2/ If 1/ is true, determine the number of ways to represent n by $f(u,v)$ (multiplicity of the representation of n)

Quadratic forms $f(u,v)$, $g(u,v)$ are equivalent if there $\exists$ $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in GL(2,\mathbb{Z})$:

$$g(u,v) = f(\alpha u + \beta v, \gamma u + \delta v) \quad \forall u,v,$$

So f is mapped to g under linear transform

$$\begin{pmatrix}u \\ v\end{pmatrix} \longmapsto \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix}u \\ v\end{pmatrix} = \begin{pmatrix}\alpha u + \beta v \\ \gamma u + \delta v\end{pmatrix}.$$

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The invertibility of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2,\mathbb{Z})$ implies

Prop: Two equivalent quadratic forms represent the same set of integers with the same multiplicity.

A redundancy in each equivalence class will be removed by considering proper equivalence of quadratic forms: f, g are properly equivalent provided $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ belongs to $SL(2,\mathbb{Z})$.

The quadratic form $f(u,v) = au^2 + buv + cv^2$ corresponds to the symmetric matrix $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$, and the action of $SL(2,\mathbb{Z})$ on the space of quadratic forms corresponds to

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} := \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

We shall now relax the condition of integrality of coefficients for quadratic form. Quadratic form $f(u,v) = au^2 + buv + cv^2$ is positive definite if $a > 0$ and $d = b^2 - 4ac < 0$, d is the discriminant of f. We shall restrict to positive definite quadratic forms; the quadratic equation

$$az^2 + bz + c = 0, \quad \text{for } z = \frac{u}{v},$$

has two distinct complex roots $z = \dfrac{-b + i\sqrt{|d|}}{2a}$, $\bar{z} = \dfrac{-b - i\sqrt{|d|}}{2a}$.

The root $z \in \mathbb{H}$ determines the form up to a positive multiple. In fact, $f(u,v) = a(u - zv)(u - \bar{z}v)$, and the coefficient a is uniquely determined by the discriminant d.

Prop: For each $d<0$ integer, denote Q_d the set of positive definite quadratic forms $f(u,v)$ with discriminant $d \in -\mathbb{N}$. Then Q_d corresponds bijectively to $\mathbb{H}$ under the map

$$f = au^2 + buv + cv^2 \longmapsto z = \frac{-b + i\sqrt{|d|}}{2a},$$

and this map is equivariant with respect to the action of $SL(2,\mathbb{Z})$.

Pf: As explained above, this map is injective. Since $\forall z \in \mathbb{H}$, $u - zv \neq 0$ for all $u,v \in \mathbb{R}$, $(u-zv)(u-\bar{z}v) = |u-zv|^2 > 0$, hence there exists a unique $a \in \mathbb{R}_+$ such that $f(u,v) = a(u-av)(u-\bar{z}v)$ is a positive def. quadr. form with discriminant d $\Rightarrow$ the map is bijective. Since $z = \frac{u}{v}$ and the action of $SL(2,\mathbb{Z})$ on the quadratic forms is given by
$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \longmapsto \begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix}\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$
the map is obviously $SL(2,\mathbb{Z})$ equivariant. $\square$

In fact, $SL(2,\mathbb{R})$ acts transitively on the space of positive definite quadratic forms of a fixed determinant, and the map above is $SL(2,\mathbb{R})$-equivariant.

Prop: The map $Q_d \to \mathbb{H}$ in the previous Proposition implies that the quadratic forms $f(u,v)$ corresponding to the exact fundamental domain $F \subseteq \Omega$ satisfy:

Either 1/ $0 \leq b \leq a = c$,
or 2/ $-a < b \leq a < c$.

Pf: By definition, $z = \frac{-b + i\sqrt{4ac - b^2}}{2a}$ and $Re(z) = -\frac{b}{2a}$.

If $z \in F$, then $-\frac{1}{2} \leq -\frac{b}{2a} < \frac{1}{2}$, hence $-a < b \leq a$.

Since $|z|^2 = \dfrac{b^2-d^2}{4a^2} = \dfrac{4ac}{4a^2} = \dfrac{c}{a}$,

$|z| \geq 1 \Rightarrow c \geqslant a$,

$|z| > 1 \Rightarrow c > a$. The quadratic forms are positive-definite, and hence $a > 0$.

if ~~|z|=1~~ $c = a$, then $|z| = 1$ and the condition $-\frac{1}{2} \leq Re(z) \leq 0$ is equivalent to $0 \leq b \leq a$. This completes the proof. □

Quadratic forms $f(u,v) = au^2 + buv + cv^2$, whose coefficients a, b, c satisfy the last proposition, are called reduced forms in number theory. Each proper equivalence class of positive definite quadratic forms contain exactly one reduced form.

For each fixed discriminant $d \in \mathbb{Z}_+$, there are only finitely many reduced positive definite integral quadratic forms, and they can be listed down explicitly. Given $n \in \mathbb{Z}$, it is easy to decide whether there exists a form of discriminant d that represents $n \in \mathbb{N}$. The precise conditions is $d = b^2 \mod 4|n|$ for some $b \in \mathbb{N}$, i.e. d is a square residue mod $4|n|$. This allows to decide if a given quadratic form represents an integer.

$\boxed{\text{Sphere packings}}$

A lattice $\Lambda \subseteq \mathbb{R}^n$ is a discrete subgroup of rank $n \in \mathbb{N}$ (i.e, a free $\mathbb{Z}$-module of rank $n \in \mathbb{N}$.) We have

$$\Lambda = \mathbb{Z} v_1 + \dots + \mathbb{Z} v_n,$$

where $v_1, \dots, v_n$ are linearly intependent vectors in $\mathbb{R}^n$ (i.e, a basis of $\mathbb{R}^n/\mathbb{R}$.) A fundamental domain for Λ acting on $\mathbb{R}^n$ is

$$\{ t_1 v_1 + t_2 v_2 + \dots + t_n v_n \mid t_i \in (0,1) \text{ for } i = 1, \dots, n \} \subseteq \mathbb{R}^n,$$

hence

$$Vol(\mathbb{R}^n/\Lambda, dx_1 \wedge \dots \wedge dx_n) = |\det A|, \quad A^T = \left(v_1 \middle| v_2 \middle| \dots \middle| v_n \right).$$

$\forall \Lambda \subseteq \mathbb{R}^n$, there is "sphere packing" given by placing (S^{n-1}, r) at each lattice point so that the spheres do not overlap and r is maximal with respect to non-overlapping property.

$(n-1)$-dim sphere of radius $r \in \mathbb{R}_+$

Since there is a one sphere at each lattice point of Λ, the density of sphere packing is

$$\frac{\sigma_n r^n}{Vol(\mathbb{R}^n/\Lambda, dx_1\wedge\dots\wedge dx_n)},$$

where σ_n is the volume of unit ball $B_n(1) \subseteq \mathbb{R}^n$ and $\sigma_n r^n$ is the volume of $B_n(r) \subseteq \mathbb{R}^n$. The density is scale ($r \to \lambda r$ for any $\lambda \in \mathbb{C}^*$) invariant.

A basic question: Find a lattice Λ with maximum density. (for sphere packing)

Focuse on $n=2$ and use properties of fundamental domaine for $SL(2,\mathbb{Z})$.

In general, let $a := \min\limits_{\substack{v\in\Lambda\\ v\neq 0}} \|v\|^2$, the minimum (euclidean) norm square of non-zero vectors in Λ, then $r = \frac{\sqrt{a}}{2}$. The sphere packing problem becomes a problem to find the global maximum of the function

$$\Lambda \mapsto \frac{a^{n/2}}{Vol(\mathbb{R}^n/\Lambda, dx_1\wedge\dots\wedge dx_n)}$$

on the space of all lattices. So we need to parametrize the space of (equivalence classes of) all lattices. Each basis $\{v_1,\dots,v_n\}$ of $\mathbb{R}^n$ determines a lattice $\Lambda = \mathbb{Z}v_1 + \dots + \mathbb{Z}v_n$. However, different bases can give rise to the same lattice. In particular, two bases $\{v_1,\dots,v_n\}$ $\{w_1,\dots,w_n\}$ generate the same lattice iff $\exists\, T \in GL(n,\mathbb{Z})$ such that

$$(w_1 \mid w_2 \dots \mid w_n) = T(v_1 \mid v_2 \dots \mid v_n).$$

This is an equivalence relation on the space of all bases, and we shall choose a representatives called reduced bases.
For each lattice $\Lambda = \mathbb{Z}v_1 + \dots + \mathbb{Z}v_n$ corresponding to the basis $v_1,\dots,v_n$, let $A^T = (v_1 \mid \dots \mid v_n)$ be its $(n\times n)$-matrix. Define a symmetric matrix $S = AA^T$ (A^T is the transpose of A.) Then

$$Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$
$$Q(X,Y) := X^T S Y, \qquad X, Y \in \mathbb{R}^n$$

is a positive definite quadratic form on $\mathbb{R}^n$.

On the other hand, Q determines A and Λ up to rotation, but the density of sphere packing problem is rotation invariant and so we can ignore this non-uniqueness. The minimum (euclidean) norm square of nonzero lattice points is equal to the minimum value of Q on non-zero integral vectors.

Let $\langle\,,\,\rangle$ be the standard (euclidean) scalar product (inducing quadratic form) on $\mathbb{R}^n$, and $\mathbb{Z}^n$ the standard lattice (for canonical basis of $\mathbb{R}^n$.) Let $\mathcal{P}_1$ be the space of pairs $(\langle\,,\,\rangle, \Lambda)$ given by standard quadratic form on $\mathbb{R}^n$ and a lattice $\Lambda \subseteq \mathbb{R}^n$, and $\mathcal{P}_2$ the space of pairs $(Q, \mathbb{Z}^n)$ of arbitrary (pos. def.) quadratic forms and the standard lattice $\mathbb{Z}^n$. The map $Q \mapsto \langle A-, A-\rangle$ induces the bijection $\mathcal{P}_1 \simeq \mathcal{P}_2$.

The above sphere packing problem can be formulated as a question to find a positive definite quadratic form of determinant 1, whose minimal value (minimum) on $\mathbb{Z}^n$ is maximal. As already said, let us solve it for $n=2$ using reduced quadratic forms.

For a reduced binary quadratic form $au^2+buv+cv^2$, the coefficients a,b,c satisfy $0<a\le c$, $-a<b\le a$.

Hence
$$\min_{\substack{(u,v)\neq 0\\ u,v\in\mathbb{Z}}} f(u,v) = a.$$

So let ρ be the sphere packing density for f. Then

$$\rho = \frac{\pi r^2}{\sqrt{ac-\frac{b^2}{4}}} \overset{r=\frac{\sqrt{a}}{2}}{=} \frac{\frac{\pi a}{4}}{\sqrt{ac-\frac{b^2}{4}}} \Rightarrow \rho^2 = \frac{\pi^2 a^2}{4(4ac-b^2)} \underset{\substack{a\le c\\ |b|\le|a|}}{\le} \frac{\pi^2 a^2}{4(4a^2-a^2)} = \frac{\pi^2}{12},$$

and the maximum density $\rho = \frac{\pi}{2\sqrt{3}}$ is realized by $a=b=c$.

Let us finally determine the corresponding lattice $\Lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2$.

Since $A^T = (v_1 \mid v_2) \Leftrightarrow A = \begin{pmatrix} v_1 \\ \hline v_2 \end{pmatrix}$, $S = AA^T = \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle \\ \langle v_1, v_2\rangle & \langle v_2, v_2\rangle \end{pmatrix}$

and so $a = |v_1|^2$, $b = 2\langle v_1, v_2\rangle$, $c = |v_2|^2$. The angle between vectors v_1 and v_2 is determined by

$$\cos\theta = \frac{\langle v_1, v_2\rangle}{\|v_1\|\,\|v_2\|} = \frac{\frac{b}{2}}{\sqrt{ac}} \overset{\substack{a\leq c \\ |b|\leq |a| \\ \Downarrow}}{\leq} \frac{1}{2},$$

and the upper bound is achieved for $a = b = c$, corresponding to the densest sphere packing as concluded above. Therefore, $\Lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2$ for $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$, and this has bigger density for sphere packing than the standard lattice $\mathbb{Z}^2$ $\left(v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \right)$

Moduli space of elliptic curves

$\Gamma \backslash \mathbb{H}$ for $\Gamma = SL(2,\mathbb{Z})$ was identified as the space of

- equivalence classes of positive definitive binary quadratic forms of determinant one,
- lattices of co-volume 1 up to rotation.

Now we identify this space with moduli (equivalence classes) of complex elliptic curves.

Def: A complex elliptic curve is a smooth $\underbrace{\text{Riemann surface}}_{\substack{\text{orientable Riemannian Kähler} \\ \text{manifold of real dimension 2.}}}$ Σ of genus 1.

Riemann uniformization theorem $\Rightarrow$ an elliptic curve Σ is of the form $\mathbb{C}/\Lambda$, $\Lambda \subseteq \mathbb{C}$ a lattice in $\mathbb{C}$ (a free $\mathbb{Z}$-module of rank 2.) Two elliptic curves Σ_1, Σ_2 (regarded as Riemann surfaces) are called equivalent if there $\exists$ a biholomorphic map $\varphi : \Sigma_1 \to \Sigma_2$. Moduli space of elliptic curves is the set of equivalence classes of elliptic curves.

Lemma: Two elliptic curves $\mathbb{C}/\Lambda_1$, $\mathbb{C}/\Lambda_2$, are equivalent iff there $\exists\ a \in \mathbb{C}^*$: $a\Lambda_1 = \Lambda_2$ (i.e., Λ_1 and Λ_2 are homothetic lattices.)

Pf: Let $\varphi: \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ be a biholomorphic map, and denote $\tilde{\varphi}: \mathbb{C} \to \mathbb{C}$ its lift. Similarly, the inverse holomorphic
$\tilde{\varphi}(0)=0$
map $\varphi^{-1}: \mathbb{C}/\Lambda_2 \to \mathbb{C}/\Lambda_1$ lifts to a map $\tilde{\varphi}^{-1}: \mathbb{C} \to \mathbb{C}$
$\tilde{\varphi}^{-1}(0)=0$.

Here $\tilde{\varphi}$ and $\tilde{\varphi}^{-1}$ are inverses each other, and $\tilde{\varphi}, \tilde{\varphi}^{-1}$ are both holomorphic. By (small) Picard theorem, the analytic map $\tilde{\varphi}$ is linear, $\tilde{\varphi}(z) = az+b$, $a, b \in \mathbb{C}$. The condition $\tilde{\varphi}(0)=0$ implies $\tilde{\varphi} = a$ for $a \in \mathbb{C}^*$ (because $\tilde{\varphi}$ is non-constant. The proof is complete. □

Proposition: The moduli space of elliptic curves can be identified with $\mathbb{H}/SL(2,\mathbb{Z})$ by the map $z \in \mathbb{H} \mapsto \mathbb{C}/(\mathbb{Z}+z\mathbb{Z})$.

Pf: Any lattice $\Lambda \subseteq \mathbb{C}$ is of the form $\Lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2$, where $v_1, v_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent. This means $\frac{v_1}{v_2} \notin \mathbb{R}$, and since $\left(\frac{v_1}{v_2}\right)\left(\frac{v_2}{v_1}\right) = 1$, one of them, say $\frac{v_1}{v_2}$, has positive imaginary part: $\frac{v_1}{v_2} \in \mathbb{H}$. Define $z = \frac{v_1}{v_2}$, and since the bases of equivalent lattices is given by $SL(2,\mathbb{Z})$, the Proposition follows. □

Monodromy groups of hypergeometric differential equations

The concept of Fuchsian groups in the theory of Riemann surfaces arises from the monodromy groups of ordinary differential equations with regular singularities, and the related uniformization.

(16) The hypergeometric differential equation is

$$z(1-z)w'' + (c-(a+b+1)z)w' - ab\,w = 0,$$

where $w = w(z)$ is a meromorphic function, $w' \equiv \left(\frac{d}{dz} w\right)(z)$, $z \in \mathbb{CP}^1$, $a, b, c \in \mathbb{C}$ constants (spectral parameters of hypergeometric differential equation). The differential equation has regular singularities at three points $z = 0, 1, \infty$.

Fix any base point $z_0 \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$, and any two linearly independent solutions $w_1 = w_1(z)$, $w_2 = w_2(z)$ in a neighborhood of z_0. Then analytic continuation of w_1, w_2 along paths in $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ defines multi-valued functions on $\mathbb{CP}^1 \setminus \{0,1,\infty\}$ and gives a monodromy representation

$$\begin{array}{l} \rho : \pi_1(\mathbb{CP}^1 \setminus \{0,1,\infty\}) \longrightarrow GL(2,\mathbb{C}), \\ \qquad\quad \shortparallel \\ \quad\;\; \pi_1(\mathbb{CP}^1 \setminus \{0,1,\infty\}; z_0) \end{array}$$

depending on the choice of z_0 and w_1, w_2. However, different choices give conjugate representations and so define an equivalence class of representations. It is clear that one can restrict to $PGL(2,\mathbb{C}) := GL(2,\mathbb{C})/\mathbb{C}^* \cdot Id_{2\times 2}$, and denote the image of ρ in $PGL(2,\mathbb{C})$ by Γ.

Consider the multi-valued complex function

$$\begin{array}{rl} u(z) := \left(\frac{w_1}{w_2}\right)(z) : \mathbb{CP}^1 \setminus \{0,1,\infty\} & \longrightarrow \mathbb{CP}^1, \\ z & \longmapsto u(z). \end{array}$$

Denote the image of u in $\mathbb{CP}^1$ by D, $Im(u) = D \subseteq \mathbb{CP}^1$.
Then u is well-defined holomorphic map

$$u : \mathbb{CP}^1 \setminus \{0, 1, \infty\} \longrightarrow \Gamma \backslash D.$$

L. Schwarz proved that for $a = b = \frac{1}{12}$, $c = \frac{2}{3}$ and suitable choice of the base point z_0 and solutions w_1, w_2 near z_0, the image D is equal to the unit disc $\{z \in \mathbb{C} \mid |z| < 1\} \subseteq \mathbb{CP}^1$, and $\Gamma \simeq SL(2,\mathbb{Z})$. In fact, Γ is isomorphic to $SL(2,\mathbb{Z})$ embedded in $SU(1,1)$ under the Cayley transform $SL(2,\mathbb{C}) \to SU(1,1)$, where $SU(1,1)/U(1) \simeq \{z \in \mathbb{C} \mid |z| < 1\}$ and the Cayley transform maps $\mathbb{H}$ biholomorphically onto the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$. This leads to new transcendental functions arising as automorphic functions (forms) important in number theory.

$\boxed{\text{Realization of discrete series}}$

An irreducible representation of G (a connected semi-simple Lie group) is called a discrete series representation if it appears as an irreducible subrepresentation of the regular representation of G in $L^2(G)$. The discrete series can be realized on non-compact symmetric space $X = G/K$ on $L^2(X)$-solutions of certain system of elliptic differential equations.

For example, for $G = SL(2,\mathbb{R})$ and $X = \mathbb{H} = SL(2,\mathbb{R})/SO(2)$, the discrete series representations $D_n^{\pm}$, $n \geqslant 2$, can be realized on the space

$$V_n := \{ f(z) \text{ holomorphic on } \mathbb{H} \mid \|f\|^2 = \int_{\mathbb{H}} |f(z)|^2 y^{n-2} \, dx\,dy < \infty \},$$

with G acting ~~as~~ unitarily on V_n as (this corresponds to D_n^+, D_n^- is analogous)

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} f \right](z) = (-bz+d)^{-n} f\left(\frac{az-c}{-bz+d} \right).$$

The elliptic operator in question is the Cauchy-Riemann operator.