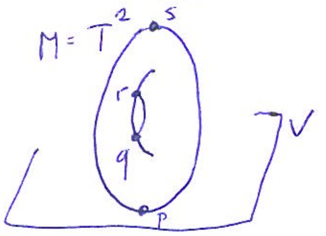
(21) Problems to solve:

1/ $f \in C^{\infty}(\mathbb{R}^n)$ is a continuous fcion, g a smooth fcion on $\mathbb{R}^n$ with compact support. Show $(f * g)(x) := \int f(y)\, g(x-y)\, d^n y$ is smooth.

2/ $C \subseteq \mathbb{R}^n$ compact set, $C \subseteq U \subseteq \mathbb{R}^n$ open; show $\exists\, f: \mathbb{R}^n \to \langle 0,1 \rangle$ smooth, $f(x) = 1$ for $x \in C$ and such that the support of f (smallest closed subset of $\mathbb{R}^n$ containing $\forall$ ~~...~~ points where $f(x) > 0$) is contained in U.

Non-degenerate smooth fcions on manifolds, Morse theory

Example:



$M = T^2$ ... torus, tangent to V (at p)

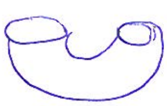
$f: M \to \mathbb{R}$ the height fcion over V

$M^a \subseteq M$ ... the set of pts $x \in M$: $f(x) \leq a$

Then:

1/ if $a < 0 < f(p)$, M^a is vacuous

2/ if $f(p) < a < f(q)$, then M^a is homeom. to a 2-cell (2-dim disk) D^2 

3/ if $f(q) < a < f(r)$, then M^a is homeom. to a cylinder 

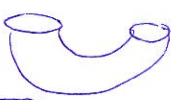
4/ if $f(r) < a < f(s)$, M^a is homeom. to a compact man. of genus 1 with a circle as a boundary. 

5/ if $f(s) < a$, then $M^a \simeq T^2$.

homotopy type

In terms of the homotopy type, the change in M^a is:

$1/ \to 2/$ - attaching a 0-cell: $\underset{f(p)<a<f(q)}{M^a} \simeq \bullet \overset{h}{\simeq}$ 

$2/ \to 3/$ - — " — 1-cell: $M^a \approx$  $\approx$ 

$3/ \to 4/$ — " — $M^a \approx$  $\approx$ 

Attaching cell (say, a k-cell): Y ... top. space,

$e^k := \{x \in \mathbb{R}^k \mid \|x\| \leq 1\}$ ... k-cell,

$\partial e^k := \dot{e}^k = \{x \in \mathbb{R}^k \mid \|x\| = 1\}$ ... boundary of e^k
$\overset{\shortparallel}{S^{k-1}}$

$g: S^{k-1} \to Y$ a cont. map, then $Y \cup_g e^k := (Y \cup e^k)/\sim$ $(g(x), x)$ $\forall x \in S^{k-1}$

The points p, q, r, s change the homotopy type of M^a, have a characterization in terms of f: they are the critical pts of the height fion on $M = \mathbb{T}^2$. A coordinate system (x, y) near these pts, $df = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$ are both zero. At p we can choose (x, y) so that $f = x^2 + y^2$, at s so that $f = \text{constant} - x^2 - y^2$, and at q, r so that $f = \text{const} + x^2 - y^2$. Notice: the number of $-$ signs at $\forall$ pt is the dimension of the cell we must attach to get from M^a to M^b, $a < f(\text{point}) < b$.

We generalize this observation to $\forall$ smooth fions on M.

M - C^∞-man, $\underset{p \in M}{T_p M}$, $g: M \to N$ smooth, $dg := g_*: TM \to TN$; $f: M \to \mathbb{R}$ smooth, $p \in M$ is critical if $f_*: T_p M \to T_{f(p)} \mathbb{R}$ is trivial. In a local chart $x_1, \dots, x_n$ on $\mathcal{U}$, $p \in \mathcal{U}$, $\frac{\partial f}{\partial x_1}(p) = \dots = \frac{\partial f}{\partial x_n}(p) = 0$; $f(p)$ = critical value of f. $M^a := \{x \in M \mid f(x) \leq a\}$; implicit fion theory implies M^a = smooth manifold with boundary, $f^{-1}(a)$ = a smooth subman. of M.

A critical point p is non-degenerate iff $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{i,j=1}^n$ is non-sing. (this notion is independent of coord. chart.)

p - a critical pt, f_{**} - Hessian of f at p $(TM \overset{sym}{\otimes} TM \to \mathbb{R})$

$v, w \in T_p M$, $\tilde{v}, \tilde{w}$ ... extended vector fields; $f_{**}(v, w) := \tilde{v}_p(\tilde{w}(f))$,
$ \cap TM$ $\qquad \tilde{v}_p = v_p$, $\tilde{w}_p = w_p$.

f_{**} is symmetric: $\tilde{v}_p(\tilde{w} f) - \tilde{w}_p(\tilde{v} f) = [\tilde{v}, \tilde{w}]_p f = 0$,

$[\tilde{v}, \tilde{w}]$ is the Poisson bracket; $[\tilde{v}, \tilde{w}]_p(f) = 0 \Leftarrow p$ is critical pt of f.
$\phantom{[\tilde{v}, \tilde{w}] \text{ is the }}$ $\shortparallel$ Lie

(23) f_{**} is well-defined, i.e. independent on extension of v to $\tilde{v}$ and w to $\tilde{w}$.

$(x^1, \dots, x^n)$ local chart, $v = \sum a_i \frac{\partial}{\partial x^i}\Big|_p$, $w = \sum b_j \frac{\partial}{\partial x^j}\Big|_p$), we take $\tilde{w} = \sum b_j \frac{\partial}{\partial x^j}$ with

b_j constant fcions $\forall j$ on a neigh of p; then

$$f_{**}(v,w) = v(\tilde{w}(f))(p) = v\Big(\sum_j b_j \frac{\partial f}{\partial x_j}\Big) = \sum_{ij} a_i b_j \frac{\partial^2 f}{\partial x^i \partial x^j}(p)$$

and $\left\{\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right\}_{i,j=1}^{n}$ represents f_{**} in p for the basis $\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p$.

- index of f_{**} on T_pM (or f at p) - maximal dimension of a subspace of T_pM on which f_{**} is negative definite;
- nullity of f_{**} on T_pM ( ) — — " — — " — f_{**} $F(v,w) = 0$ $\forall w \in T_pM$.

Critical points of f in Morse theory : determined by index of f_{**}.

Lemma: $f \in C^\infty(V)$, $0 \in V \subseteq \mathbb{R}^n$ a neigh, $f(0) = 0$. Then

$f(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \dots, x_n)$ for suitable

$g_i \in C^\infty(V)$ such that $g_i(0) = \frac{\partial f}{\partial x_i}(0)$, $i = 1, \dots, n$.

Pf: $f(x_1, \dots, x_n) = \int_0^1 \frac{d}{dt} f(tx_1, \dots, tx_n)\,dt$

(diff of the comp. of smooth fcions) $= \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n)\, x_i\, dt$,

and we define $g_i(x_1, \dots, x_n) := \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n)\,dt$.

Lemma (Morse): let $p \in M$ be a non-degenerate pt, which is critical for f. Then $\exists$ a local coordinate system $(y^1, \dots, y^n)$ in a neigh U of p with $y^i(p) = 0$ $\forall i = 1, \dots, n$ and such that $f = f(p) - (y^1)^2 - \dots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \dots + (y^n)^2$ is true on U with λ the index of f at p.

(24) Pf: If there is such expression for f, then λ must be the index of f at p: $(z^1, \dots, z^n)$ coordinate system, $f(q) = f(p) - (z^1(q))^2 - \dots - (z^\lambda(q))^2 + (z^{\lambda+1}(q))^2 + \dots + (z^n(q))^2$,

we have $\frac{\partial^2 f}{\partial z^i \partial z^j}(p) = \begin{cases} -2 & \text{for } i=j\le\lambda \\ 2 & \text{if } i=j>\lambda \\ 0 & \text{otherwise} \end{cases} \Rightarrow f_{**} = \begin{pmatrix} -2 & & & & \\ & \ddots & & & \\ & & -2 & & \\ & & & 2 & \\ & & & & \ddots \; 2 \end{pmatrix}$

in the basis $\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p$

therefore λ is the index of f_{**}.

We now show a suitable coordinate system does exist, denote it $(y^1, \dots, y^n)$. Assume p is the origin of the map to $\mathbb{R}^n$, and that $f(p) = f(0) = 0$. By previous lemma, we can write $f(x_1, \dots, x_n) = \sum_{j=1}^n x_j g_j(x_1, \dots, x_n)$ for $(x_1, \dots, x_n)$ in the neigh. of 0. Since 0 is assumed to be the critical pt, $g_j(0) = \frac{\partial f}{\partial x^j}(0) = 0$.

Applying the lemma, $g_j(x_1, \dots, x_n) = \sum_{i=1}^n x_i h_{ij}(x_1, \dots, x_n)$ for smooth h_{ij} so $f(x_1, \dots, x_n) = \sum_{i,j=1}^n x_i x_j h_{ij}(x_1, \dots, x_n)$. We can assume $h_{ij} = h_{ji}$ (since $\tilde{h}_{ij} := \frac{1}{2}(h_{ij} + h_{ji})$, for which we have $\tilde{h}_{ij} = \tilde{h}_{ji}$ and $f = \sum \tilde{h}_{ij} x_i x_j$.) The matrix $\tilde{h}_{ij}(0)$ equals to $\frac{1}{2}\frac{\partial^2 f}{\partial x^i \partial x^j}(0)$, hence is non-singular.

There is a non-singular coordinate change (a diffeomorphism), which implies the desired expression for f, in a smaller neigh. of 0. (it immitates the usual diagon. process for quadratic forms)

By induction: assume $\exists$ coordinates $u_1, \dots, u_n$ in $U_1 \subseteq U$ so that

$$f = \pm(u_1)^2 \pm \dots \pm (u_{r-1})^2 + \sum_{i,j\ge r} u_i u_j H_{ij}(u_1, \dots, u_n)$$

(the matrices H_{ij} are symmetric) The claim is true for $r=1$ and r $\overset{?}{\Rightarrow}$ true for $r \ge 1$, $r \le n$.

In the linear change $n-r+1$ last coordinates, we may assume $H_{rr}(0) \ne 0$; $g(u_1, \dots, u_n) := \sqrt{|H_{rr}(u_1, \dots, u_n)|}$ – this is a smooth

(25)

non-zero fcn of $u_1, \dots, u_n$ for $U_2 \subseteq U_1$. Now introduce new coordinates $v_1, \dots, v_n$ by $v_i = u_i$ for $i \neq r$ and

$$v_r(u_1, \dots, u_n) = g(u_1, \dots, u_n)\Big[u_r + \sum_{i>r} u_i H_{ir}(u_1, \dots, u_n) / H_{rr}(u_1, \dots, u_n)\Big].$$

By inverse fcn theorem, $\{v_1, \dots, v_n\}$ is a coordinate system on $U_3 \subseteq U_2$, and f is of the form

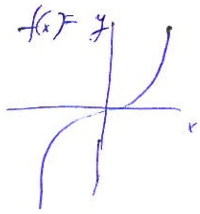
$$f = \sum_{i \leq r} \pm (v_i)^2 + \sum_{i,j > r} v_i v_j H'_{ij}(v_1, \dots, v_n),$$

which completes the induction (this is smooth analogon of Gramm-Schmidt ON-process.)

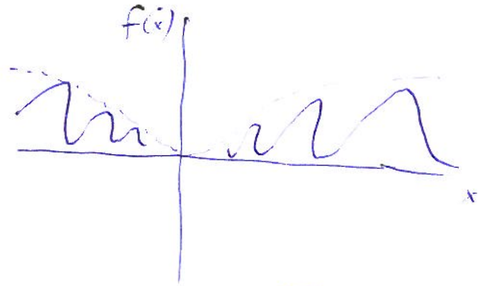
Corollary: Non-degenerate critical points are isolated.

Examples (of degenerate critical points, for fcns on $\mathbb{R}$ and $\mathbb{R}^2$.)

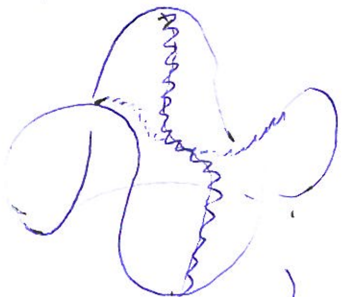
1/ $f(x) = x^3$ (the origin is degen. crit. pt)



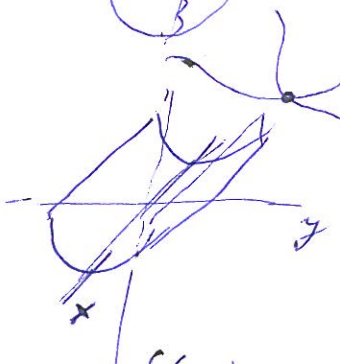
2/ $f(x) = e^{-\frac{1}{x^2}} \sin^2\left(\frac{1}{x}\right)$ (the origin is a deg., non-isolated, critical pt.)



3/ $f(x,y) = x^3 - 3xy^2 = Re\,(x+iy)^3$, ((0,0) is a degenerate critical pt, "monkey saddle")



4/ $f(x,y) = x^2$ (the set of critical pts, $\forall$ are degenerate, is the x-axis $\subseteq \mathbb{R}^2$)



5/ $f(x,y) = x^2 y^2$ (the set of critical pts, $\forall$ are degenerate, consist of the union of x and y-axis, not submanifolds)

(26) M smooth man., 1-parameter group of diff. of M is a C^∞-map
$\varphi: \mathbb{R}\times M \to M$ s.t. 1/ $\forall t\in\mathbb{R}$, $\varphi_t: M\to M$

$\varphi_t(q) = \varphi(t,q)$ is a diff. $M\to M$, $\forall q\in M$

2/ $\forall t,s\in\mathbb{R}$: $\varphi_{t+s} = \varphi_t\circ\varphi_s = \varphi_s\circ\varphi_t$.

1-par. group of diffs of M $\rightsquigarrow$ vector fields on M

$$\varphi \longmapsto X_1(f) = \lim_{t\to 0}\frac{f(\varphi_t(q)) - f(q)}{t} \quad \forall f\in C^\infty(M)$$

(X_q generates φ)

<u>Lemma</u>: A smooth vector field on M, vanishing outside of a compact set $K\subseteq M$, generates a unique 1-parameter group of diff. of M.

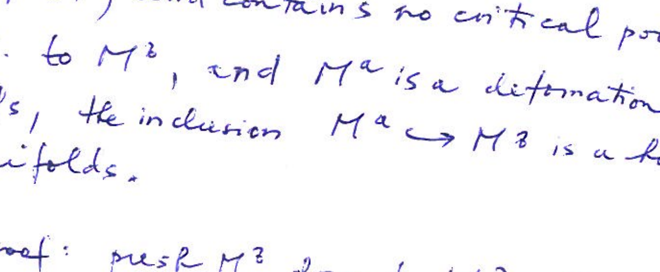
<u>Remark</u>: The hypothesis that X vanishes outside of compact set cannot be omitted. For $M=(0,1)\subseteq\mathbb{R}$, $X=\frac{d}{dt}\Big|_{(0,1)}$ does not generate any 1-par. group of diff. of $(0,1)$.

$\boxed{\text{Homotopy type in terms of critical values}}$

$f\in C^\infty(M,\mathbb{R})$, $M^a = f^{-1}(-\infty,a\rangle = \{p\in M \mid f(p)\le a\}$

<u>Theorem</u>: M, f smooth. Let $a<b$, suppose $f^{-1}\langle a,b\rangle := \{p\in M \mid a\le f(p)\le b\}$ is compact, and contains no critical points of f. Then M^a is diff. to M^b, and M^a is a deformation retract M^b. In other words, the inclusion $M^a \hookrightarrow M^b$ is a homotopy equivalence of manifolds.

The idea of proof: push M^b down to M^a along the orthogonal traject of hypersurfaces $f=$ const :



Choose a Riem. metric on M, $\langle\,,\,\rangle = g$, $\operatorname{grad}(f)$ is the vector field
$\operatorname{grad}(f) = (df)g^{-1}$: $\langle Y, \operatorname{grad} f\rangle := Y(f)$ $\forall Y\in\mathfrak{X}(M)$.

$$(\operatorname{Grad} f)_i = \sum_j g^{ij}\frac{\partial f}{\partial x_j}$$

(27) Zeroes of $\operatorname{grad}(f) \Leftrightarrow$ critical points of f; $c: \mathbb{R} \to M$ a curve, tangent vector $\frac{dc}{dt}$, $\langle \frac{dc}{dt}, \operatorname{grad} f \rangle = \frac{d(f \circ c)}{dt}$.

Let $\rho: M \to \mathbb{R}$ be a smooth function, equal to $\dfrac{1}{\langle \operatorname{grad} f, \operatorname{grad} f \rangle}$ on the compact set $f^{-1}\langle a,b\rangle$ and vanishing outside a compact set containing $f^{-1}\langle a,b\rangle$.

Then the vector field X, $X_q := \rho(q)(\operatorname{grad} f)_q$ for all $q \in M$ generates (by the last lemma) a 1-parameter group of diffeomorphisms φ_t of M: fix $q \in M$, $t \to f(\varphi_t(q))$ a smooth fcion. If $\varphi_t(q)$ lies in $f^{-1}\langle a,b\rangle$, then

$$\frac{df(\varphi_t(q))}{dt} = \left\langle \frac{d\varphi_t(q)}{dt}, \operatorname{grad}(f) \right\rangle = \langle X, \operatorname{grad}(f) \rangle = +1,$$

and so $t \to f(\varphi_t(q))$ is linear with derivative $+1$ for $f(\varphi_t(q)) \in \langle a,b\rangle$.

The diff. $\varphi_{b-a}: M \to M$ carries M^a diff. onto M^b ($\Rightarrow$ proof of first part of theorem)

Now define a 1-param. family of smooth maps:

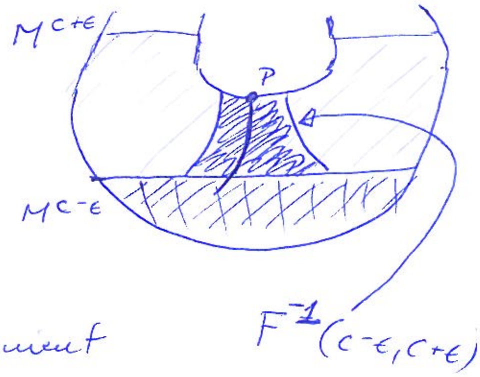
$r_t: M^b \to M^b$ by $r_t(q) = \begin{cases} q & \text{if } f(q) \le a, \\ \varphi_{t(a-f(q))}(q) & \text{if } a \le f(q) \le b. \end{cases}$

Then $r_0 = Id_{M^b}$, r_1 is retraction from M^b to M^a $\Rightarrow$ M^a is deform retract of M^b by smooth homotopy. ▨

Remark: Compactness of $f^{-1}\langle a,b\rangle$ can not be omitted.

Theorem: Let f be a smooth fcion, p a non-degenerate critical point of f with index λ. Set $f(p) = c$, suppose $f^{-1}\langle c-\epsilon, c+\epsilon\rangle$ is compact and for some $\epsilon > 0$ does not contain a critical point of f other than p. Then the set $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell attached (for ϵ sufficiently small.)

(28)



structure of proof: The idea is indicated on the right picture. A new smooth fcon is introduced, $F: M \to \mathbb{R}$, which coincides with f except that $F < f$ in a small neigh. of $p \in M$, $H \subseteq M$. Thus $F^{-1}(-\infty, c-\epsilon) = M^{c-\epsilon} \cup H$. Choosing a convenient cell $e^\lambda \subseteq H$, an argument shows $M^{c-\epsilon} \cup e^\lambda$ is a defomation retract of $M^{c-\epsilon} \cup H$. The previous theorem applied to F and the region $F^{-1}\langle c-\epsilon, c+\epsilon\rangle$ implies that $M^{c-\epsilon} \cup H$ is a defomation retract of $M^{c+\epsilon}$.

Choose a coordinate chart $u^1, \dots, u^m$ in U, $p \in U$, so that we have on U:

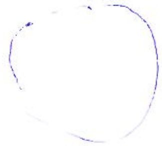
$$f = c - (u^1)^2 - \dots - (u^\lambda)^2 + (u^{\lambda+1})^2 + \dots + (u^m)^2$$

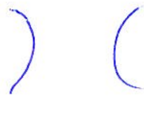
(the critical point p has the coordinates $u^1(p) = \dots = u^m(p) = 0$.)

For $\epsilon > 0$ sufficiently small

- the region $f^{-1}\langle c-\epsilon, c+\epsilon\rangle$ is compact, and contains no critical pt. other than p
- the image of U under the diff embedding $(u^1, \dots, u^m): U \to \mathbb{R}^m$ contains the closed ball $\{(u^1, \dots, u^m) \mid \sum_{j=1}^m (u^j)^2 < 2\epsilon\}$.

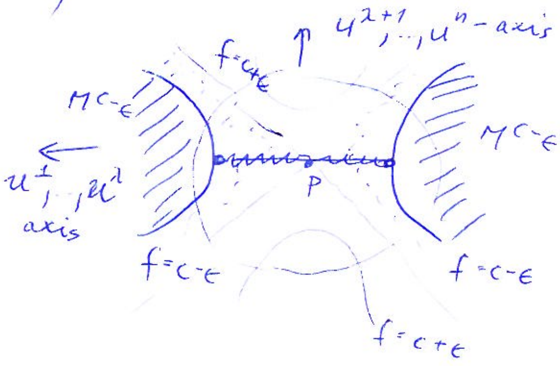
Define $e^\lambda \subseteq U$ such that $(u^1)^2 + \dots + (u^\lambda)^2 \le \epsilon$, $u^{\lambda+1} = \dots = u^n = 0$.

 - the circle = boundary of ball of radius $\sqrt{2\epsilon}$

 - hyperbolas = $f = c - \epsilon$

 - hyperbolas = $f = c + \epsilon$

 - the line represents the cell e^λ, $\partial e^\lambda = e^\lambda \cap M^{c-\epsilon}$ is the boundary, so e^λ is attached to $M^{c-\epsilon}$



We have to prove $M^{c-\epsilon} \cup e^\lambda$ is a defomation retract of $M^{c+\epsilon}$.
For that, we introduce a new smooth function $F: M \to \mathbb{R}$. Let $\mu: \mathbb{R} \to \mathbb{R}$ be a smooth fcion: $\mu(0) > \epsilon$, $\mu(r) = 0$ for $r \geq 2\epsilon$, $-1 < \mu'(r) \le 0$ $\forall r \in \mathbb{R}$.

$\mu'(r) = \frac{d\mu}{dr}$

(29) $F = f$ on $M \setminus \mathcal{U}$, $p \in \mathcal{U}$, and $F = f - \mu\left((u^1)^2 + \dots + (u^\lambda)^2 + 2(u^{\lambda+1})^2 + \dots + 2(u^m)^2\right)$, on $\mathcal{U} \subseteq M$.

Moreover, $\xi : \mathcal{U} \to \langle 0, \infty)$, $\xi = (u^1)^2 + \dots + (u^\lambda)^2$; $\eta = (u^{\lambda+1})^2 + \dots + (u^m)^2$, $\eta : \mathcal{U} \to \langle 0, \infty)$ $\}$ smooth fun's

Then $f = c - \xi + \eta$, $F(q) = c - \xi(q) + \eta(q) - \mu\left(\xi(q) + 2\eta(q)\right)$ $\forall q \in \mathcal{U} \subseteq M$.

<u>Lemma</u>: The locus $F^{-1}(-\infty, c+\epsilon) \subseteq M$ is $M^{c+\epsilon} = f^{-1}(-\infty, c+\epsilon\rangle$.

<u>Pf</u>: Let $\xi + 2\eta \leq 2\epsilon$ be the ellipsoid.
- outside the ellipsoid $\xi + 2\eta > 2\epsilon$, f, F coincide ;
- inside — " — : $F \leq f = c - \xi + \eta \leq c + \frac{1}{2}\xi + \eta \leq c + \epsilon$ ∎

<u>Lemma</u>: The critical points of F are the same as the critical points of f.

<u>Pf</u>: We have $\frac{\partial F}{\partial \xi} = -1 - \mu'(\xi + 2\eta) < 0$, $\frac{\partial F}{\partial \eta} = 1 - 2\mu'(\xi + 2\eta) \geq 1$.
Since $dF = \frac{\partial F}{\partial \xi} d\xi + \frac{\partial F}{\partial \eta} d\eta$ and the forms $d\xi, d\eta$ are simultaneously zero only at the origin, F has no critical points in $\mathcal{U}$ other than in p (the origin in $\mathcal{U}$.) ∎

<u>Lemma</u>: The locus $F^{-1}(-\infty, c-\epsilon\rangle$ is a deformation retract of $M^{c+\epsilon}$.

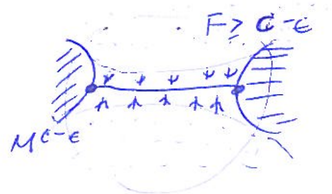
<u>Pf</u>: The first lemma above together with $F \leq f$ implies $F^{-1}\langle c-\epsilon, c+\epsilon\rangle \subseteq f^{-1}\langle c-\epsilon, c+\epsilon\rangle$. Therefore $F^{-1}\langle c-\epsilon, c+\epsilon\rangle$ is compact and can contain no critical pts of F except (possibly) p. We have $F(p) = c - \mu(0) < c - \epsilon$, hence $F^{-1}\langle c-\epsilon, c+\epsilon\rangle$ contains no critical points. The claim follows by the theorem on deformation retract and homotopy equivalence. ∎

We denote the locus $F^{-1}(-\infty, c-\epsilon\rangle$ by $M^{c-\epsilon} \cup H$, where $H = \overline{F^{-1}(-\infty, c-\epsilon) \setminus M^{c-\epsilon}}$

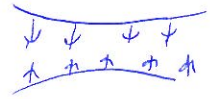
($H \equiv$ a handle attached to $M^{c-\epsilon}$, $M^{c-\epsilon} \cup H \overset{\text{diffeom}}{\underset{\text{man. with bound.}}{\simeq}} M^{c+\epsilon}$)

We introduce the cell e^λ, given by $\forall q \in \mathcal{U} : \xi(q) \leq \epsilon,\ \eta(q) = 0$, so $e^\lambda \subseteq H$: since $\frac{\partial F}{\partial \xi} < 0$, $F(q) \leq F(p) < c - \epsilon$; but $f(q) \geq c - \epsilon$ for $q \in e^\lambda$.

  ... $M^{c-\epsilon}$

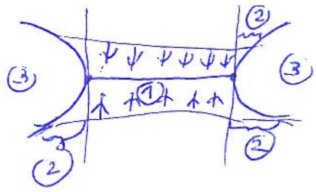


... $F^{-1}\langle c-\epsilon, c+\epsilon\rangle$

 ... H the handle

Lemma: $M^{c-\epsilon} \cup e^{\lambda}$ is a deformation retract of $M^{c-\epsilon} \cup H$.

Pf: We follow in the construction of def. retr. $r_t : M^{c-\epsilon} \cup e^{\lambda} \to M^{c-\epsilon} \cup H$ the picture with three regions: ①, ②, ③



Let r_t be identity out of $U \ni p$, and defined within U as:

① $\xi \leq \epsilon$, r_t is defined by $(u^1, \dots, u^m) \mapsto (u^1, \dots, u^\lambda, t u^{\lambda+1}, \dots, t u^m)$
$\Rightarrow$ r_1 is identity, $Im(r_0) \subseteq e^{\lambda}$, $r_t : F^{-1}(-\infty, c-\epsilon\rangle \to F^{-1}(-\infty, c-\epsilon\rangle$ $\left(\Leftarrow \frac{\partial F}{\partial \eta} > 0.\right)$

② $\epsilon \leq \xi \leq \eta + \epsilon$, r_t is defined $(u^1, \dots, u^m) \mapsto (u^1, \dots, u^\lambda, s_t u^{\lambda+1}, \dots, s_t u^m)$
with $s_t \in [0,1]$: $s_t = t + (1-t)\left(\frac{\xi - \epsilon}{\eta}\right)^{\frac{1}{2}}$.
$\Rightarrow$ r_1 is the identity, r_0 maps the locus ② to the hypersurface $f^{-1}(c-\epsilon)$;
$s_t u^i$ are continuous for $\xi \to \epsilon$, $\eta \to 0$, equals to ① for $\xi = \epsilon$.

③ $\eta + \epsilon \leq \xi$ $(\subseteq M^{c-\epsilon})$, r_t is the identity; coincides with ② for $\xi = \eta + \epsilon$.

This completes the proof, and together with the previous lemma proves second theorem of this section. ∎

Remark: M^c is a def. retract of $F^{-1}(-\infty, c\rangle$, which is a def. retr. of $M^{c+\epsilon}$ $\Rightarrow$ $M^{c-\epsilon} \cup e^{\lambda}$ is a def. retr. of M^c.

Theorem: If f is a smooth fun on M with no degenerate critical pts, and if $\forall$ M^a is compact, then M has the homotopy type of a CW-complex, with one cell of dimension λ $\forall$ critical pt of index λ.

(31) The proof is based on the following lemmas: $\quad X \underset{\varphi}{\cup} Y = X \times Y /\sim_\varphi$ for $\varphi : Y \to X$

<u>Lemma</u> (Whitehead): $\varphi_0, \varphi_1 : \partial e^\lambda \to X$ homotopic maps. Then $Id : X \to X$ extends to a homotopy equivalence $k : X \underset{\varphi_0}{\cup} e^\lambda \to X \underset{\varphi_1}{\cup} e^\lambda$.

Pf: Define k by

$$k(x) = x \qquad x \in X,$$
$$k(tu) = 2tu \text{ for } 0 \le t \le \tfrac{1}{2},\ u \in \partial e^\lambda,$$
$$k(tu) = \varphi_{2-2t}(u) \text{ for } \tfrac{1}{2} \le t \le 1,\ u \in \partial e^\lambda.$$

φ_t ... homotopy between φ_0, φ_1, tu ... t-multiple of the unit vector $u \in \partial e^\lambda$;

Analogously for $l : X \underset{\varphi_1}{\cup} e^\lambda \to X \underset{\varphi_0}{\cup} e^\lambda$, then we see $k \circ l$ and $l \circ k$ are homotopic to the identity map $\Rightarrow$ k is homot. equiv. ∎

<u>Lemma</u>: Let $\varphi : \partial e^\lambda \to X$ be the attaching map. Any homotopy equivalence $f : X \to Y$ extends to a homotopy equivalence $F : X \cup_\varphi e^\lambda \to \cancel{Y} \underset{f \circ \varphi}{\cup} e^\lambda$.

Pf: Define F by $\quad F/_X = f$, $F/_{e^\lambda} = Id$.

Let $g : Y \to X$ be a homot. inverse of f, define
$G : Y \underset{f \circ \varphi}{\cup} e^\lambda \to X \underset{g \circ f \circ \varphi}{\cup} e^\lambda$ by $G/_Y = g$, $G/_{e^\lambda} = Id$.

Since $g \circ f \circ \varphi$ is homot. to φ, the previous lemma gives homot. equiv. $\quad k : X \underset{g \circ f \circ \varphi}{\cup} e^\lambda \to X \cup_\varphi e^\lambda$;

we prove $\quad k \circ G \circ F : X \cup_\varphi e^\lambda \to X \cup_\varphi e^\lambda$ is homotop. to the identity map.

Let h_t : a homot. $gf \sim Id$, and notice

$$(k \circ G \circ F)(x) = (gf)(x) \qquad \text{for } x \in X,$$
$$(k \circ G \circ F)(tu) = 2tu \qquad \text{for } 0 \le t \le \tfrac{1}{2},\ u \in \partial e^\lambda,$$
$$(k \circ G \circ F)(tu) = \varphi_{2-2t}(u) \quad \text{for } \tfrac{1}{2} \le t \le 1,\ u \in \partial e^\lambda.$$

The required homotopy $\quad q_\tau : X \cup_\varphi e^\lambda \to X \underset{\varphi}{\cup} e^\lambda$

is defined by

$$q_\tau(x) = h_\tau(x) \qquad \text{for } x \in X,$$
$$q_\tau(tu) = \frac{2}{1+\tau}\, tu \quad \text{for } 0 \le t \le \frac{1+\tau}{2},\ u \in \partial e^\lambda,$$
$$q_\tau(tu) = (h_{2-2t+\tau}\, \varphi)(u) \quad \text{for } \frac{1+\tau}{2} \le t \le 1,\ u \in \partial e^\lambda.$$

 $\Rightarrow$ F has a left homot. inverse. F is a homotopy equivalence, because

<u>Claim</u>: If a map F has a left homotopy inverse L and a right homotopy inverse R, then F is a homot. equivalence (then R resp. L is a 2-sided homot. inverse.)

<u>Pf</u>: $\left.\begin{array}{l} LF \simeq Id \\ FR \simeq Id \end{array}\right\} \Rightarrow L \simeq L(FR) = (LF)R \simeq R \Rightarrow RF \simeq LF \simeq Id \Rightarrow$ R is 2-sided inverse

The relation $kGF \simeq Id$ implies - F has a left homot. inverse,
- analogously, G has a left homot. inverse.

Then
- since $k(GF) \simeq Id$, and k has a left inverse, it follows $(GF)k \simeq Id$.
- since $G(Fk) \simeq Id$, and G is known to have a left inverse, it follows $(Fk)G \simeq Id$.
- since $F(kG) \simeq Id$, and F has kG as a left inverse as well, it follows F is a homotop. equiv.

The proof of lemma is complete. ▨

<u>Pf (of theorem)</u>: Assume $c_1 < c_2 < c_3 < \dots$ critical values of $f: M \to \mathbb{R}$.
M^a is compact $\Rightarrow$ $\{c_i\}_i$ has no limit point, M^a is empty for $a < c_1$.
Assume $a \neq c_i\ \forall i$, and M^a is of homotopy type of CW-complex. For c smallest in $\{c_i\}_i$ s.t. $c_i > a$. By previous Theorems, $M^{c+\epsilon}$ has homot. type $M^{c-\epsilon} \cup_{\varphi_1} e^{\lambda_1} \cup \dots \cup_{\varphi_{j(c)}} e^{\lambda_{j(c)}}$ for ϵ small enough, and $h: M^{c-\epsilon} \to M^a$ is a homotop. equivalence (M^a is a CW-complex, i.e. $\exists$ homot. equiv. $h': M^a \to K$ for CW-complex K.)

Then $h' \circ h \circ \varphi_j$ is homotopic by cell. approx. to a map $\psi_j : \partial e^{\lambda_j} \to (\lambda_j - 1)$-skeleton of K, $j = 1, \dots, j(c)$,

so that $K \cup_{\psi_1} e^{\lambda_1} \cup \dots \cup_{\psi_{j(c)}} e^{\lambda_{j(c)}}$ is a CW-complex homot. equiv to $M^{c+\epsilon}$.

By induction, $M^{a'}$ is of homot. type of a CW-complex. If M is compact, or M non-compact but $\forall$ crit. pts lie in one of the compact sets M^a, the proof is complete.

If there are infinitely many critical points, then the above gives us infinite chain of homotopy equivalences:

$$\begin{Bmatrix} M^{a_i} \\ \downarrow \\ K_i \end{Bmatrix}_i \quad \begin{matrix} M^{a_1} & \subseteq & M^{a_2} & \subseteq & M^{a_3} & \subseteq & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ K_1 & \subseteq & K_2 & \subseteq & K_3 & \subseteq & \dots \end{matrix} \quad \Big\}\text{ the next extending the previous one.}$$

direct system

$K := \bigcup_{i=1}^{\infty} K_i$ with direct limit topology, $g: M \to K$ the limit map into universal object

Then g induces isomorphism on the total homotopy groups.
By Whitehead theorem, g is homotopy equivalence.
The proof is complete. ∎

Examples/Applications of Morse theory

Theorem (Reeb) If M is a compact manifold, f a smooth fcion on M with only 2 critical pts (both non-degenerate). Then M is homeomorphic to sphere of dimension m.

Pf: The two points have to be minimum/maximum pts. Say $f(p)=0$ is the minimum resp. $f(q)=1$ maximum. For $\epsilon>0$ small enough, $M^\epsilon = f^{-1}\langle 0,\epsilon\rangle$ and $f^{-1}\langle 1-\epsilon, 1\rangle$ are closed m-cells. Because $M^\epsilon \underset{\text{homeom.}}{\simeq} M^{1-\epsilon}$, $M = M^{1-\epsilon} \cup f^{-1}\langle 1-\epsilon, 1\rangle$ glued along their common boundary. Now one constructs a homeom. of M and S^m. ∎

Let $\mathbb{CP}_m$ be a complex projective space: $(m+1)$-tuples $(z_0, \dots, z_m) \in \mathbb{C}^{m+1}$ $\sum_{j=0}^{m} |z_j|^2 = 1$; a class of $(z_0, \dots, z_m)$ is denoted by $(z_0 : z_1 : \dots : z_m)$.

Define a smooth fcion $f: \mathbb{CP}_m \to \mathbb{R}$

$$(z_0 : \dots : z_m) \mapsto \sum_{j=0}^{m} c_j |z_j|^2, \quad c_0, \dots, c_m \in \mathbb{R}, \quad c_i \neq c_j \ \forall i,j$$

Critical pts of f: $U_0 = \{(z_0 : \dots : z_m) \mid z_0 \neq 0\}$, set $|z_0| \frac{z_j}{z_0} = x_j + i y_j$

$\{x_1, y_1, \dots, x_m, y_m\}: U_0 \to \mathbb{R}$ coordinate chart, Image $\underset{\text{diff.}}{\simeq}$ open ball in $\mathbb{R}^{2m}$.

$|z_j|^2 = x_j^2 + y_j^2$, $|z_0|^2 = 1 - \sum (x_j^2 + y_j^2)$, so $f\Big|_{U_0} = c_0 + \sum_{j=1}^{m} (c_j - c_0)(x_j^2 + y_j^2)$.

So $p_0 = (1:0:\ldots:0)$ is the only critical point of $f|_{U_0}$; p_0 is non-degenerate, index is equal to twice the number of j : $c_j < c_0$.

Analogously: $U_1, \ldots, U_m$ with (the only) critical points
$p_1 = (0:1:0:\ldots:0), \ldots, p_m = (0:0:\ldots:0:1)$.

The index of f at p_k $(k = 0, \ldots, m)$ is the twice the number of j : $c_j < c_k$. Consequently, $\forall$ possible even index between $0, \ldots, 2m$ occurs just one. By the main theorem, $\mathbb{CP}_m$ has the homotopy type of a CW-complex : $e^0 \cup e^2 \cup e^4 \cup \ldots \cup e^{2m}$.

Thus $H_i(\mathbb{CP}_m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 2, \ldots, 2m, \\ 0 & \text{otherwise.} \end{cases}$

The $\boxed{\text{Morse inequalities}}$

Before the theorems of the previous part were known, the relationship $\{$topology of $M\} \leftrightarrow \{$critical pts of $f\}$ was described in terms of inequalities.

Let S be a fcion to a pair of topological spaces (certain manifolds) to $\mathbb{Z}$ (integers).
S is subadditive if whenever $X \supseteq Y \supseteq Z$ we have $S(X,Z) \leq S(X,Y) + S(Y,Z)$.
If the equality holds, S is called additive.

Example: $\mathbb{F}$... a field $b_\lambda(X,Y) = \lambda$-th Betti number of (X,Y)
(= rank$/\mathbb{F}$ of $H_\lambda(X,Y;\mathbb{F})$)

A SES for (X,Y,Z) implies
$$\cdots \to H_\lambda(Y,Z,\mathbb{F}) \to H_\lambda(X,Z,\mathbb{F}) \to H_\lambda(X,Y) \to H_{\lambda-1}(Y,Z) \to \cdots$$

that the Euler charact. $\chi(X,Y) := \sum (-1)^\lambda H_\lambda(X,Y,\mathbb{F})$ is additive.

<u>Lemma</u>: Let S be subadditive and $X_0 \subseteq X_1 \subseteq \dots \subseteq X_m$. Then

$$S(X_m, X_0) \leq \sum_{i=1}^{m} S(X_i, X_{i-1}),$$

and if S is additive then equality holds.

<u>Pf</u>: By induction. ☒

For M compact, f smooth on M with isolated non-degenerate critical points. Let $a_1 < \dots < a_k$ be such that M^{a_i} contains just i-critical pts, $M^{a_k} = M$. Then

$$\begin{aligned} H_*(M^{a_i}, M^{a_i-1}) &= H_*(M^{a_{i-1}} \cup e^{\lambda_i}, M^{a_i-1}) \\ &\overset{\text{by excision}}{=} H_*(e^{\lambda_i}, \partial e^{\lambda_i}) \\ &= \begin{cases} \text{coefficient group in dim } \lambda_i, \\ 0 \quad \text{otherwise.} \end{cases} \end{aligned}$$

$\lambda_i \equiv$ index of the critical point

The application of this concept to $\emptyset = M^{a_0} \subseteq \dots \subseteq M^{a_k} = M$, $S = b_\lambda$, we have

$$b_\lambda(M) \leq \sum_{j=1}^{k} \cancel{P} b_\lambda(M^{a_j}, M^{a_j-1}) = C_\lambda, \quad C_\lambda = \text{\# critical points of index } \lambda$$

and with $S = \chi$, we get

$$\chi(M) = \sum_{j=1}^{k} \chi(M^{a_j}, M^{a_j-1}) = C_0 - C_1 + C_2 - \dots \pm C_m.$$

We proved

<u>Theorem</u> (Weak Morse inequalities) If C_λ denotes # of critical points of index λ on the compact manifold M, then

$$b_\lambda(M) \leq C_\lambda, \quad \text{and} \quad \sum (-1)^\lambda b_\lambda(M) = \sum (-1)^\lambda C_\lambda.$$

In fact, the function S_λ defined by

$$S_\lambda(X,Y) = b_\lambda(X,Y) - b_{\lambda-1}(X,Y) + b_{\lambda-2}(X,Y) - \dots \pm b_0(X,Y)$$

is subadditive $\forall \lambda$. It satisfies the Morse inequalities:

$$S_\lambda(M) \leq \sum_{j=1}^{k} S_\lambda(M^{a_j}, M^{a_j-1}) = C_\lambda - C_{\lambda-1} + \dots \pm C_0, \quad \text{or}$$

$$b_\lambda(M) - b_{\lambda-1}(M) + \dots \pm b_0(M) \leq C_\lambda - C_{\lambda-1} + \dots \pm C_0.$$