# Algebraic Topology

## Lecture Notes for Algebraic Topology MAT008

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# Contents

# 1. Advanced Homology Theory

## 1.1 Direct and Inverse Limits

**1 Definition.** *A* **partial order** *is a relation $\leq \subset S \times S$ that satisfies the following three properties:*

*(1) Reflexivity: $a \leq a$ for all $a \in S$*
*(2) Antisymmetry: If $a \leq b$ and $b \leq a$ for any $a, b \in S$, then $a = b$*
*(3) Transitivity: If $a \leq b$ and $b \leq c$ for any $a, b, c \in S$, then $a \leq c$*

*A* **total order** *is a partial order that satisfies the fourth property*

*(4) Comparability: For all $a, b \in S$ it holds either $a \leq b$ or $b \leq a$.*

*A set with partial order is called a* **partially ordered set** *- shortly* **poset**. *A set with total order is called a* **totally ordered set**.

**2 Lemma.** (Zorn) *Let $S$ be a partially ordered set. Suppose that every chain in $A$ has an upper bound. Then $S$ has a maximal element $a$, in the sense that for all $b \in S$, $b \not> a$.*

**3 Definition.** *A* **directed set** *is a partially ordered set $(S, \leq)$ such that whenever $a, b \in S$ there is $c \in S$ such that $a \leq c$ and $b \leq c$.*

*Note: Many authors do not require $\leq$ to be antisymmetric, so that $(S, \leq)$ is only a suborder with the above property.*

*A* **direct system** *of sets $\{A^\alpha, f_\alpha^\beta\}_{\alpha,\beta \in S}$ consists of*

*(1) collection of sets $\{A^\alpha\}_{\alpha \in S}$*
*(2) collection of functions $\{f_\alpha^\beta : A^\alpha \to A^\beta\}_{\alpha,\beta \in S : \alpha \leq \beta}$ such that*
  *(a) $f_\alpha^\alpha = \mathrm{Id}_{A^\alpha} : A^\alpha \hookrightarrow A^\alpha$ for all $\alpha \in S$,*
  *(b) $f_\alpha^\gamma = f_\beta^\gamma \circ f_\alpha^\beta : A^\alpha \to A^\gamma$ for all $\alpha, \beta, \gamma \in S : \alpha \leq \beta \leq \gamma$.*

*A* **direct limit** *of the direct system, denoted by $\varinjlim A^\alpha{}_{\alpha \in S}$, is a set of equivalence classes of elements of disjoint union $\bigcup_{\alpha \in S} A^\alpha := \bigcup_{\alpha \in S} (\alpha \times A^\alpha)$ w.r. to equivalence relation*

$$a^\alpha \sim a^\beta \iff \exists \gamma \in S : \alpha, \beta \leq \gamma \text{ and } f_\alpha^\gamma a^\alpha = f_\beta^\gamma a^\beta.$$

*For each $\alpha \in S$ there evidently exists a map $i_\alpha : A^\alpha \to \varinjlim \{A^\alpha\}_{\alpha \in S}$ and if $\alpha, \beta \in S$ : $\alpha \leq \beta$ then $i_\alpha = i_\beta \circ f_\alpha^\beta$.*

**4 Observation.** A **universal property** holds for direct limit.

For given direct system of sets $\{A^\alpha, f_\alpha^\beta\}_{\alpha,\beta \in S}$ and any set $B$ together with functions $g_\alpha : A^\alpha \to B$ for all $\alpha \in S$ such that $g_\alpha = g_\beta \circ f_\alpha^\beta$ for $\alpha \leq \beta$, there exists unique function $g : \varinjlim \{A^\alpha\}_{\alpha \in S} \to B$ such that $g \circ i_\alpha = g_\alpha$ for all $\alpha \in S$.

The mapping $g$ is a bijection iff

(1) $B = \bigcup g_\alpha(A^\alpha)$

(2) $g_\alpha a^\alpha = g_\beta a^\beta$ iff $\exists \gamma : \alpha, \beta \leq \gamma$ and $f_\alpha^\gamma a^\alpha = f_\beta^\gamma a^\beta$

**5 Aplication.** Let us consider the direct limit $\varinjlim\{G_i\}$ of a sequence of abelian groups

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} G_3 \xrightarrow{\alpha_3} \cdots$$

i.e $A^\alpha = G$, $i \in \mathbb{N}$ where the maps $\{\alpha_i\}_{i\in\mathbb{N}}$ are homomorphisms of abelian groups and

$$f_i^j \stackrel{\text{def}}{=} \alpha_{j-1} \circ \cdots \circ \alpha_i : g_i \in G_i \mapsto \alpha_{j-1} \circ \cdots \circ \alpha_i(g_i) \in G_j$$

The direct limit is a quotient of $\bigoplus\limits_i G_i$ by a subgroup of elements of the form

$$(g_1, g_2 - \alpha_1(g_1), g_3 - \alpha_2(g_2), \dots).$$

**6 Example.** Let $p \in \mathbb{N}$ be a prime. Consider

$$\mathbb{Z} \xrightarrow{p\times} \mathbb{Z} \xrightarrow{p\times} \mathbb{Z} \xrightarrow{p\times} \dots \quad \text{(all maps are multiplication by p)}$$

Then $\varinjlim\{G_i\} = \varinjlim\{\mathbb{Z}\} = \mathbb{Z}[\frac{1}{p}]$, i.e. subgroup of $\mathbb{Q}$.

An element of $\varinjlim\{G_i\}$ is a finite series $a_0 + \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots \frac{a_n}{p^n}$, $a_0, \dots, a_n \in \mathbb{Z}$. If $a_1 = pa_0'$ then we have again $(a_0 + a_0') + \frac{0}{p} + \frac{a_2}{p^2} + \cdots \frac{a_n}{p^n}$, $a_i \in G_i$.

**7 Example.** Let $p \in \mathbb{N}$ be a prime. Consider the sequence of injections

$$\mathbb{Z}_p \xrightarrow{p\times} \mathbb{Z}_{p^2} \xrightarrow{p\times} \mathbb{Z}_{p^3} \xrightarrow{p\times} \dots \quad \text{(all maps are multiplication by p)}$$

Realizing $\mathbb{Z}_p$ as a cyclotomic units of order $p$ (i.e. p-th root of unity) we can identify the direct limit $\mathbb{Z}_{p^\infty}$ with an obvious isomorphism $\varinjlim\{G_i\} \simeq \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$.

**8 Definition.** *The* **mapping telescope** *of the sequence*

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

*for $\{f_i\}_{i\in\mathbb{N}_0}$ homeomorphisms of corresponding topological spaces $\{X_i\}_{i\in\mathbb{N}_0}$ ($X_i$ is a simplicial complex of finite type or CW and $\{f_i\}_{i\in\mathbb{N}_0}$ are simplicial or cellular maps for all $i \in \mathbb{N}_0$), is defined as a quotient space*

$$T \stackrel{\text{def}}{=} \coprod_{i\in\mathbb{N}_0} (X_i \times [i, i+1])/\sim,$$

*where*

$$\sim \stackrel{\text{def}}{=} (x_i, i+1) \in X_i \times [i, i+1] \sim (f_i(x_i), i+1) \in X_{i+1} \times [i+1, i+2],$$

*equipped with quotient topology w.r. to* $\sim$ .

Let $T_i \overset{\text{def}}{=}$ be a union of first i-mapping cylinders. Then $T_i$ deformation retracts on $X_i$ (because $T_1$ def. retracts on $X_1$, $T_2$ def. retracts on $X_2$, etc.). In the case $f_i$ are simplicial (or cellular) maps, the mapping cylinder is simplicial complex (or CW) of finite type. In other words, we have $T_i \simeq X_i$ ($\simeq$ means homotopy equivalence). Then we can write

$$H_n(T) = \varinjlim H_n(T_i) \simeq \varinjlim H_n(X_i) \quad \text{i.e.} \quad G_i = H_n(X_i), f_i^j = (f_i)_*.$$

**9 Definition.** *An* **inverse system** *of sets* $\{A^\alpha, f_\alpha^\beta\}_{\alpha,\beta\in S}$ *(S is a directed set) consists of*

(1) *collection of sets* $\{A^\alpha\}_{\alpha\in S}$

(2) *collection of functions* $\{f_\alpha^\beta : A^\beta \to A^\alpha\}_{\alpha,\beta\in S:\alpha\leq\beta}$ *such that*

  (a) $f_\alpha^\alpha = \mathrm{Id}_{A^\alpha} : A^\alpha \hookrightarrow A^\alpha$ *for all* $\alpha \in S$,

  (b) $f_\alpha^\gamma = f_\beta^\alpha \circ f_\gamma^\beta : A^\gamma \to A^\alpha$ *for all* $\alpha, \beta, \gamma \in S : \alpha \leq \beta \leq \gamma$.

*An* **inverse limit** *of the inverse system, denoted by* $\varprojlim A^\alpha{}_{\alpha\in S}$, *is a subset of direct product* $\prod_{\alpha\in S} A^\alpha$ *consisting of* $a_\alpha$ *such that if* $\alpha \leq \beta$ *then* $a_\alpha = f_\alpha^\beta a_\beta$ *for some* $a_\beta \in A_\beta$.

*For each* $\alpha \in S$ *there evidently exists a map* $\pi_\alpha : \varprojlim\{A^\alpha\}_{\alpha\in S} \to A^\alpha$ *and if* $\alpha, \beta \in S$ : $\alpha \leq \beta$ *then* $\pi_\alpha = f_\alpha^\beta \circ \pi_\beta$.

**10 Observation.** A **universal property** holds for inverse limit.

For given direct system of sets $\{A^\alpha, f_\alpha^\beta\}_{\alpha,\beta\in S}$ and any set $B$ together with functions $g_\alpha : B \to A^\alpha$ for all $\alpha \in S$ such that $g_\alpha = f_\alpha^\beta \circ g_\beta$ for $\alpha \leq \beta$, there exists unique function $g : B \to \varprojlim\{A^\alpha\}_{\alpha\in S}$ such that $\pi_\alpha \circ g = g_\alpha$ for all $\alpha \in S$.

The mapping $g$ is a bijection iff

(1) $g_\alpha(b) = g_\alpha(b') \quad \forall\alpha \in S;\ b, b' \in B \ \Rightarrow\ b = b'$

(2) for any $(a_\alpha)_{\alpha\in S} \in \prod_{\alpha\in S}$ such that $a_\alpha = f_\alpha^\beta a_\beta : \alpha \leq \beta$ there exists $b \in B$ such that $g_\alpha(b) = a_\alpha$ for all $\alpha \in S$.

**11 Aplication.** Let us consider the inverse limit $\varprojlim\{G_i\}$ of a sequence of abelian groups

$$\cdots \xrightarrow{f_3} G_2 \xrightarrow{f_2} G_1 \xrightarrow{f_1} G_0$$

i.e $A^\alpha = G$, $i \in \mathbb{N}_0$ where the maps $\{f_i\}_{i\in\mathbb{N}_0}$ are homomorphisms of abelian groups and

$$f_i^j \overset{\text{def}}{=} f_{i-1} \circ \cdots \circ f_j : g_j \in G_j \mapsto f_{i-1} \circ \cdots \circ f_j(g_j) \in G_i$$

The inverse limit is a subgroup of $\prod_i G_i$ consisting of elements of the form $(\ldots, g_2, g_1, g_0)$, such that $f_i(g_i) = g_{i-1}$ for all $i \in \mathbb{N}_0$.

**12 Example.** Let $p \in \mathbb{N}$ be a prime. Consider

$$\xrightarrow{\text{mod } p^3} \mathbb{Z}_{p^3} \xrightarrow{\text{mod } p^2} \mathbb{Z}_{p^2} \xrightarrow{\text{mod } p} \mathbb{Z}_p \xrightarrow{\text{mod } p} \mathbb{Z} \quad \text{(all maps are modulo by } p^i\text{)}.$$

Then $\varprojlim\{G_i\} = \varprojlim\{\mathbb{Z}_{p^i}\}$ is denoted by $\widehat{\mathbb{Z}}_p$ and called **p-adic integers**.

$\widehat{\mathbb{Z}}_p$ consists of elements of the form $(\ldots, a_2, a_1, a_0)$, $a_i \in \mathbb{Z}_{p^i}$ such that $a_{i+1} \equiv a_i \mod p^i$ for all $i \in \mathbb{N}_0$. Evidently this set is infinite since for every $a_i$ there exist $p$ choices for $a_{i+1}$. There is also a natural inclusion $\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}}_p$ given by $n \mapsto (\ldots, n, n, n)$ — a constat sequence.

Another way of realizing $\widehat{\mathbb{Z}}_p$ : An element $(\ldots, a_2, a_1, a_0) \in \mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}}_p$ has a unique representation such that $0 \leq a_i \leq p^i$ for all $i \in \mathbb{N}_0$. We can write $a_i$ in a p-adic factorization i.e $a_i = b_{i-1}p^{i-1} + \cdots + b_1 p + b_0 : 0 \leq b_i \leq p^i$. Because $a_{i+1} \equiv a_i \mod p^i$, $b_j$ depends only on $(\ldots, a_2, a_1, a_0) \in \mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}}_p$. Therefore, elements of $\widehat{\mathbb{Z}}_p$ : are in 1-1 correspondence with their p-adic coefficients $\ldots, b_2, b_1, B_0$. In this case, group $\mathbb{Z}$ is realized as a subgroup of finite length elements $b_n, \ldots, b_2, b_1, b_0$.

**13 Theorem.** *The homology functor commutes with direct limits.*

*Proof.* Let $\{C^\alpha, \tau_\alpha^\beta\}_{\alpha,\beta\in S}$ be a direct system of chain complexes and let $\{C, i_\alpha\}_{\alpha\in S}$ be the direct limit of this system together with the functions $i_\alpha : C^\alpha \to C = \varinjlim\{C^\alpha\}_{\alpha\in S}$. Then $\{H(C^\alpha), (\tau_\alpha^\beta)_*\}_{\alpha,\beta\in S}$ is a direct system of graded groups. Set $\psi : \varinjlim\{H(C^\alpha), (\tau_\alpha^\beta)_*\} \to \{H(C), (i_\alpha)_*\}_{\alpha\in S}$.

(1) surjectivity: $\{z \in H_q(C)\} \overset{?}{\Rightarrow} (i_\beta)_*\{\tau_\alpha^\beta c^\alpha\} = \{z\}$.

Let $\{z\}$ be any element of $H_q(C)$. Then $z = i_\alpha c^\alpha$ for some $c^\alpha \in (C^\alpha)_q$. Since $0 = \partial_q z = \partial_q i_\alpha c^\alpha = i_\alpha \partial_q^\alpha c^\alpha$, there exists $\beta : \alpha \leq \beta$, such that $\tau_\alpha^\beta \partial_q^\alpha c^\alpha = 0$. Then $\partial_q^\beta \tau_\alpha^\beta c^\alpha = 0$, i.e. $\tau_\alpha^\beta c^\alpha \in Z_q(C^\beta)$. Moreover, $i_\beta \tau_\alpha^\beta c^\alpha = i_\alpha c_\alpha = z$, i.e. $(i_\beta)_*\{\tau_\alpha^\beta c^\alpha\} = \{z\}$.

(2) injectivity: $\{z^\alpha\} \in H_q(C^\alpha) : (i_\alpha)_*\{z^\alpha\} = 0 \overset{?}{\Rightarrow} \exists\gamma : \alpha \leq \gamma : \{z^\alpha\} \in \mathrm{Ker}(\tau_\alpha^\gamma)_*$.

Let $\{z^\alpha\} \in H_q(C^\alpha) : (i_\alpha)_*\{z^\alpha\} = 0$. Then $i_\alpha z^\alpha = \partial_{q+1} c$ for some $c \in C_{q+1}$. Because $c = i_\beta c^\beta$ for some $\beta$, we have $i_\alpha z^\alpha = \partial_{q+1} i_\beta c^\beta = i_\beta \partial_{q+1}^\beta c^\beta$. Choose $\gamma'$ s.t. $\alpha, \beta \leq \gamma$. Then $i_{\gamma'}(\tau_\alpha^{\gamma'} z^\alpha - \tau_\beta^{\gamma'} \partial_{q+1}^\beta c^\beta) = i_\alpha z^\alpha - i_\beta \partial_{q+1}^\beta c^\beta = 0$. Hence, there exists $\gamma : \gamma \leq \gamma$ such that $\tau_{\gamma'}^\gamma(\tau_\alpha^{\gamma'} z^\alpha - \tau_\beta^{\gamma'} \partial_{q+1}^\beta c^\beta) = 0$. Then $\tau_\alpha^\gamma z^\alpha - \tau_\beta^\gamma \partial_{q+1}^\beta c^\beta = 0$. It implies $\tau_\alpha^\gamma z^\alpha = \tau_\beta^\gamma \partial_{q+1}^\beta c^\beta$. Thus, $(\tau_\alpha^\gamma)_*\{z^\alpha\} = 0$.

**14 Example.** In general, the homology functor doesn't commutes with inverse limits. This follows from the following example. Let us consider chain complexes $C_n : n \in \mathbb{N}_0$ s.t.

$$(C_n)_q = \begin{cases} \mathbb{Z}, & q = 0, 1, \\ \emptyset, & \text{otherwise}, \end{cases}$$

where the non trivial part is $\mathbb{Z} \xrightarrow{2\times} \mathbb{Z}$ for all of them. Let $\tau_n : C_{n+1} \to C_n$ be a chain map defined on nontrivial part by $\mathbb{Z} \xrightarrow{3\times} \mathbb{Z}$ for all $n \in \mathbb{N}_0$.

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d, "3\times"] & {} \arrow[d, "3\times"] & {} \arrow[d] & \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & (\mathbb{Z})_{n+1} \arrow[r, "2\times"] \arrow[d, "3\times"] & (\mathbb{Z})_{n+1} \arrow[r] \arrow[d, "3\times"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & (\mathbb{Z})_{n} \arrow[r, "2\times"] \arrow[d, "3\times"] & (\mathbb{Z})_{n} \arrow[r] \arrow[d, "3\times"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & (\mathbb{Z})_{n-1} \arrow[r, "2\times"] \arrow[d, "3\times"] & (\mathbb{Z})_{n-1} \arrow[r] \arrow[d, "3\times"] & 0 \arrow[r] \arrow[d] & \cdots \\
 & {} & {} & {} & {} &
\end{tikzcd}$$

Then for $n \leq m$ define $\tau_n^m : C_m \to C_n$ by $\tau_n^m \overset{\text{def}}{=} \tau_n \circ \tau_{n+1} \circ \cdots \circ \tau_{m-1}$. So, $\{(C_n)_q, \tau_n^m\}_{n \in \mathbb{N}_0}$ is an inverse system which is non-trivial for $q = 1, 0$.

$$\ldots \xrightarrow{3\times} \mathbb{Z} \xrightarrow{3\times} \mathbb{Z} \xrightarrow{3\times} \mathbb{Z}$$

Evidently, the inverse limit of this system is an empty set since there is no natural number containing infinite power of three. Hence, $H_q(\varprojlim(C_n, \tau_n^m)) = \emptyset$ for all $q \in \mathbb{N}_0$. However, $H_0((C_n)) = \mathbb{Z}_2$ for all $n \in \mathbb{N}_0$, and the induced mapping $(\tau_n^m)_* : H_0(C_m) \to C_n$ is defined on nontrivial part by $\mathbb{Z}_2 \xrightarrow{3^{n-1} \equiv 1 \mod 2} \mathbb{Z}_2$, i.e it is an isomorphism. Hence, $\varprojlim\{H_0(C_n), (\tau_n^m)_*\} \simeq \mathbb{Z}_2$. Therefore, $\varprojlim\{H_0(C_n), (\tau_n^m)_*\} \neq H_0(\varprojlim((C_n)), \tau_n^m)$.

Because $\Delta^q$ is compact, every singular simplex $\sigma : \Delta^q \to X$ maps $\Delta^q$ into some compact subspace of $X$. Then if $\{X_\alpha\}$ is a collection of compact subsets of $X$ directed by inclusion, we have $\Delta(X) = \varinjlim \Delta(X_\alpha)$. Because $\varinjlim$ commutes with the singular homology functor we immediately get following:

**15 Theorem.** *The singular homology group of a space is isomorphic to the direct limit of singular homology groups of its compact subspaces.*

## 1.2 Application of Homology Theory in Functional Analysis

**16 Lemma.** *If $n \neq m$; $n, m \in \mathbb{N}_0$ then $\mathbb{S}^n$ and $\mathbb{S}^m$ are not of the same homotopy type.*

*Proof.* For such $m$ and $n$ we know $\widetilde{H}_n(\mathbb{S}^n) \neq 0$ and $\widetilde{H}_n(\mathbb{S}^m) = 0$. Therefore, $\mathbb{S}^n$ and $\mathbb{S}^m$ can not be of the same homotopy type.

**17 Theorem.** *If $n \neq m; n, m \in \mathbb{N}_0$ then $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.*

*Proof.* Assume they are. Fix a point $x \in \mathbb{R}^n$ and $f(x) \in \mathbb{R}^m$ where $f : \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism. Then $f/\mathbb{R}^n \setminus \{x\} : \mathbb{R}^n \setminus \{x\} \to \mathbb{R}^m \setminus \{f(x)\}$ is again a homeomorphism. But these spaces are deformation retractions $r_1 : \mathbb{R}^n \setminus \{x\} \to \mathbb{S}^{n-1}$ resp. $r_2 : \mathbb{R}^m \setminus \{f(x)\} \to \mathbb{S}^{m-1}$. It follows from the previous Lemma that $\widetilde{H}_{n-1}(\mathbb{R}^n \setminus \{x\}) \neq \widetilde{H}_{n-1}(\mathbb{R}^m \setminus \{f(x)\})$. Hence, there is no such homeomorphism $f$.

These two result were applications of the homology theory to the problem of classification of topological spaces. The next one concerns the problem of an extension.

**18 Theorem.** *Let $(X, A)$ be a topological pair such that $A$ is a retract of $X$. Then $H(X) \simeq H(A) \oplus H(X, A)$.*

*Proof.* Let $i : A \subset X$, $j : (X, \emptyset) \subset (X, A)$ and $r : X \to A$ be a retraction s.t. $ri = \mathrm{Id}_A$. Then the induced map on homology satisfy $r_* i_* = \mathrm{Id}_{H(A)}$. Since $i_*$ is a homomorphism, its image is a subgroup of $H(X)$. Surjectivity of $i_*$ follows from the fact that $i$ is an inclusion. Therefore, $i_*$ is a monomorphism of $H(A)$ on the first of two direct summands of $H(X)$. The second summand is $\mathrm{Ker}(r_*)$. From the fact $\mathrm{Ker}(i_*) = 0$ and the LES for (X,A):

$$\ldots \xrightarrow{\partial_*} H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial_*} H_{q-1}(A) \xrightarrow{i_*} \ldots$$

we get $\mathrm{Im}(\partial_*) = 0$. Thus $\partial_*$ is a trivial map. Than the exactness on relative group implies $j_*$ is an epimorphism. Since $\mathrm{Ker}(j_*) \simeq \mathrm{Im}(i_*)$, $j_*$ induces an isomorphism between $\mathrm{Ker}(r_*)$ and $H(X, A)$.

**19 Remark.** Note that the previous Theorem holds also in the case when $A$ is a weak retract of $X$. Note also that the only "non-trivial" part of the proof concerns the direct sum $\oplus$. The rest follows from the LES of $(X, A)$.

**20 Lemma.** *For $n \geq 0$; $n \in N_0$, $\mathbb{S}^n$ is not a (weak) retract of $\mathbb{E}^{n+1}$.*

*Proof.* We know $\widetilde{H}_n(\mathbb{S}^n) \neq 0$ and $\widetilde{H}_n(\mathbb{E}^{n+1}) = 0$. So, $\widetilde{H}_n(\mathbb{S}^n)$ is not a direct summand of $\widetilde{H}_n(\mathbb{E}^{n+1})$. The previous Theorem implies that there is no (weak) retract $r : \mathbb{E}^{n+1} \to \mathbb{S}^n$.

**21 Theorem.** (Brower fixed-point Theorem) *For $n \leq 0$ every continuous map from $\mathbb{E}^n$ to itself has a fixed point.*

*Proof.* In the case $n = 0$ there is nothing to prove. For $n \geq 0$ let $f : \mathbb{E}^n \to \mathbb{E}^n$ be a homomorphism. Assume it has no fixed point and define a map $g : \mathbb{E}^n \to \mathbb{S}^{n-1}$ given by $x \mapsto g(x)$ where $g(x) \stackrel{\text{def}}{=} z \in |\overleftrightarrow{xf(x)}| \cap \mathbb{S}^{n-1}$ s.t. $f(x)$ is closer to $z$ then $x$ is. The map $g$ is evidently continuous and so gives the retraction $r : \mathbb{E}^n \to \mathbb{S}^{n-1}$. This is however in contradiction with the previous Lemma.

**22 Definition.** *Let $C$ be a finitely generated graded group and let $h : C \to C$ ($\equiv \{h_q : C_q \to C_q\}$) be an endomorphism of $C$ of degree 0. The* **Lefschitz number** *$\lambda(h)$ is defined by $\lambda(h) \stackrel{\text{def}}{=} \sum\limits_q (-1)^q \,\mathrm{Tr}(h_q)$.*

**23 Theorem.** (Hopf trace formula) *Let $C$ be a finitely generated chain complex, and let $\tau : C \to C$ ($\equiv \{\tau_q : C_q \to C_q\}$) be a chain map. Then $\lambda(\tau) = \lambda(\tau_*)$.*

*Proof.* Analogous to the prof of the case $\{h_q = \mathrm{Id}_q\}$ corresponding to Euler characteristic.

**24 Definition.** *Let topological space $X$ has finitely generated homology, and let $f : X \to X$ be a homeomorphism. The Lefschitz number $\lambda(f)$ is defined to be the Lefschitz number of the endomorphism $f_* : H(X) \to H(X)$ induced by $f$. It counts (in a graded sense) the algebraic number of fixed homology classes of $f_*$.*

**25 Theorem.** *Let $X$ be a compact polyhedron and let $f : X \to Y$ be a map. If $\lambda(f) \neq 0$, then $f$ has a fixed point.*

*Proof.* Can be found in [AT], Chapter 4, Section 7, page 195.

**26 Theorem.** *Every continuous map from a compact contractible polyhedron to itself has a fixed point.*

*Proof.* $X$ is contractible gives $\widetilde{H}(X) = 0$. Then $f_*$ is the identity on $H_0 \simeq \mathbb{Z}$ for any $f : X \to X$, and we get $\lambda(f) = 1$.

**27 Example.** The last Theorem fails in the case of non-compact polyhedron. E.g. $\mathbb{R}$ is a contractible polyhedron, and any map $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x + a$; $a \in \mathbb{R}$ fails to have a fixed point.

**28 Definition.** *Given a continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$, the* **degree** *of $f$ is the unique integer $\deg(f)$ such that $f_*(z) = \deg(f) \cdot z$ for all $z \in \widetilde{H}_n(\mathbb{S}^n)$.*

**29 Observation.** The antipodal map of $\mathbb{S}^n$ has degree $(-1)^{n+1}$.

*Proof.* For any $f : \mathbb{S}^n \to \mathbb{S}^n$ we have $\lambda(f) = 1 + (-1)^n \deg(f)$. In the case $f$ is antipodal (i.e.$x \in \mathbb{S}^n \subset \mathbb{R}^{n+1} \mapsto -x \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$), $f$ has no fixed point. Thus $\lambda(f) = 0$ which gives $\deg(f) = (-1)^{n+1}$.

**30 Observation.** If $n \in 2\mathbb{N}$, then there is no continuous map $\mathbb{S}^n \to \mathbb{S}^n$ such that $x$ and $f(x)$ are orthogonal for all $x \in \mathbb{S}^n$.

*Proof.* Assume existence of such a $f$. Then we can define homotopy

$$F : f \simeq \mathrm{Id}_{\mathbb{S}^n} \text{ by } F(x,t) = \frac{(1-t)f(x) + tx}{||(1-t)f(x) + tx||}, \quad t \in [0,1];\ x, f(x) \in \mathbb{S}^n.$$

This is well defined because $< x, f(x) >= 0$ implies $||(1-t)f(x)+tx||^2 = (1-t)^2+t^2 \geq 0$ for $t \in [0,1]$. Hence $f$ must have a fixed point which is the contradiction to the orthogonality of $x$ and $f(x)$ for all $x \in \mathbb{S}^n$.

This observation is equivalent to the assertion that an even dimension sphere has no continuous tangent vector field which is nowhere zero on $\mathbb{S}^n$. Well know as a statement that one cannot comb the even dimensional sphere.

For an odd dimensional sphere, such a vector field exists. For example $f : \mathbb{S}^{2m-1} \subset \mathbb{R}^{2m} \to \mathbb{S}^{2m-1} \subset \mathbb{R}^{2m}$ defined by $(x_1, \ldots, x_{2m}) \to (-x_2, x_1, \ldots, -x_{2m}, x_{2m-1})$ is a continuous and $x$ is orthogonal to $f(x)$ for all $x \in \mathbb{S}^n$.

**31 Definition.** *A **flow** on $X$ is a homeomorphism $\varphi : \mathbb{R} \times X \to X$ such that*

*(1) $\varphi(t_1 + t_2, x) = \varphi(t_1, \varphi(t_2, x)$ for all $t_1, t_2 \in \mathbb{R}$, $x \in X$,*
*(2) $\varphi(0, x) = x$ for all $x \in X$.*

*For $t \in \mathbb{R}$ let $\varphi_t : X \to X$ be defined by $\varphi_t(x) = \varphi(t, x)$. Conditions 1) and 2) imply $\varphi_{-t} = (\varphi_t)^{-1}$ which gives $\varphi_t$ is a homeomorphism for every $t \in \mathbb{R}$. A **fixed point** of the flow is a point $x_0 \in X$ such that $\varphi_t(x_0) = x_0$ for all $t \in \mathbb{R}$.*

**32 Theorem.** *If $X$ is a compact polyhedron with $\chi(X) \neq 0$, then every flow $\varphi_t : X \to X$ has a fixed point.*

*Proof.* For every $t \in \mathbb{R}$ there exists homotopy $F_{t'} : \varphi_t \simeq \mathrm{Id}_X$ defined by $F(x, t') = \varphi((1 - t')t, x)$. Therefore, $\lambda(\varphi_t) = \lambda(\mathrm{Id}_X) = \chi(X) \neq 0$ which implies $\varphi_t$ has a fixed point for every $t \in \mathbb{R}$. For $n \geq 1$, let $A_n \stackrel{\text{def}}{=} \{x \in X : x \text{ is a fixed point of } \varphi_{\frac{1}{2^n}}\}$. Then $A_n$ is a closed set and $A_{n+1} \subset A_n$ because for $x_0$ fixed point of $\varphi_{\frac{1}{2^{n+1}}}$ we have $\varphi_{\frac{1}{2^n}}(x_0) = \varphi(\frac{1}{2^n}, x_0) = \varphi(\frac{1}{2^{n+1}}, \varphi(\frac{1}{2^{n+1}}, x_0)) = \varphi(\frac{1}{2^{n+1}}, x_0) = x_0$. Hence $\{A_n\}_{n \in \mathbb{N}}$ is decreasing sequence of nonempty closed subsets of the compact space $X$. Let $F = \bigcap A_n$. It follows from topology that $F$ is nonempty, and any point from $F$ is a fixed point of $\varphi_t$ for all $t$ of form $\frac{1}{2^n}$. The same considerations imply that $F$ is fixed under $\varphi_t$ for all dyadic rational $t = \frac{m}{2^n}$. Since dyadic rational numbers are dense in $\mathbb{R}$, it follows that every $x_0 \in F$ is fixed under any $\varphi_t$ for $t \in \mathbb{R}$.

We shall now discuss the separation properties for spheres. They are based on the following Lemma.

**33 Lemma.** *Let $A \subset \mathbb{S}^n$ be homeomorphic to $I^k \stackrel{\text{def}}{=} [0, 1] >^k$ for some $0 \leq k \leq n$. Then $\widetilde{H}(\mathbb{S}^n \setminus A) = 0$.*

*Proof.* Prove it by induction on $k$. If $k = 0$, the set $A$ is just a point and $\mathbb{S}^n \setminus A \stackrel{\text{homeom}}{\simeq} \mathbb{R}^n$. So, $\widetilde{H}(\mathbb{S}^n \setminus A) = 0$. Assume the result holds for $k \leq m$, and let $A$ be homeomorphic to $I^m$. Regard $A \stackrel{\text{homeom}}{\simeq} B \times I$, where $B \stackrel{\text{homeom}}{\simeq} I^{m-1}$ by a homeomorphism $h : B \times I \to A$. Let $A' = h(B \times [0, \frac{1}{2}])$ and $A'' = h(B \times [\frac{1}{2}, 1])$. Then $A = A' \cup A''$ and $A = A' \cap A''$ is homeomorphic to $B \times \{\frac{1}{2}\}$, i.e. to $B$ itself. By the induction assumption we have $\widetilde{H}(\mathbb{S}^n \setminus (A' \cap A'')) = 0$. Because $\mathbb{S}^n \setminus A'$ and $\mathbb{S}^n \setminus A''$ are open sets, they are excessive and LES of Mayer-Vietoris implies

$$i_* : \widetilde{H}_q(\mathbb{S}^n \setminus A) \simeq \widetilde{H}_q(\mathbb{S}^n \setminus A') \oplus \widetilde{H}_q(\mathbb{S}^n \setminus A'')$$

If $z \in \widetilde{H}_q(\mathbb{S}^n \setminus A)$ is a non-zero, then either $i'_* z \neq 0$ in $\widetilde{H}_q(\mathbb{S}^n \setminus A')$ or $i''_* z \neq 0$ in $\widetilde{H}_q(\mathbb{S}^n \setminus A'')$, where $i'_* : \mathbb{S}^n \setminus A \hookrightarrow \mathbb{S}^n \setminus A'$ and $i'_* : \mathbb{S}^n \setminus A \hookrightarrow Sn \setminus A'$. Assume $i'_* z \neq 0$. Now we can repeat the construction for $A'$. Hence, we get a sequence of sets $A \supset A_1 \supset A_2 \supset \ldots$ such that

(1) $i : \mathbb{S}^n \setminus A \hookrightarrow \mathbb{S}^n \setminus A_j$ maps $z$ into a non-zero element of $\widetilde{H}_q(\mathbb{S}^n \setminus A_j)$ for every $j$,
(2) $\bigcap A_i$ is homeomorphic to $I^{m-1}$.

Because for all compact subspaces $\mathbb{S}^n \setminus \bigcap A_i$ is contained in $\mathbb{S}^n \setminus A_j$ for some $j$, it follows from the Theorem 15 that $\varinjlim(\widetilde{H}_q(\mathbb{S}^n \setminus A_j)) \simeq \widetilde{H}_q(\mathbb{S}^n \setminus \bigcap A_i)$. But this is a contradiction since the element $z$ determines a non-zero element of $\varinjlim(\widetilde{H}_q(\mathbb{S}^n \setminus A_j))$ which should be empty by the induction assumption $\widetilde{H}_q(\mathbb{S}^n \setminus \bigcap A_i) = 0$.

From this we can get with some extra work:

**34 Theorem.** (Jordan-Brower separation Theorem) *An $(n-1)$– sphere embedded in $\mathbb{S}^n$ separates $\mathbb{S}^n$ into two components of which it is their common boundary.*

It is a generalization of Jordan separation Theorem for a Jordan curve in the complex plane.

**35 Theorem.** (Brower Theorem on the invariance of domains) *If $U, V \subset \mathbb{S}^n$ are homeomorphic subsets and $U$ is open in $\mathbb{S}^n$, then $V$ is also open in $\mathbb{S}^n$.*

## 1.3 Homology with Coefficients

In following text $R$ will always denote a commutative ring with a unit if not stated otherwise. And we will work mostly in category of $R$-modules and homomorphisms of $R$-modules where epimorphisms coincide with surjective maps and monomorphism with injective maps.

**36 Definition.** *A* **chain complex** $C \equiv \{C_q, \partial_q\}_{q \in \mathbb{Z}}$ *over $R$ consists of a sequence $C_q$ of $R$-mod's and homomorphisms $\partial_q : C_q \to C_{q-1}$ of $R$-mod's s.t. $\partial_{q-1}\partial_q = 0$ for all $q \in \mathbb{Z}$. The* **graded homology module** $H(C) = \{H_q(C) = \mathrm{Ker}\partial_q / \mathrm{Im}\partial_{q+1}\}$ *is also an $R$-mod.*

*All standard properties for $\mathbb{Z}$-mod carry over to the case of general $R$-mod. i.e for the category of SES's $0 \to C' \to C \to C'' \to 0$ there is a functorial* **connecting homomorphism** $\partial_* : H_q(C'') \to H_{q-1}(C')$ *and the corresponding functorial exact sequence*

$$\ldots \xrightarrow{\partial_*} H_q(C') \to H_q(C) \to H_q(C'') \xrightarrow{\partial_*} H_{q-1}(C') \to \ldots.$$

*If $G'$ is an $R$-mod, an* **augmentation** *of $C$ over $G'$ is an epimorphism $\varepsilon : C_0 \to G'$ s.t. $\varepsilon \circ \partial_1 = 0$. If $C \equiv \{C_q, \partial_q\}_{q \in \mathbb{Z}}$ is a chain complex over $R$ and $G$ is an $R$-mod, then $C \otimes G \equiv \{C_q \otimes G, \partial_q \otimes \mathrm{Id}\}_{q \in \mathbb{Z}}$ is also a chain complex over $R$. Moreover, if $C$ is augmented over $G'$, then $C \otimes G$ is augmented over $G' \otimes G$. The graded homology module $H(C \otimes G)$ is called* **homology module of $C$ with coefficients $G$,** *denoted $H(C, G)$. If $\tau : C \to C'$ is a chain map, $\tau \otimes \mathrm{Id} : C \otimes G \to C' \otimes G$ is also a chain map and the induced map of*

*$\tau \otimes \mathrm{Id} : C \otimes G \to C' \otimes G$ is denoted by $\tau_* \equiv (\tau \otimes \mathrm{Id})_* : H(C,G) \to H(C',G)$. For a homomorphism $\varphi : G \to G'$, there is a chain map $\mathrm{Id} \otimes \varphi : C \otimes G \to C \otimes G'$ inducing a homomorphism $\varphi_* : H(C,G) \to H(C,G')$.*

**37 Theorem.** *There is a covariant functor from the product category of a category of graded $R$-mods and a category of $R$-mods to the category of graded $R$-mods :*

$$F_R : (C,G) \mapsto H(C,G) \text{ where } C \equiv \{C_q, \partial_q, \}_{q\in\mathbb{Z}}.$$

*Proof.* If $c \in C_q$ and $g \in G$, then $c \otimes q \in Z_q \otimes G$ is a cycle of $C \otimes G$, and if $b \in B_q$, then $b \otimes q \in B_q \otimes G$. Therefore, there exists a bilinear map $\mu \equiv \{\mu_q\}_{q\in\mathbb{Z}} : H_q(C) \times G \to H_q(C,G)$ defined by $(c,g) \mapsto C \otimes g$. The rest follows from preceding discussion.

Let $C(K)$ denote oriented chain complex of $K$. For abelian group $G$ and simplicial pair $(K,L)$, the oriented homology group of $(K,L)$ with coefficients $G$, denoted by $H(K,L,G)$, is defined as a graded homology group $[C(K)/C(L)] \otimes G$. Then there exists a covariant functor $H_{(K,L,G)}$ from the product category of a category of simplicial pairs and a category of abelian groups to the category of graded abelian groups.

For abelian group $G$ and topological pair $(X,A)$, the singular homology group of $(X,A)$ with coefficients $G$, denoted by $H(X,A,G)$, is defined as a graded homology group $[\Delta(X)/\Delta(A)] \otimes G$. Then there is a covariant functor $H_{(X,A,G)}$ from the product category of a category of topological pairs and a category of abelian groups to the category of graded abelian groups.

**38 Lemma.** *The tensor product of two epimorphisms is an epimorphism.*

*Proof.* Let $\alpha : A \to A''$ and $\beta : B \to B''$ be epimorphisms. Then $A'' \otimes B''$ is generated by elements of the form $a'' \otimes b''$ where $a'' \in A''$ and $b'' \in B''$. Since $\alpha$ and $\beta$ are epimorphisms, $A'' \otimes B''$ is generated by elements of the form $\alpha(a) \otimes \beta(b) = (\alpha \otimes \beta)(a \otimes b)$ where $a \in A$ and $b \in B$. Therefore, $A'' \otimes B''$ is generated by $(\alpha \otimes \beta)(A \otimes B)$. Hence, $\alpha \otimes \beta$ is epimorphism.

**39 Example.** The tensor product of an exact sequence and a module need not be an exact sequence (tensor product of two monomorphism need not be a monomorphism.)

Consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}_2 \to 0$ where $\beta \stackrel{\text{def}}{=} \mod 2$ and $\alpha \stackrel{\text{def}}{=} 2\times$. Looking at generators we have $\alpha : 1 \mapsto 2$ and $\beta : 1 \mapsto \bar{1}$ where $\bar{1}$ is a generator of $\mathbb{Z}_2$. The tensor product of this SES with $\mathbb{Z}_2$ is not SES because $(\alpha \otimes \mathrm{Id}) : \mathbb{Z} \otimes \mathbb{Z}_2 \to \mathbb{Z} \otimes \mathbb{Z}_2$ is not a monomorphism for $\mathbb{Z} \otimes \mathbb{Z}_2 \simeq \mathbb{Z}_2 \neq 0$, but $(\alpha \otimes \mathrm{Id})(1 \otimes \bar{1}) = 2 \otimes \bar{1} = 1 \otimes 2 \cdot \bar{1} = 1 \otimes \bar{0} = 0$.

**40 Lemma.** *If $\alpha$ and $\beta$ are epimorphisms, then $\mathrm{Ker}(\alpha \otimes \beta)$ is generated by elements of the form $a \otimes b$, where $a \in \mathrm{Ker}(\alpha)$ or $b \in \mathrm{Ker}(\beta)$.*

*Proof.* Let $\alpha : A \to A''$ and $\beta : B \to B''$ be epimorphisms and let $D \subset A \otimes B$ a submodule generated by elements of the form $a \otimes b$ s.t. $a \in \mathrm{Ker}(\alpha)$ or $b \in \mathrm{Ker}(\beta)$. Next let $p : A \otimes B \to (A \otimes B)/D$ be the projection. Then there is well-defined bilinear map

$A''\times B'' \to (A\otimes B)/D$ defined by $(a'',b'')\mapsto p(a\otimes b)$ where $a\otimes b\in A\otimes B$ s.t. $\alpha(a)=a''$ and $\beta(b)=b''$. This bilinear map correspond to a homomorphism $\psi: A''\otimes B''\to (A\otimes B)/D$ defined by $(a''\otimes b'')\mapsto p(a\otimes b)$ where $a\otimes b\in A\otimes B$ s.t. $\alpha(a)=a''$ and $\beta(b)=b''$. Evidently the map $p$ is equal to the composition $A\otimes B\xrightarrow{\alpha\otimes\beta} A''\otimes B''\xrightarrow{\psi}(A\otimes B)/D$. Hence, $\mathrm{Ker}(\alpha\otimes\beta)\subset D$. The other inclusion holds by definition. So, we get $\mathrm{Ker}(\alpha\otimes\beta)=D$.

**41 Lemma.** *Given an exact sequence $A'\to A\to A''\to 0$ and a module $B$, there exists an exact sequence $A'\otimes B\to A\otimes B\to A''\otimes B\to 0$.*

*Proof.* We already proved in Lemma 38 that the map $A\otimes B\to A''\otimes B$ is epimorphism. Therefore, the sequence is exact at $A''\otimes B$. Let $\tilde{A}\subset A$ be an image of the map $A'\to A$. By the same Lemma we get that $A'\otimes B\to \tilde{A}\otimes B$ is an epimorphism. Hence, $\mathrm{Im}(A'\otimes B\to \tilde{A}\otimes B)=\tilde{A}\otimes B$. On the other hand, the previous Lemma tells us that $\mathrm{Ker}(A\otimes B\to A''\otimes B)=\mathrm{Ker}(A\to A'')\otimes B=\tilde{A}\otimes B$ for the sequence is exact in $A$. So, the sequence is exact at $A\otimes B$.

**42 Lemma.** *The functor of tensor product commutes with direct sums, resp. direct limits. In other words, $A\otimes B\simeq\bigoplus\limits_{j,k}(A_j\otimes B_k)$ for $A=\bigoplus\limits_j A_j$ and $B=\bigoplus\limits_k B_k$, resp. $A\otimes B\simeq\varinjlim\{A^\alpha\otimes B^\beta\}$ for $A=\varinjlim\{A^\alpha\}$ and $B=\varinjlim\{B^\beta\}$.*

*Proof.* The proof is standard. It's based on the universal properties of tensor product and direct sums, resp. direct limits.

Let $A=\varinjlim\{A^\alpha\}$. Consider a bilinear map $A\times B\to\varinjlim\{A^\alpha\otimes B\}$ given by $(\{a\},b)\mapsto\{a\otimes b\}$ where $a\in A^\alpha, b\in B$. Then the characteristic properties of tensor product and direct limits imply commutative triangles:

$$\begin{tikzcd}
A\times B \arrow[d] \arrow[dr] & \\
A\otimes B \arrow[r, "\varphi"] & \varinjlim A^\alpha\otimes B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & A^\alpha\otimes B \arrow[dl] \arrow[d] \\
A\otimes B & \varinjlim A^\alpha\otimes B \arrow[l, "\psi"']
\end{tikzcd}$$

Clearly $\varphi$ and $\psi$ are inverses. Hence $A\otimes B\simeq\varinjlim\{A^\alpha\otimes B\}$ for $A=\varinjlim\{A^\alpha\}$.

**43 Definition.** *A SES $0\to A'\xrightarrow{\alpha}A\xrightarrow{\beta}A''\to 0$ is said to* **split** *if $\beta$ has a right inverse, i.e. there exists homomorphism $\beta': A''\to A$ s.t. $\beta\circ\beta'=\mathrm{Id}_{A''}$.*

**44 Observation.** If $A''$ is free, then a SES $0\to A'\xrightarrow{\alpha}A\xrightarrow{\beta}A''\to 0$ splits.

*Proof.* Let $\{a''_j\}$ be a basis for $A''$. For every $j$ choose $a_j\in A$ s.t. $\beta(a_j)=a''_j$. Define a homomorphism $\beta': A''\to A$ s.t. $\beta'(a''_j)=a_j$ for all $j$. Then $\beta'$ is a right inverse of $\beta$.

**45 Lemma.** *For a given SES $0\to A'\xrightarrow{\alpha}A\xrightarrow{\beta}A''\to 0$ define an exact sequence*

$$A'\xrightarrow{i}A'\oplus A''\xrightarrow{p}A''\to 0$$

*by $i : a' \mapsto (a', 0)$ and $p : (a', a'') \mapsto a''$ for all $a' \in A''$ and all $a'' \in A''$. Then the following claims are equivalent.*

*(1) The first sequence splits.*
*(2) There exists $\gamma' : A' \oplus A'' \to A$ and the following left diagram commutes.*

$$\begin{tikzcd}
& A \arrow[dr, "\beta"] & \\
A' \arrow[ur, "\alpha"] \arrow[dr, "i"'] & & A'' \\
& A' \oplus A'' \arrow[uu, "\gamma'"] \arrow[ur, "p"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[dr, "\beta"] \arrow[dd, "\gamma"] & \\
A' \arrow[ur, "\alpha"] \arrow[dr, "i"'] & & A'' \\
& A' \oplus A'' \arrow[ur, "p"'] &
\end{tikzcd}$$

*(3) There exists $\gamma : A \to A' \oplus A''$ and the preceding right diagram commutes.*
*(4) The map $\alpha$ has a left inverse, i.e. there exists homomorphism $\alpha' : A \to A'$ s.t. $\alpha' \circ \alpha = \mathrm{Id}_{A'}$.*

*Proof.*

$1 \Leftrightarrow 2$ Let $\beta' : A'' \to A$ is a right inverse of $\beta$. Define $\gamma' : A' \oplus A'' \to A$ by $\gamma' : (a', a'') \mapsto \alpha(a') + \beta'(a'')$. Then $(\gamma' \circ i)(a') = \gamma'(a', 0) = \alpha(a') + \beta'(0) = \alpha(a')$ and $(\beta \circ \gamma')(a', a'') = \beta\alpha(a') + \beta\beta'(a'') = 0 + a'' = p(a', a'')$. Conversely, for a given map $\gamma'$, define $\beta' : A'' \to A$ by $\beta' : a'' \mapsto \gamma'(0, a'')$ for all $a'' \in A''$. Then $(\beta \circ \beta')(a'') = \beta\gamma'(0, a'') = p(0, a'') = a''$. Hence $\beta'$ is a right inverse of $\beta$.

$3 \Leftrightarrow 4$ Let $\alpha' : A \to Aa$ is a left inverse of $\alpha$. Define $\gamma : A \to A' \oplus A''$ by $\gamma : a \mapsto (\alpha'(a), \beta(a))$. Then $(\gamma \circ \alpha)(a') = (\alpha'\alpha(a'), \beta\alpha(a')) = (a', 0) = i(a')$ and $(p \circ \gamma)(a) = p(\alpha'(a), \beta(a)) = \beta(a)$. Conversely, for a given map $\gamma$, define $\alpha' : A \to A'$ by $\alpha' : a \mapsto a'$ for all $a \in A$ where $\gamma(a) = (a', a'')$. Then $(\alpha' \circ \alpha)(a') = \tilde{a}'$ where $\gamma(\alpha(a')) = (\tilde{a}', \tilde{a}'')$. From the commutativity $\gamma \circ \alpha = i$ we get $\gamma(\alpha(a')) = (a', 0)$. Therefore $\tilde{a}' = a'$ and hence $\alpha'$ is a left inverse of $\alpha$.

$2 \Leftrightarrow 3$ The assertion follows from the Lemma about five isomorphisms which implies $\gamma^{-1} = \gamma'$.

**46 Lemma.** *For a given splitting SES $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$ and a module $B$, the sequence $0 \to A' \otimes B \xrightarrow{\alpha \otimes \mathrm{Id}} A \otimes B \xrightarrow{\beta \otimes \mathrm{Id}} A'' \otimes B \to 0$ is also a splitting SES.*

*Proof.* To be a SES it suffices by the Lemma 41 to show exactness at $A' \otimes B$. The previous Lemma reduces the problem to the existence of a left inverse of $\alpha \otimes \mathrm{Id}$. This would guarantee that the sequence splits, moreover it would force $\alpha \otimes \mathrm{Id}$ to be monomorphic and thus the sequence to be exact at $A' \otimes B$. Let $\alpha'$ be a left inverse of $\alpha$. Then clearly $\alpha' \otimes \mathrm{Id}$ is a left inverse of $\alpha \otimes \mathrm{Id}$. So, the claim is proved.

**47 Corollary.** *Suppose $0 \to C' \to C \to C'' \to 0$ (e.g. $C'' \equiv C/C'$) is a splitting SES of chain complexes. Then for any module $G$, $0 \to C' \otimes G \to C \otimes G \to C'' \otimes G \to 0$ is a SES of chain complexes.*

**48 Theorem.** *For a given splitting SES of chain complexes $0 \to C' \to C \to C'' \to 0$ (e.g. $C'' \equiv C/C'$) and a given module G, there exists functorial LES*

$$\cdots \to H_q(C', G) \to H_q(C, G) \to H_q(C'', G) \to H_{q-1}(C', G) \to \ldots$$

*Proof.* Follows from the preceding discussion and Theorem [12.2].

**49 Observation.** If $\{(X_1, A_1), (X_2, A_2)\}$ is an excessive couple of topological pairs, then the SES of singular chain complexes

$$0 \to \delta(X_1 \cap X_1)/\delta(A_1 \cap A_1) \to \delta(X_1)/\delta(A_1) \oplus \delta(X_2)/\delta(A_2) \to \delta(X_1 \cup X_1)/\delta(A_1 \cup A_1) \to 0$$

is a splitting sequence because all groups are free abelian groups.

**50 Theorem.** *If $\{(X_1, A_1), (X_2, A_2)\}$ is an excessive couple of topological pairs and $G$ is an $R$-mod, there exists an exact relative Mayer-Vietoris sequence of $\{(X_1, A_1), (X_2, A_2)\}$ with coefficients $G$.*

*Proof.* Follows from the preceding observation and Theorem [12.2].

**51 Definition.** *Let $G$ be $R$-module. A* **homology theory with coefficients** *$G$ consists of a covariant functor $H_G$ :* **Top** $\to$ **graded** $R$**-mod** *and a natural transformation $\partial : H(X, A) \to H(A)$ of degree $-1$ satisfying axioms of homotopy, exactness and excision together with the axiom of dimension - in the category of 1-pt spaces there exists a natural equivalence of $H_G$ with the constant functor assigning to every 1-pt space the graded module isomorphic to $G$ in degree zero and trivial otherwise.*

## 1.4 Universal Coefficient Theorem

The aim of this section is to express $H_*(C, G)$ in terms of $H_*(C)$ and $G$ using certain functors of modules that are associated to the tensor product functors. We assure G is PID (principal ideal domain).

**52 Definition.** *Let $A$ be an $R$-module. A* **resolution** *of $A$ is an exact sequence of $R$-modules $C_i$:*

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} A \to 0.$$

*If each $C_q$ is a free $R$-mod, the resolution is said to be* **free**.

**53 Observation.** Any $R$-mod $A$ has a free resolution.

*Proof.* Denote $F(B)$ the free module over elements of $B$, and let $\varepsilon : F(B) \to B$ be the canonical map. The canonical free resolution of $A$ can be defined inductively

$$\dots \to \mathcal{F}(\mathrm{Ker}\partial_q) \xrightarrow{\partial_{q+1}} \mathcal{F}(\mathrm{Ker}\partial_{q-1}) \xrightarrow{\partial_q} \dots \xrightarrow{\partial_1} \mathcal{F}(\mathrm{Ker}\varepsilon) \xrightarrow{\partial_1} \mathcal{F}(A) \xrightarrow{\varepsilon} A \to 0.$$

The application of the method of acyclic models to the category of chain complex over $R$ implies the following result:

**54 Theorem.** *Let $C$ be a free non-negative chain complex augmented over $A$, and let $C'$ be a resolution of $A'$. Then any homomorphism $\varphi : A \to A'$ can be extended to a chain map*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{q+1} \arrow[r] \arrow[d] & C_q \arrow[r] \arrow[d] & \cdots \arrow[r] & C_0 \arrow[r] \arrow[d] & A \arrow[r, "\varepsilon"] \arrow[d, "\varphi"] & 0 \\
\cdots \arrow[r] & C'_{q+1} \arrow[r] & C'_q \arrow[r] & \cdots \arrow[r] & C'_0 \arrow[r] & A' \arrow[r] & 0
\end{tikzcd}$$

*preserving augmentation, and two such chain maps are chain homotopic.*

In the case $\varphi = \mathrm{Id}_A : A \to A$ we get

**55 Corollary.** *If $C, C'$ are two free resolutions of $A$, then $C \simeq C'$, i.e $C$ and $C'$ are canonically chain homotopic.*

**56 Corollary.** *Let $A, B$ be $R$-modules, and $C$ free resolution of $A$. Then the graded module $H_*(C, B)$ depends only on $A$ and $B$.*

**57 Definition.** *For given $R$-modules $A$ and $B$, choose a canonical free resolution $C$. For $q \geq 0$ define $\mathrm{Tor}_q(A, B) \overset{\mathrm{def}}{=} H_q(C, B)$, a covariant functor of $A$ and $B$.*

For a resolution $C$ we can construct SES $0 \to \partial_1 C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} A \to 0$. The Lemma 41 implies that for a module B, there exists an exact sequence

$$\partial_1 C_1 \otimes_R B \xrightarrow{\partial_1} C_0 \otimes_R B \xrightarrow{\varepsilon} A \otimes_R B \to 0.$$

Since $\mathrm{Tor}_0(A, B)$ was defined to be the $0^{th}$ homology $R$-mod of the chain complex

$$\dots \to C_2 \otimes B \xrightarrow{\partial_2 \otimes \mathrm{Id}} C_1 \otimes B \xrightarrow{\partial_1 \otimes \mathrm{Id}} C_0 \otimes B \to 0,$$

$\mathrm{Tor}_0(A, B) = C_0 \otimes B / \mathrm{Im}(\partial_1 \otimes \mathrm{Id})$, while $\mathrm{Im}(\partial_1 \otimes \mathrm{Id}) = \mathrm{Ker}(\varepsilon \otimes \mathrm{Id})$. Therefore

$$\mathrm{Tor}_0(A, B) = C_0 \otimes B / Ker(\varepsilon \otimes \mathrm{Id}) \simeq A \otimes B.$$

In the case $R$ is the commutative ring with unity having the property of PID, any submodule of free module is a free module. In particulary, any $A$ has a free resolution $0 \to C_1 \to C_0 \to A \to 0$ where $C_0 = \mathcal{F}(A)$ and $C_1 = \mathrm{Ker}(\mathcal{F}(A) \to A) \subset \mathcal{F}(A)$.

**58 Definition.** *This resolution is called* **free presentation** *of $R$-module $A$.*

Because $\{C_q\}_{q\geq 2}$ are trivial $R$-modules, $\mathrm{Tor}_q(A,B)=\emptyset$ for $q\geq 2$.

**59 Definition.** *The* **torsion product** *is defined by* $A\star B \stackrel{\text{def}}{=} \mathrm{Tor}_1(A,B)$.

It is characterized by the property, that for a given free presentation of $A$

$$0\to C_1\to C_0\to A\to 0$$

there is an exact sequence

$$0\to A\star B\to C_1\otimes B\to C_0\otimes B\to A\otimes B\to 0$$

for $A\star B=H_1(C\otimes B)=\mathrm{Ker}(C_1\otimes B\to C_0\otimes B)$. Because the tensor product commutes with direct sums and direct limits and the direct limit of exact sequences is an exact sequence, the torsion product commutes with direct sums and direct limits.

**60 Observation.** If $A$ is a free module, there is a free presentation $0\to 0\to A\to A\to 0$ and so $A\star B=0$ for every $R$-module $B$.

**61 Observation.** If $A$ is a cyclic $R$-mod such that $A\simeq R/RI$ for some principal ideal $I\subset R$, there is a free presentation $0\to R\xrightarrow{\alpha} R\to A\to 0$ given by $\alpha: r\mapsto rr'$, where $r'\in I$ is a generator of $I$. Then for any $R$-mod $B$ there is an isomorphism $R\otimes_R B\simeq B$ given by $1\otimes b\mapsto b$. Under this isomorphism the mapping $\alpha\otimes\mathrm{Id}: R\otimes_R B\to R\otimes_R B$ corresponds to the mapping $\alpha': B\to B$ given by $b\mapsto r'b$ for all $b\in B$. Therefore $\mathrm{Ker}\alpha'\simeq\{b\in B: r'b=0\}$ is an $R$-mod of $B$ and so $A\star B\simeq (R/RI)\star B\simeq\{b\in B: r'b=0\}$.

**62 Lemma.** *If $A$ or $B$ is a torsion free $R$-mod, then $A\star B=0$.*

*Proof.* Because the torsion product commutes with direct limits, it suffices to consider the case of finitely generated $A$ and $B$. In this case, torsion free is equivalent to free. If $A$ is free, the result $A\star B=0$ follows from the previous Observation. If $B$ is free and finitely generated $R$-mod, it is isomorphic to $R^{\oplus n}$ for some $n\in\mathbb{N}$. Let $0\to C_1\to C_0\to A\to 0$ be free presentation of A. Then

$$C_1\otimes_R B\to C_0\otimes_R B\to A\otimes_R B\to 0$$

is isomorphic to a direct sum of $n$ copies of the sequence

$$C_1\otimes_R R\to C_0\otimes_R R\to A\otimes_R R\to 0.$$

Since $C_1\otimes_R R\to C_0\otimes_R R$ is monomorphism, so is $C_1\otimes_R B\to C_0\otimes_R B$. Hence $A\star B=0$.

**63 Corollary.** *If $R$ is a field, $A\star B=0$ for all $A,B$.*

**64 Lemma.** *For a given SES of $R$-modules*

$$0 \to A' \to A \to A'' \to 0,$$

*and an $R$-mod $B$, if $A''$ or $B$ is torsion free, there exists a SES*

$$0 \to A' \otimes_R B \to A \otimes_R B \to A'' \otimes_R B \to 0.$$

*Proof.* We can reduce the problem to the case of finitely generated (hence free) $R$-modules. If $A''$ is torsion free (hence free), then the given SES splits. The assertion follows from the Lemma 46. If $B$ is torsion free (hence free), the map $A' \otimes_R B \to A \otimes_R B$ is a (finite) direct sum of copies of $A' \otimes_R R \to A \otimes_R R$, which is a monomorphism. So is $A' \otimes_R B \to A \otimes_R B$. The rest follows from the Lemma 38.

**65 Theorem.** *In the product category of torsion free chain complexes and SES of $R$-modules of a form $0 \to G' \xrightarrow{f} G \xrightarrow{g} G'' \to 0$, there exists a natural connecting homomorphism $\partial_* : H_*(C, G'') \to H_{*-1}(C, G')$ such that $|\partial_*| = -1$ and a functorial exact sequence*

$$\cdots \to H_q(C, G') \xrightarrow{f_*} H_q(C, G) \xrightarrow{g_*} H_q(C, G'') \xrightarrow{\partial_*} H_{q-1}(C, G') \to \cdots$$

*Proof.* By previous Lemma, there exists a SES of chain complexes

$$0 \to C \otimes_R G' \xrightarrow{\mathrm{Id}\otimes f} C \otimes_R G \xrightarrow{\mathrm{Id}\otimes g} C \otimes_R G'' \to 0$$

which is functorial both in $C$ and SES $0 \to G' \to G \to G'' \to 0$. The rest follows from the Theorem [12.2].

**66 Definition.** *The connecting homomorphism $\partial_*$ is usually called* **Bochstein homomorphism** *of the SES $0 \to G' \to G \to G'' \to 0$.*

**67 Remark.** The Theorem 65 is also valid for $R$-modules where $R$ is only a commutative ring with unit provided $C$ is free chain complex.

**68 Theorem.** (Universal Coefficient Theorem for Homology) *Let $C$ be a free chain complex and let $G$ be a module. Then there exists a functorial SES*

$$0 \to H_q(C) \otimes G \to H_q(C \otimes G) \to H_{q-1}(C) \star G \to 0$$

*and the sequence splits.*

*Proof.* Let $Z \subset C$ and $B \subset C$ be subcomplexes of $C$ defined by $Z_q = Z_q(C)$ and $B_q = B_{q-1}(C)$ with trivial boundary operations. Both $Z, B$ are free chain subcomplexes.

Evidently there exists a SES $0 \to Z \xrightarrow{\alpha} C \xrightarrow{\beta} B \to 0$ defined by $\alpha : z \mapsto z$ and $\beta : c \in C_q \mapsto \partial_q c$. So there is a SES $0 \to Z \otimes G \xrightarrow{\alpha \otimes \mathrm{Id}} C \otimes G \xrightarrow{\beta \otimes \mathrm{Id}} B \otimes G \to 0$ giving a LES of homology groups

$$\cdots \to H_q(Z, G) \xrightarrow{\alpha_*} H_q(C, G) \xrightarrow{\beta_*} H_q(B, G) \xrightarrow{\partial_*} H_{q-1}(Z, G) \to \cdots$$

where $\partial_* \overset{\text{def}}{=} \{\alpha_{q-1}^{-1} \partial_q \beta^{-1} b\} = \{\alpha_{q-1}^{-1}(z)\}$ for $b \in B_{q-1}$. Since $Z, B$ have trivial boundary operators, so do $Z \otimes G$ and $B \otimes G$. Therefore $H_q(Z, G) = Z_q \otimes G$ and $H_q(B, G) = B_q \otimes G = B_{q-1}(C) \otimes G$. So the LES above becomes

$$\cdots \to B_q(C) \otimes G \xrightarrow{\gamma_q \otimes \mathrm{Id}} Z_q(C) \otimes G \to H_q(C, G) \to B_{q-1}(C) \otimes G \xrightarrow{\gamma_{q-1} \otimes \mathrm{Id}} Z_{q-1}(C) \otimes G \to \cdots$$

where $\gamma_q : B_q(C) \hookrightarrow Z_q(C)$. From this LES we can obtain a set of SES

$$0 \to \mathrm{Coker}(\gamma_q \otimes \mathrm{Id}) \to H_q(C, G) \to \mathrm{Ker}(\gamma_{q-1} \otimes \mathrm{Id}) \to 0.$$

It remains to interpret the modules at both sides of $H_q(C, G)$. Since $Z_q(C)$ is free for every $q$ (hence, $B_q(C)$ is free either), the free resolution of $H_q(C)$ is

$$0 \to B_q(C) \xrightarrow{\gamma_q} Z_q(C) \to H_q(C) \to 0.$$

Therefore, there exists a SES

$$0 \to H_q(C) \star G \to B_q(C) \otimes G \xrightarrow{\gamma_q \otimes \mathrm{Id}} Z_q(C) \otimes G \to H_q(C) \otimes G \to 0$$

which immediately implies
$$\begin{aligned} \mathrm{Coker}(\gamma_q \otimes \mathrm{Id}) &\simeq H_q(C) \otimes G \\ \mathrm{Ker}(\gamma_q \otimes \mathrm{Id}) &\simeq H_q(C) \star G. \end{aligned}$$

Substituting back we get the first assertion of the Theorem.

Let $\tau : C \to C'$ be a chain map. Then it defines a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Z \arrow[r, "\alpha"] \arrow[d, "\tau'"] & C \arrow[r, "\beta"] \arrow[d, "\tau"] & B \arrow[r] \arrow[d, "\tau''"] & 0 \\ 0 \arrow[r] & Z' \arrow[r, "\alpha'"] & C' \arrow[r, "\beta'"] & B' \arrow[r] & 0 \end{tikzcd}$$

from which we obtain the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H_q(C) \otimes G \arrow[r] \arrow[d, "\tau_* \otimes \mathrm{Id}"'] & H_q(C, G) \arrow[r] \arrow[d, "\tau_*"] & H_{q-1}(C) \star G \arrow[r] \arrow[d, "\tau_* \star \mathrm{Id}"] & 0 \\ 0 \arrow[r] & H_q(C') \otimes G \arrow[r] & H_q(C', G) \arrow[r, "\beta'"] & H_{q-1}(C') \star G \arrow[r] & 0 \end{tikzcd}$$

Thus, the SES for $H_q(C, G)$ is functorial.

We now prove that the SES splits (but not functorially!). Because $B_{q-1}(C)$ is free and $\partial_q C_q = B_{q-1}(C)$, there exists homomorphism $h_q : B_{q-1}(C) \to C_q$ for every $q$ such that $\partial_q h_q = \mathrm{Id} : B_{q-1} \to C_q \to B_{q-1}$. Then $h_q \otimes \mathrm{Id} : B_{q-1}(C) \otimes G \to C_q \otimes G$ maps $\mathrm{Ker}(\gamma_{q-1} \otimes \mathrm{Id})$ into cycles of $C_q \otimes G$ (see the LES) and so induces a homomorphism $H_{q-1}(C) \star G \to H_q(C, G)$ which is the promised right inverse of the homomorphism $H_q(C, G) \to H_{q-1}(C) \star G$ in the SES of the Theorem.

The following results are closely connected with various properties of torsion product. We shall start with six-term exact sequence connecting tensor and torsion product.

**69 Lemma.** *Let $0 \to B' \xrightarrow{\alpha'} B \xrightarrow{\beta'} B'' \to 0$ be a SES of $R$-modules, and let $A$ be an $R$-mod. Then there exists a six-term exact sequence*

$$0 \to A \star B' \xrightarrow{\mathrm{Id}\star\alpha'} A \star B \xrightarrow{\mathrm{Id}\star\beta'} A \star B'' \to A \otimes B' \xrightarrow{\mathrm{Id}\otimes\alpha'} A \otimes B \xrightarrow{\mathrm{Id}\otimes\beta'} A \otimes B'' \to 0.$$

*Proof.* Let $0 \to C_1 \to C_0 \to A \to 0$ be a free presentation of $A$. Denote $C$ the chain complex $\cdots \to 0 \to C_1 \to C_0 \to 0$. Then $H_q(C) = 0$ for all $q \neq 0$ and $H_0(C) = A$. Since $C$ is free, there exists a SES of chain complexes

$$0 \to C \otimes B' \xrightarrow{\mathrm{Id}\otimes\alpha'} C \otimes B \xrightarrow{\mathrm{Id}\otimes\beta'} C \otimes B'' \to 0$$

The nontrivial part of corresponding LES is

$$0 \to H_1(C\otimes B') \to H_1(C\otimes B) \to H_1(C\otimes B'') \to H_0(C\otimes B') \to H_0(C\otimes B) \to H_0(C\otimes B'') \to 0.$$

From UCT we can conclude that $H_1(C \otimes B') \simeq A \star B'$, $H_1(C \otimes B) \simeq A \star B$, $H_1(C \otimes B'') \simeq A \star B''$, $H_0(C \otimes B') \simeq A \otimes B'$, $H_0(C \otimes B) \simeq A \otimes B$, $H_0(C \otimes B'') \simeq A \otimes B''$. These isomorphisms together with the LES above give the result.

**70 Lemma.** *There exists a functorial isomorphism $\psi : A \star B \simeq B \star A$ of $R$-modules.*

*Proof.* Let $0 \to C_1 \to C_0 \to B \to 0$ be a free presentation of $B$. Since $C_0$ is free $R$-mod, $A \star C_0 = 0$ (see the Observation 60). By the previous Lemma there exists an exact sequence

$$0 \to A \star B \to A \otimes C_1 \to A \otimes C_0 \to A \otimes B \to 0.$$

By definition of torsion product we have

$$0 \to B \star A \to C_1 \otimes A \to C_0 \otimes A \to B \otimes A \to 0.$$

Because we are working over commutative ring, we have also following functorial isomorphisms of tensor product $A \otimes C_1 \simeq C_1 \otimes A$, $A \otimes C_0 \simeq C_0 \otimes A$ and $A \otimes B \simeq B \otimes A$.

This give us

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A \star B & \longrightarrow & A \otimes C_1 & \longrightarrow & A \otimes C_0 & \longrightarrow & A \otimes B & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{iso}} & & & & \downarrow{\scriptstyle \mathrm{iso}} & & \downarrow{\scriptstyle \mathrm{iso}} & & \downarrow{\scriptstyle \mathrm{iso}} & & \downarrow{\scriptstyle \mathrm{iso}} \\
0 & \longrightarrow & B \star A & \longrightarrow & C_1 \otimes A & \longrightarrow & C_0 \otimes A & \longrightarrow & B \otimes A & \longrightarrow & 0
\end{array}$$

where iso denotes mentioned isomorphisms. Using the Lemma about five isomorphisms we get the result.

**71 Lemma.** *Let $A, B$ be $R$-modules and denote $i : \mathrm{Tor}(A) \hookrightarrow A$ and $j : \mathrm{Tor}(B) \hookrightarrow B$ where $\mathrm{Tor}(X)$ are the torsion elements of $X$. Then $i \star j : \mathrm{Tor}(A) \star \mathrm{Tor}(B) \simeq A \star B$.*

*Proof.* Clearly there exists a SES $0 \to \mathrm{Tor}(B) \xrightarrow{j} B \to B/\mathrm{Tor}(B) \to 0$ where $B/\mathrm{Tor}(B)$ is a free $R$-mod. This implies (see the Observation 60) $A \star (B/\mathrm{Tor}(B)) = 0$. Then the Lemma 69 gives us sequence

$$0 \to A \star \mathrm{Tor}(B) \xrightarrow{\mathrm{Id} \star j} A \star B \to 0 \to A \otimes \mathrm{Tor}(B) \to A \otimes \mathrm{Tor}(B) \to A \otimes (B/\mathrm{Tor}(B)) \to 0.$$

Hence $\mathrm{Id} \star j : A \star \mathrm{Tor}(B) \simeq A \star B$. Similarly we get $i \star \mathrm{Id} : \mathrm{Tor}(A) \star \mathrm{Tor}(B) \simeq A \star \mathrm{Tor}(B)$. Their composition gives the result.

**72 Definition.** *For a given chain complex $C$ over $R$ a* **free approximation** *of $C$ is a chain complex $\bar{C}$ with a chain map $\tau : \bar{C} \to C$ such that*

*(1) $\bar{C}$ is a free chain complex over $R$,*
*(2) $\tau$ is an epimorphism,*
*(3) $\tau$ induces an isomorphism $\tau_* : H(\bar{C}) \simeq H(C)$ of $R$-modules.*

Note that the result of the UCT (Theorem 68) can be extended using the following Theorem.

**73 Theorem.** *Any chain complex $C$ has a free approximation, uniquely determined up to homotopy equivalence.*

## 1.5 Künneth Formula

We are going to prove Eilenberg-Zilber Theorem saying that the singular complex of a product of spaces is a chain equivalent to the tensor product of the singular complexes of the factor spaces.

If $C, C'$ are graded $R$-modules, then $C \otimes C' = \{(C \otimes C)_q\}_q$, where $(C \otimes C)_q$ is defined by $(C \otimes C)_q \stackrel{\text{def}}{=} \bigoplus_{i+j=q} C_i \otimes C_j$, is also a graded $R$-module. Similarly, their torsion product $C \star C'$ is a graded $R$-module $\{(C \star C)_q\}_q$, where $(C \star C)_q$ is defined by $(C \star C)_q \stackrel{\text{def}}{=} \bigoplus_{i+j=q} C_i \star C_j$.

**74 Observation.** If $C, C'$ are chain complexes, then their tensor product $\{(C \otimes C)_q, \partial_q^{\otimes}\}_q$, where $\partial_q^{\otimes} = \sum_{i+j=q} (\partial_i \otimes \mathrm{Id}_j + (-1)^i \mathrm{Id}_i \otimes \partial'_j)$, and torsion product $\{(C \star C)_q, \partial_q^{\star}\}_q$ where $\partial_q^{\star} = \sum_{i+j=q} (\partial_i \star \mathrm{Id}_j + (-1)^i \mathrm{Id}_i \star \partial'_j)$, are chain complexes.

*Proof.* By definition, $\partial_q^{\otimes}(c \otimes c') = \partial_i c \otimes c' + (-1)^i c \otimes \partial'_j c'$ and $\partial_q^{\star}(c \otimes c') = \partial_i c \star c' + (-1)^i c \star \partial'_j c'$ for any $c \in C, c' \in C'$. Applying these again we can see that both $\partial^{\otimes}$ and $\partial^{\star}$ yield zero.

If $C' \equiv \{C'_q\}_q$, where $C'_q = 0$ for $q \neq 0$, then $C \otimes C' = C \otimes C'_0$. Therefore, we can reasonable expect the extension of the UCT (Theorem 68) to express the homology of $C \otimes C'$ in terms of the homology of $C$ and $C'$.

Define (functorial) homomorphism $\mu$ of degree 0 from $H(C) \otimes H(C')$ to $H(C \otimes C')$ by

$$\begin{matrix} \mu : c \in Z_i(C) \otimes c' \in Z_j(C') \mapsto c \otimes c' \in Z_i + j(C \otimes C'), \\ \mu : b \in B_i(C) \otimes b' \in B_j(C') \mapsto b \otimes b' \in B_i + j(C \otimes C') \end{matrix} \rightsquigarrow \mu : \{c\} \otimes \{c'\} \mapsto \{c \otimes c'\}.$$

**75 Lemma.** *Let $C, C'$ be chain complexes with $C'$ free. Then there exists a functorial SES*

$$0 \to [H(C)] \otimes H(C')]_q \xrightarrow{\mu} H_q(C \otimes C') \to [H(C) \star H(C')]_{q-1} \to 0,$$

*where $[-]_q$ denotes the free sum of $i$-components and $j$-components such that $i + j = q$. If $C$ is also free, then the above SES splits.*

*Proof.* Let $Z'$, $B' \subset C'$ be subcomplexes defined by $Z'_q = Z_q(C')$, $B'_q = B_{q-1}(C')$. Then there exists SES $0 \to Z' \to C' \to B' \to 0$. Since $C'$ is free and $Z', B'$ are submodules of $C'$. (By assumption $R$ is PID. Hence $Z', B'$ are free.) So, there exists a SES

$$0 \to C \otimes Z' \to C \otimes C' \to C \otimes B' \to 0$$

and the corresponding LES is

$$\cdots \to H_q(C \otimes Z') \to H_q(C \otimes C') \to H_q(C \otimes B') \xrightarrow{\partial_*} H_{q-1}(C \otimes Z') \to \dots,$$

where $C \otimes Z' = \bigoplus_j C^j$, where $(C^j)_q \overset{\text{def}}{=} C_{q-j} \otimes Z_j(C')$, and $C \otimes B' = \bigoplus_j \bar{C}^j$, where $(\bar{C}^j)_q \overset{\text{def}}{=} C_{q-j} \otimes B_{j-1}(C')$. Since $Z_j(C')$ and $B_j(C')$ are free, we get

$$H_q(C \otimes Z') \simeq \bigoplus_j (C^j) = \bigoplus_{i+j=q} H_i(C) \otimes Z_j(C'),$$

$$H_q(C \otimes B') \simeq \bigoplus_j (\bar{C}^j) = \bigoplus_{i+j=q-1} H_i(C) \otimes B_j(C').$$

The mapping $\partial_*$ corresponds to a homomorphism $(-1)^i \mathrm{Id} \otimes \gamma_j$, where $\gamma_j : B_j(C') \hookrightarrow Z_j(C')$. Therefore, we have a SES

$$0 \to \bigoplus_{i+j=q} [\mathrm{Coker}((-1)^i \mathrm{Id} \otimes \gamma_j)] \to H_q(C \otimes C') \to \bigoplus_{i+j=q-1} [\mathrm{Ker}((-1)^i \mathrm{Id} \otimes \gamma_j)] \to 0.$$

In order to compute both sides of the above sequence, we consider a SES

$$0 \to B_j(C') \xrightarrow{\gamma_i} Z_j(C') \to H_j(C') \to 0.$$

Since $B_j(C')$ and $Z_j(C')$ are free, there exists a SES

$$0 \to H_i(C) \star H_j(C') \to H_i(C) \otimes B_j(C') \xrightarrow{(-1)^i \mathrm{Id} \otimes \gamma_i} H_i(C) \otimes Z_j(C') \to H_i(C) \otimes H_j(C') \to 0$$

which yields

$$\bigoplus_{i+j=q} [\mathrm{Coker}((-1)^i \mathrm{Id} \otimes \gamma_j)] = \bigoplus_{i+j=q} H_i(C) \otimes H_j(C')$$

and

$$\bigoplus_{i+j=q-1} [\mathrm{Ker}((-1)^i \mathrm{Id} \otimes \gamma_j)] = \bigoplus_{i+j=q-1} H_i(C) \star H_j(C').$$

Substituting back, we arrive at

$$0 \to [H(C) \otimes H(C')]_q \xrightarrow{\nu} H_q(C \otimes C') \to [H(C) \star H(C')]_{q-1} \to 0.$$

Now we must verify that $\nu$ is identical to the mapping $\mu$. For given $\{c\} \in H(C)$ and $\{c'\} \in H(C')$, we have $\{c\} \otimes c' \in H(C) \otimes Z(C')$ and $\{c\} \otimes c' = \{c \otimes c'\}_{C \otimes Z(C')}$. Therefore, $\nu(\{c\} \otimes \{c'\}) = \{c \otimes c'\}_{C \otimes C'} = \mu(\{c\} \otimes \{c'\})$. Hence, we get the desired functorial SES. Assume $C$ is free. By Lemma 45, it suffices to find left inverse of $\mu$. Because $C, C'$ are free modules, so are $B(C), B(C'), Z(C)$, and $Z(C')$. Therefore, there exist homomorphisms $p : C \to Z(C)$ and $p' : C' \to Z(C)$ such that $p(c) = c$ and $p(c') = c'$ for all $c \in Z(C)$ and $c' \in Z(C')$. Then the homomorphism $p \otimes p' : C \otimes C' \mapsto Z(C) \otimes Z(C')$ maps $B(C \otimes C')$ (which is contained in the union of $\mathrm{Im}[B(C) \otimes C' \to C \otimes C']$ and $\mathrm{Im}[C \otimes B(C') \to C \otimes C'])$ into the union of $\mathrm{Im}[B(C) \otimes Z(C') \to Z(C) \otimes Z(C')]$ and $\mathrm{Im}[Z(C) \otimes B(C') \to Z(C) \otimes Z(C')]$. Therefore, the composite

$$\varphi : Z(C \otimes C') \subseteq C \otimes C' \xrightarrow{p \otimes p'} Z(C) \otimes Z(C') \to H(C) \otimes H(C')$$

maps $B(C \otimes C') \subset Z(C \otimes C') \xrightarrow{\varphi} 0$. So, it induces homomorphism $H(C \otimes C') \to H(C) \otimes H(C')$ as a left inverse of $\mu$.

**76 Corollary.** *If $C'$ is a free chain complex and either $C$ or $C'$ is acyclic, then $C \otimes C'$ is acyclic.*

There is a generalization of the previous Lemma.

**77 Theorem.** *In the subcategory of the product category of chain complexes $C, C'$ such that $C \star C'$ is acyclic there exists a functorial SES*

$$0 \to [H(C)] \otimes H(C')]_q \xrightarrow{\mu} H_q(C \otimes C') \to [H(C) \star H(C')]_{q-1} \to 0$$

*which splits.*

*Proof.* The proof is omitted. It's based on the free approximation of chain complexes and their acyclicity properties.

**78 Definition.** *Let $C, C'$ be chain complexes over $R$ and let $G, G'$ be $R$-modules ($R$ having the property of PID). The composite of mappings*

$$H(C \otimes G) \otimes H(C' \otimes G') \xrightarrow{\mu} H[(C \otimes G) \otimes (C' \otimes G')] \to H[(C \otimes C') \otimes (G' \otimes G')],$$

*where the RHS homomorphism is induced by the canonical isomorphism*

$$(C \otimes G) \otimes (C' \otimes G') \simeq (C \otimes C') \otimes (G \otimes G'),$$

*induces a functorial homomorphism*

$$\mu' : H(C \otimes G) \otimes H(C' \otimes G') \to H(C \otimes C', G' \otimes G')$$

*called* **cross product**. *For $z \in H(C, G)$ and $z' \in H(C', G')$ we write*

$$z \boxtimes z' \stackrel{\text{def}}{=} \mu'(z \otimes z') \in H[(C \otimes C', G' \otimes G').$$

**79 Lemma.** *For given torsion-free chain complexes $C, C'$ and modules $G, G'$ such that $G \star G' = 0$, there exists a functorial SES*

$$0 \to [H(C, G) \otimes H(C', G')]_q \xrightarrow{\mu'} H_q(C \otimes C', G \otimes G') \to [H(C, G) \star H(C', G')]_{q-1} \to 0$$

*and the sequence splits.*

*Proof.* The proof will follow from the previous Theorem once we verify that $(C \otimes G) \star (C' \otimes G')$ is a trivial complex. Let $0 \to F' \to F \to G \to 0$ be free presentation of $G$. Because we assure $G \star G' = 0$ the corresponding SES has a form

$$0 \to F' \otimes G' \to F \otimes G' \to G \otimes G' \to 0.$$

Since $C, C'$ are free chain complexes, there exists a SES

$$(*) \qquad 0 \to (C \otimes F') \otimes (C' \otimes G') \to (C \otimes F) \otimes (C' \otimes G') \to (C \otimes G) \otimes (C' \otimes G') \to 0$$

There is also a SES

$$0 \to C \otimes F' \to C \otimes F \to C \otimes G \to 0.$$

Since $C \otimes F'$ and $C \otimes F$ are free $R$-modules the corresponding SES has a form

$$0 \to (C \otimes G) \star (C' \otimes G') \to (C \otimes F') \otimes (C' \otimes G') \to (C \otimes F) \otimes (C' \otimes G') \to (C \otimes G) \otimes (C' \otimes G') \to 0.$$

Thus,

$$(C \otimes G) \star (C' \otimes G') \simeq \mathrm{Ker}[(C \otimes F') \otimes (C' \otimes G') \to (C \otimes F) \otimes (C' \otimes G')]$$

but this is trivial as follows from (*).

The cross product defined above commutes, in graded sense, with connecting homomorphism $\partial_*$.

**80 Theorem.** *Let $0 \to \overline{\overline{C}} \to \bar{C} \to C \to 0$ be a splitting SES of free chain complexes, and let $z \in H(C, G), z' \in H(C', G')$. Then*

$$\partial_*(z \boxtimes z') = \partial_*(z) \boxtimes z' \quad \textit{and} \quad \partial_*(z' \boxtimes z) = (-1)^{\deg z'} z \boxtimes \partial_*(z').$$

*Proof.* We have a commutative diagram of chain maps

$$\begin{tikzcd}
0 \arrow[r] & \overline{\overline{C}} \otimes G \arrow[r] \arrow[d, "\overline{\overline{\tau}}"] & \bar{C} \otimes G \arrow[r] \arrow[d, "\bar{\tau}"] & C \otimes G \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & (\overline{\overline{C}} \otimes G) \otimes (C' \otimes G') \arrow[r] & (\bar{C} \otimes G) \otimes (C' \otimes G') \arrow[r] & (C \otimes G) \otimes (C' \otimes G') \arrow[r] & 0
\end{tikzcd}$$

with exact rows and with the vertical maps defined by forming the tensor product on the right with $c' \in Z(C' \otimes G')$, while $z' = \{c'\}$ (i.e. $\tau(c) = c \otimes c'$ for $c \in C \otimes G$). Because $c'$ is a cycle, each vertical map is a chain map. Since the connecting homomorphism is functorial, we obtain a commutative diagram

$$\begin{tikzcd}
H(C \otimes G) \arrow[r, "\tau_*"] \arrow[d, "\partial_*"] & H[(C \otimes G) \otimes (C' \otimes G')] \arrow[r, phantom, "\simeq"] \arrow[d, "\partial_*"] & H[(C \otimes C') \otimes (G \otimes G')] \arrow[d, "\partial_*"] \\
H(\overline{\overline{C}} \otimes G) \arrow[r, "\overline{\overline{\tau}}_*"] & H[(\overline{\overline{C}} \otimes G) \otimes (C' \otimes G')] \arrow[r, phantom, "\simeq"] & H[(\overline{\overline{C}} \otimes C') \otimes (G \otimes G')]
\end{tikzcd}$$

where each vertical is a suitable connecting homomorphism. The top row maps $z \mapsto z \boxtimes z'$, the bottom one maps $\partial_* z \mapsto \partial_*(z) \boxtimes z'$. This proved the first part of the claim. The second one can be proven similarly. The only difference is that the tensor product with $c'$ on the left is not a chain map, but either commutes or anticommutes with the boundary operators for the presence of the factor $(-1)^{\deg z'}$ in the second equation above.

**81 Theorem.** (Eilenberg-Zilber) *In the category of ordered pairs of topological spaces $X, Y$ there exists a natural chain equivalence of the functor $\Delta(X \times Y)$ with the functor $\Delta(X) \otimes \Delta(Y)$.*

*Proof.* It is based on the method of acyclic models and can be found in [Sp], Th.6,p.232-233.

Because $\Delta(X)/\Delta(A)$ and $\Delta(Y)/\Delta(B)$ are free chain complexes, we deduce the following **Künneth formula** for singular homology.

**82 Theorem.** *If $\{X \times B, A \times Y\}$ is an excessive couple in $X \times Y$ and $G, G'$ are modules over PID such that $G \star G' = 0$, then there exists a functorial SES*

$$0 \to [H(X, A, G) \otimes H(Y, B, G')]_q \xrightarrow{\mu'} H_q((X, A) \times (Y, B), G \otimes G') \to [H(X, A, G) \star H(Y, B, G')]_{q-1} \to 0$$

*and the sequence splits.*

# 2. Cohomology Theory

## 2.1 Cohomology with Coefficients

**83 Definition.** *Let $R$ be a commutative ring with unit. A* **cochain complex** *over $R$, usually denoted $C^* \equiv \{C^q, \delta^q\}$, is a graded $R$-module together with a differential $\delta$ which is an $R$-linear homogenous mapping such that $|\delta| = 1$, so called* **coboundary operator.**

*From the definition we have $\delta^q : C^q \to C^{q+1}$ with $\delta^{q+1}\delta = 0$ for all $q$. We denote $Z(C^*)$ the $R$-module of* **cocycles***, i.e $Z(C^*) \overset{\text{def}}{=} \text{Ker}(\delta)$, and $B(C^*)$ the $R$-module of* **coboundaries***, i.e $B(C^*) \overset{\text{def}}{=} \text{Im}(\delta)$. Then clearly $B(C^*) \subset Z(C^*)$.*

*The* **cohomology** *of cochain complex is defined to be $H(C^*) \overset{\text{def}}{=} Z(C^*)/B(C^*)$.*

**84 Observation.** For a cochain complex $C^*$, there exists a chain complex $C_*$ defined by $C_q \overset{\text{def}}{=} C^{-q}$ and $\partial_q : C_q \to C_{q-1} \overset{\text{def}}{=} \delta^{-q} : C^{-q} \to C^{-q+1}$. Then $H_q(C_*) = H^{-q}(C^*)$ and the cohomology a $R$-module $C^*$ corresponds to the homology of the $R$-module $C_*$.

In this way, the results about chain complexes imply the results about cochain complexes. Thus the cohomology module is a covariant functor from the category of graded abelian groups and their homeomorphisms. For a given SES of cochain complexes

$$0 \to \overline{\overline{C}}^* \xrightarrow{\alpha} \bar{C}^* \xrightarrow{\beta} C^* \to 0$$

there exists a functorial **connecting homomorphism**

$$\delta^* H(C^*) \to H(\overline{\overline{C}}^*) \quad \text{such that} \quad |\delta^*| = 1$$

and a functorial exact sequence

$$\cdots \to H^q(C^*) \xrightarrow{\delta^*} H^{q+1}(\overline{\overline{C}}^*) \xrightarrow{\alpha^*} H^{q+1}(\bar{C}^*) \xrightarrow{\beta^*} H^{q+1}(C^*) \xrightarrow{\delta^*} \dots$$

Taking into account the shift in degrees give the following result.

**85 Theorem.** (from the Theorem 68) *For a given cochain complex $C^*$ and a module $G$ such that $C^* \star G$ is an acyclic cochain complex, there exists a functorial SES*

$$0 \to H^q(C^*) \otimes G \xrightarrow{\mu} H^q(C^* \otimes G) \to H^{q+1}(C^*) \star G \to 0$$

*and the sequence splits.*

**86 Theorem.** (from the Theorem 77) *For a given cochain complexes $C^*, \bar{C}^*$ such that $C^* \star G$ is an acyclic cochain complex, there exists a functorial SES*

$$0 \to [H^*(C^*) \otimes H^*(\bar{C}^*)]^q \xrightarrow{\mu} H^q(C^* \otimes \bar{C}^*) \to [H^*(C^*) \star H^*(\bar{C}^*)]^{q+1} \to 0$$

*and the sequence splits.*

**87 Definition.** *An* **augmented cochain complex** *over $A$ consists of a non-negative cochain complex $C^*$, i.e $C^q \equiv 0$ for $q < 0$, and an augmentation over $A$, i.e a monomorphism $\mu : A \to C^0$ such that $\delta^0 \circ \mu = 0$.*

*The* **reduced cochain complex** *$\tilde{C}^*$ of an augmented cochain complex $C^*$ is defined to be the cochain complex $\tilde{C}^q \stackrel{\text{def}}{=} C^q$ for $q \neq 0$, and $\tilde{C}^0 \stackrel{\text{def}}{=} \mathrm{Coker}(\mu)$ together with $\tilde{\delta}^q$ induced by $\delta^q$. Moreover, we define $\tilde{H}(C^*) \stackrel{\text{def}}{=} H(\tilde{C}^*)$.*

Evidently there is a SES

$$0 \to A \xrightarrow{\mu} C^* \to \tilde{C}^* \to 0$$

inducing the isomorphisms $H^q(C^*) \simeq \tilde{H}^q(C^*)$ for $q \neq 0$ and a SES for $q = 0$

$$0 \to A \to H^0(C^*) \to \tilde{H}^0(C^*) \to 0.$$

**88 Definition.** *Let $C_*$ be a chain complex over $R$ and let $G$ be $R$-mod. Then there is a cochain complex $\mathrm{Hom}(C, G) \equiv \{\mathrm{Hom}(C_q, G), \delta^q\}$, where*

$$\delta^q : f \in \mathrm{Hom}(C_q, G) \mapsto \delta^q f \in \mathrm{Hom}(C_{q+1}, G)$$

*is defined by $(\delta^q f)(c) \stackrel{\text{def}}{=} f(\partial_{q+1} c)$ for all $c \in C_{q+1}$.*

We shall write $< f, c > \equiv f(c)$ and set $< f, c >= 0$ for $\deg(f) \neq \deg(c)$. Then $< \delta^q f, c >=< f, \partial_{q+1} c >$ .

If $C_*$ is an augmented chain complex over $A$ with $\varepsilon : C_0 \to A$, then evidently $\mathrm{Hom}(C, G)$ is augmented by $\mu : \mathrm{Hom}(A, G) \to \mathrm{Hom}(C_0, G)$, where $(\mu(f))(c) \stackrel{\text{def}}{=} f\varepsilon(c)$ for $f \in \mathrm{Hom}(A, G)$ and $c \in C_0$.

$$\begin{tikzcd}
C_1 \arrow[r] & C_0 \arrow[r, "\varepsilon"] \arrow[d, "\mu f"'] \arrow[dr, phantom, "\overset{\mu}{\Longleftarrow}"] & A \arrow[r] \arrow[d, "f"] & 0 \\
G \arrow[r, leftrightarrow, "\mathrm{Id}"] & G \arrow[r, leftrightarrow, "\mathrm{Id}"] & G
\end{tikzcd}$$

**89 Theorem.** *There exists a functor from the product category of the opposite category of the chain complexes and the category of modules to the category of graded abelian groups denoted $\mathrm{Hom}(C, G)$ which is contravariant in chain complexes $C$ and covariant in modules $G$.*

**90 Definition.** *For a given chain complex $C_*$ and a module $G$, we define a homology module to be $H^*(C,G) \equiv \{H^q(C,G)\}_q$ by $H^q(C,G) \stackrel{\text{def}}{=} H^q(\text{Hom}(C,G))$.*

For a chain map $\tau : C \to C'$ we get a cochain map

$$\text{Hom}(\tau, \text{Id}) : \text{Hom}(C'G) \to \text{Hom}(C,G)$$

inducing $\tau^* : H^*(C',G) \to H^*(C,G)$. For a homomorphism $\varphi : G \to G'$ we get a homomorphism

$$\text{Hom}(\text{Id}, \varphi) : \text{Hom}(C,G) \to \text{Hom}(C,G')$$

inducing $\varphi^* : H^*(C,G) \to H^*(C,G')$.

**91 Observation.** For an abelian group $G$ and a simplicial pair $(K,L)$, the oriented cohomology group of $(K,L)$ with coefficients $G$, denoted $H^*(K,L,G)$, is defined to be the graded cohomology group of the cochain complex $\text{Hom}(C(K)/C(L),G)$ augmented over $\text{Hom}(\mathbb{Z},G) \simeq G$. Then $H^*(K,L,G)$ is a contravariant functor in $(K,L)$ and a covariant functor in $G$ to the category of graded abelian groups.

**92 Observation.** Let $(X,A)$ be a topological pair and let $G$ be an abelian group. The singular cohomology group of $(X,A)$ with coefficients $G$, denoted $H^*(X,A,G)$, is defined to be the graded cohomology group of the cochain complex $\text{Hom}(\Delta(X)/\Delta(A),G)$ augmented over $G$. Then $H^*(X,A,G)$ is a contravariant functor in $(X,A)$ and a covariant functor in $G$ to the category of graded abelian groups. If $i : (X',A') \hookrightarrow (X,A)$ and $u \in H^*(X,A,G)$, we denote $u/(X',A')$ the element of $H^*(X',A',G)$ equal to $i^*u$. We also denote the image of $1 \in R$ under the augmentation map $\mu : R \to H^0(X,R)$ by $1 \in H^0(X,R)$.

Now we will introduce two Lemmas about properties of the functor Hom.

**93 Lemma.** *For a given sequence of $R$-modules $A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$ and an $R$-mod $B$, there exists an exact sequence*

$$0 \to \text{Hom}(A'',B) \to \text{Hom}(A,B) \to \text{Hom}(A',B).$$

*Proof.* Let $f,g \in \text{Hom}(A'',B)$ s.t. $\text{Hom}(\beta,\text{Id})(f) = \text{Hom}(\beta,\text{Id})(g)$. Then $f\beta = g\beta$. Since $\beta$ is epimorphism, we get $f = g$. Hence $\text{Hom}(\beta,\text{Id})$ is mono. The rest is analogous therefore it is omitted.

If $A' \to A$ is a monomorphism (i.e. if $0 \to A' \to A$ is an exact sequence), it need not be the case $\text{Hom}(A,B) \to \text{Hom}(A',B)$ is an epimorphism (i.e. $\text{Hom}(A,B) \to \text{Hom}(A',B) \to 0$ is an exact sequence).

**94 Lemma.** *For a given a splitting SES of $R$-modules $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$ and a $R$-mod $B$, there exists a SES*

$$0 \to \text{Hom}(A'',B) \to \text{Hom}(A,B) \to \text{Hom}(A',B) \to 0$$

*and the sequence splits.*

*Proof.* $\mathrm{Hom}(\alpha, \mathrm{Id})(g)$ is epi because for any $g \in \mathrm{Hom}(A', B)$ there exists $\alpha' g \in \mathrm{Hom}(A, B)$ s.t. $\alpha\alpha' g = g$.

In the case $0 \to C' \to C \to C'' \to 0$ is a splitting SES of chain complexes, it follows for any $R$-mod $G$ there exists a SES of cochain complexes

$$0 \to \mathrm{Hom}(C'', G) \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(C', G) \to 0.$$

**95 Theorem.** *For a given splitting SES of chain complexes $0 \to C' \to C \to C'' \to 0$ and a $R$-mod $G$, there exists a functorial LES of cohomology groups*

$$\dots \to H^q(C'', G) \to H^q(C, G) \to H^q(C', G) \xrightarrow{\partial^*} H^{q+1}(C'', G) \to \dots$$

There is also an analog of Mayer-Vietoris sequence for cohomology.

**96 Theorem.** *If $\{(X_1, A_1), (X_2, A_2)\}$ is an excessive couple of pairs of topological spaces and $G$ is $R$-mod, there exists a functorial LES of cohomology groups*

$$\dots \xrightarrow{\partial^*} H^q(X_1 \cup X_2, A_1 \cup A_2, G) \xrightarrow{j^*} H^q(X_1, A_1, G) \oplus H^q(X_2, A_2, G) \xrightarrow{i^*} H^q(X_1 \cap X_2, A_1 \cap A_2, G) \xrightarrow{\partial^*} \dots,$$

*where $j^*(u) = (j_1^*(u), j_2^*(u))$ and $i^*(u_1 + u_2) = i_1^* u_1 - i_2^* u_2$, where $i_1, i_2, j_1, j_2$ are suitable inclusion maps.*

If $\{X_i\}_i$ is the set of path components of $X$, then $\Delta(X) = \bigoplus_j \Delta(X_j)$. Therefore, $\mathrm{Hom}(\Delta(X), G) = \prod_j \mathrm{Hom}(\Delta(X_j), G)$, and the singular cohomology module of the space $X$ is a direct product of singular cohomology modules of its path components $X_j$.

There is also a similar result for a SES of coefficients modules as in the case of homology (See Theorem 65).

**97 Theorem.** *In the product category of free chain complexes over $R$ and SES of $R$-modules of a form $0 \to G' \xrightarrow{f} G \xrightarrow{g} G'' \to 0$, there exists a natural connecting homomorphism $\partial^* : H^*(C, G'') \to H^{*+1}(C, G')$ such that $|\partial^*| = 1$ and a functorial exact sequence of cohomology groups*

$$\cdots \to H^q(C, G') \xrightarrow{f^*} H^q(C, G) \xrightarrow{g^*} H^q(C, G'') \xrightarrow{\partial^*} H^{q+1}(C, G') \to \cdots$$

*Proof.* Because $C$ is a free $R$-module, there exists a SES of cochain complexes

$$0 \to \mathrm{Hom}(C, G') \xrightarrow{\bar{f}} \mathrm{Hom}(C, G) \xrightarrow{\bar{g}} \mathrm{Hom}(C, G'') \to 0,$$

where $\bar{f} \equiv \mathrm{Hom}(\mathrm{Id}, f)$ and $\bar{g} \equiv \mathrm{Hom}(\mathrm{Id}, g)$.

**98 Definition.** *The connecting homomorphism $\partial^*$ is usually called* **Bochstein cohomology homomorphism** *of the SES $0 \to G' \to G \to G'' \to 0$.*

### Axiomatic Cohomology Theory

**99 Definition.** *Let $G$ be an $R$-module. A* **cohomology theory** *$(H^*, \delta^*)$ with coefficients $G$ consists of a covariant functor $H^* \equiv \{H^q\}$ from the category of topological pairs to the category of graded $R$-modules and a natural transformation $\delta^*$ of degree $1$ from the functor $H^* : (A, \emptyset) \mapsto H(A)$ to the functor $H^* : (X, A) \mapsto H(X, A)$ such that the following axioms hold:*

*(1)* **Homotopy axiom:** *If $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic, then*

$$H^*(f_0) = H^*(f_1) : H^*(Y, B) \to H^*(X, A).$$

*(2)* **Exactness axiom:** *For any pair $(X, A)$ with inclusions $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$, there exists a LES*

$$\ldots \xrightarrow{\partial^{q-1}} H^q(X, A) \xrightarrow{H^q(j)} H^q(X) \xrightarrow{H^q(i)} H^q(A) \xrightarrow{\partial^q} \ldots$$

*(3)* **Excision axiom:** *For any pair $(X, A)$ and ant open set $U \subset X$ such that $\overline{U} \subset \mathrm{int}(A)$, there exists an excision map $j : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ inducing isomorphism*

$$H^*(j) : H(X, A) \simeq H(X \setminus U, A \setminus U).$$

*(4)* **Dimension axiom:** *In the full category of one-point spaces, there is a natural equivalence of the constant functor $G$ with the functor $H^*$.*

There is a uniqueness theorem valid for cohomology theories stating that a homomorphism from one cohomology theory to another cohomology theory, which is an isomorphism for one-point spaces, is an isomorphism for compact polyhedral pairs.

The reduced cohomology modules $\tilde{H}^*$ of a cohomology theory are defined as follows.

**100 Definition.** *For $X$ a non empty topological space, let $c : X \to P$ be a unique mapping to the one-point space $P$. We define $\tilde{H}^*(X) \stackrel{\mathrm{def}}{=} \mathrm{Coker}(H^*(c))$, where $H^*(c) : H^*(P) \to H^*(X)$. Because $c$ has a right inverse, $H^*(C)$ has a left inverse. Therefore,*

$$H^*(X) \simeq \tilde{H}^*(C) \oplus H^*(P)$$

*with properties similar to the reduced singular cohomology modules.*

## 2.2 Universal Coefficient Theorem for Cohomology

Let $C$ be a free resolution of the module $A$ and let $B$ be a module over $R$. Then there exists a cochain complex $\mathrm{Hom}(C,B) \equiv \{\mathrm{Hom}(C_q,B),\delta^q\}$ where the cohomology modules depends only on $A, B$ (i.e any other free resolution $C'$ is homotopic equivalent to $C$).

**101 Definition.** *For given $R$-modules $A$ and $B$, choose $C$ to be a canonical free resolution of $A$. For $q \geq 0$ define $\mathrm{Ext}^q(A,B) \overset{\text{def}}{=} H^q(C,B) = H^q(\mathrm{Hom}(C,B))$. Then $\mathrm{Ext}^*(A,B)$ is a functor covariant in $A$ and contravariant $B$.*

For a resolution $C$ and for a module $B$ the Lemma 93 implies that there exists an exact sequence

$$0 \to \mathrm{Hom}(A,B) \xrightarrow{\mathrm{Hom}(\varepsilon,\mathrm{Id})} \mathrm{Hom}(C_0,B) \xrightarrow{\mathrm{Hom}(\partial_1,\mathrm{Id})} \mathrm{Hom}(C_1,B).$$

Since $\mathrm{Ext}^0(A,B)$ was defined to be the $0^{th}$ cohomology $R$-mod of the cochain complex

$$0 \to \mathrm{Hom}(C_0,B) \xrightarrow{\delta^0} \mathrm{Hom}(C_1,B) \xrightarrow{\delta^1} \mathrm{Hom}(C_2,B) \to \cdots,$$

$\mathrm{Ext}^0(A,B) = \mathrm{Ker}(\delta^0)$, while $\mathrm{Ker}(\delta^0) = \mathrm{Im}(\mathrm{Hom}(\varepsilon,\mathrm{Id}))$. Therefore,

$$\mathrm{Ext}^0(A,B) = \mathrm{Im}(\mathrm{Hom}(\varepsilon,\mathrm{Id})) \simeq \mathrm{Hom}(A,B).$$

In the case $R$ is the commutative ring with unity having the property of PID, any submodule of free module is a free module. In particulary, any $A$ has a free resolution $0 \to C_1 \to C_0 \to A \to 0$ where $C_0 = \mathcal{F}(A)$ and $C_1 = \mathrm{Ker}(\mathcal{F}(A)) \to A) \subset \mathcal{F}(A)$. Because $\{C_q\}_{q\geq 2}$ are trivial $R$-modules, $\mathrm{Ext}^q(A,B) = \emptyset$ for $q \geq 2$.

**102 Definition.** *An* **extension** *of $A, B$ is defined by* $\mathrm{Ext}(A,B) \overset{\text{def}}{=} \mathrm{Ext}^1(A,B)$.

It is characterized by the property, that for a given free presentation of $A$

$$0 \to C_1 \to C_0 \to A \to 0$$

there is an exact sequence

$$0 \to \mathrm{Hom}(A,B) \to \mathrm{Hom}(C_0,B) \to \mathrm{Hom}(C_1,B) \to \mathrm{Ext}(A,B) \to 0$$

for

$$\mathrm{Ext}(A,B) = H^1(C,B) = \mathrm{Hom}(C_1,B)/\mathrm{Im}[\mathrm{Hom}(\partial_1,\mathrm{Id})] = \mathrm{Coker}[\mathrm{Hom}(\partial_1,\mathrm{Id})].$$

**103 Observation.** If $A$ is a free module, there is a free presentation $0 \to 0 \to A \to A \to 0$ and so $\mathrm{Ext}(A,B) = 0$ for every $R$-module $B$.

**104 Observation.** If $A$ is a cyclic $R$-mod such that $A \simeq R/RI$ for some principal ideal $I \subset R$, there is a free presentation $0 \to R \xrightarrow{\alpha} R \to A \to 0$ given by $\alpha : r \mapsto rr'$, where $r' \in I$ is a generator of $I$. Then for any $R$-mod $B$ there is an isomorphism $\mathrm{Hom}_R(R, B) \simeq B$ given by $h \in \mathrm{Hom}(R, B) \mapsto h(1) \in B$. Under this isomorphism the mapping $\mathrm{Hom}(\alpha, \mathrm{Id}) : \mathrm{Hom}(R, B) \to \mathrm{Hom}(R, B)$ corresponds to the mapping $\alpha' : B \to B$ given by $b \mapsto r'b$ for all $b \in B$. Therefore, $\mathrm{Coker}[\mathrm{Hom}(\alpha, \mathrm{Id})] \simeq B/r'B$, and so $\mathrm{Ext}(A, B) \simeq B/r'B \simeq R/r'R \otimes_R B$.

**105 Corollary.** *If $A$ is a finitely generated torsion module, then there is an isomorphism (not functorial)* $\mathrm{Ext}(A, B) \simeq A \otimes B$.

*Proof.* It follows from the fact that Hom commutes with finite direct sums and the fact that any such $A$ is a finite sum of cyclic modules.

**106 Definition.** *The* **extension** *$E$ of $A$ by $B$ is given by the SES*

$$0 \to B \to E \to A \to 0$$

*with corresponding equivalences of extensions* $(E, E')$

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal, "\mathrm{Id}"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r] \arrow[d, leftrightarrow] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[r] \arrow[d, equal, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & B \arrow[r] & E' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

**107 Observation.** The set of equivalences of extensions $(E, E')$ is in fact an $R$-mod, isomorphic to $\mathrm{Ext}(A, B)$.

*Proof.* Let $0 \to B \to E \to A \to 0$ be an extension and let $0 \to C_1 \to C_0 \to A \to 0$ be a free presentation of $A$. By the Theorem 54, there exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C_1 \arrow[r] \arrow[d, "\varphi_1"] \arrow[dr, phantom, "\circlearrowleft"] & C_0 \arrow[r] \arrow[d, "\varphi_0"] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[r] \arrow[d, equal, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & B \arrow[r] & E \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

unique up to a chain homotopy. Therefore, the element $\varphi_1 \in \mathrm{Hom}(C_1, B)$ is unique up to $\mathrm{Im}(\mathrm{Hom}(C_0, E) \to \mathrm{Hom}(C_1, B))$ and determines an element of $\mathrm{Ext}(A, B)$ This assignment induces an isomorphism of the ($R$-mod) of equivalence classes of extensions with $\mathrm{Ext}(A, B)$.

For a graded $R$-mod $C \equiv \{C_q\}_q$, there is a graded $R$-mod $\mathrm{Ext}(C, B) \equiv \{\mathrm{Ext}(C_q, B)\}_q$. If $C$ is a chain complex, $\mathrm{Ext}(C, B)$ is a cochain complex with (co)differential

$$\delta^q \stackrel{\text{def}}{=} \mathrm{Ext}(\delta^{q+1}, \mathrm{Id}) : \mathrm{Ext}(C_q, B) \to \mathrm{Ext}(C_{q+1}, B).$$

A homomorphism $h : H^q(C, G) \to \mathrm{Hom}(H_q(C, G'), G \otimes G')$ natural in $C, G$ is defined by

$$(h\{f\})\{\sum_i c_i \otimes g'_i\} = \sum_i f(c_i) \otimes g'_i$$

for $\{f\} \in H^q(C,G)$ and $\{\sum_i c_i \otimes g_i'\} \in H_q(C,G)$ (after verification that $\sum_i f(c_i) \otimes g_i'$ is independent of the choice of $f$ in its cohomology class and $\sum_i c_i \otimes g_i'$ in its homology class). For $u \in H^q(C,G)$ and $z \in H_q(C,G')$, we define

$$< u, z > \in G \otimes G' \quad \text{to be} \quad \begin{cases} 0 & p \neq q, \\ (h(u))(z) & p = q. \end{cases}$$

We write $< \{f\}, \{\sum_i c_i \otimes g_i'\} > = \sum_i < f, c_i > \otimes g_i'$.

The homomorphism $h$ enters into the following Universal coefficient Theorem for Cohomology.

**108 Theorem.** (Universal Coefficient Theorem for Cohomology) *Let $C$ be a chain complex and let $G$ be a module such that* $\mathrm{Ext}(C,G)$ *is an acyclic cochain complex. Then, there exists a functorial SES*

$$0 \to \mathrm{Ext}(H_{q-1}(C),G) \to H^q(C,G) \xrightarrow{h} \mathrm{Hom}(H_q(C),G) \to 0$$

*and the sequence splits.*

*Proof.* Let us first consider the case of a free chain complex $C$. Let $Z \subset C$ and $B \subset C$ be subcomplexes of $C$ defined by $Z_q = Z_q(C)$ and $B_q = B_{q-1}(C)$ with trivial boundary operations. Both $Z, B$ are free chain subcomplexes. Evidently there exists a SES $0 \to Z \xrightarrow{\alpha} C \xrightarrow{\beta} B \to 0$ defined by $\alpha : z \mapsto z$ and $\beta : c \in C_q \mapsto \partial_q c$. So there is a SES

$$0 \to \mathrm{Hom}(Z,G) \xrightarrow{\mathrm{Hom}(\alpha,\mathrm{Id})} \mathrm{Hom}(C,G) \xrightarrow{\mathrm{Hom}(\beta,\mathrm{Id})} \mathrm{Hom}(B,G) \to 0$$

giving a LES of cohomology groups

$$\cdots \to H^q(B,G) \xrightarrow{\beta^*} H^q(C,G) \xrightarrow{\alpha^*} H^q(Z,G) \xrightarrow{\delta^q} H^{q+1}(B,G) \to \cdots$$

Since both chains $Z$ and $B$ have trivial boundary operators, so do both cochains $\mathrm{Hom}(Z,G)$ and $\mathrm{Hom}(B,G)$. Therefore, $H^q(Z,G) = \mathrm{Hom}(Z_q,G) = \mathrm{Hom}(Z_q(C),G)$ and $H^q(B,G) = \mathrm{Hom}(B_q,G) = \mathrm{Hom}(B_{q-1}(C),G)$. So the LES above becomes

$$\cdots \to \mathrm{Hom}(B_{q-1}(C),G) \to H^q(C,G) \to \mathrm{Hom}(Z_q(C),G) \xrightarrow{\mathrm{Hom}(\gamma_q,\mathrm{Id})} \mathrm{Hom}(B_q(C),G) \to \cdots$$

where $\gamma_q : B_q(C) \hookrightarrow Z_q(C)$. From this LES we can obtain a functorial SES

$$0 \to \mathrm{Coker}[\mathrm{Hom}(\gamma_q,\mathrm{Id})] \to H^q(C,G) \to \mathrm{Ker}[\mathrm{Hom}(\gamma_q,\mathrm{Id})] \to 0.$$

It remains to interpret the modules at both sides of $H^q(C,G)$. Since $Z_q(C)$ is free for every $q$ (hence, $B_q(C)$ is free either), the free resolution of $H_q(C)$ is

$$0 \to B_q(C) \xrightarrow{\gamma_q} Z_q(C) \to H_q(C) \to 0.$$

Therefore, there exists a SES

$$0 \to \mathrm{Hom}(H_q(C), G) \to \mathrm{Hom}(Z_q(C), G) \xrightarrow{\mathrm{Hom}(\gamma,\mathrm{Id})} \mathrm{Hom}(B_q(C), G) \to \mathrm{Ext}(H_q(C), G) \to 0$$

which immediately implies
$$\begin{aligned} \mathrm{Coker}[\mathrm{Hom}(\gamma, \mathrm{Id})] &\simeq \mathrm{Hom}(H_q(C), G) \\ \mathrm{Ker}[\mathrm{Hom}(\gamma, \mathrm{Id})] &\simeq \mathrm{Ext}(H_q(C), G). \end{aligned}$$

Substituting back we get the desired SES with the homomorphism $H^q(C, G) \to \mathrm{Hom}(H_q(C), G)$ corresponding to $h$.

The sequence splits because the SES $0 \to Z \xrightarrow{\alpha} C \xrightarrow{\beta} B \to 0$ splits. For arbitrary $C$ such that $\mathrm{Ext}(C, G)$ is acyclic, the result follows from the Theorem 73 about a free approximation of $C$.

**109 Lemma.** *If (X,A) is a pair of topological spaces such that $H_q(X, A, R)$ is finitely generated for all $q$, then the free submodules of $H^q(X, A, R)$ and $H_q(X, A, R)$ are isomorphic and similarly the torsion submodules of $H^q(X, A, R)$ and $H_{q-1}(X, A, R)$ are isomorphic.*

*Proof.* Let $H^q(X, A, R) \simeq F_q \oplus T_q$, where $F_q$, resp. $T_q$, denotes free, resp. torsion, modules of $H_q$. Then

$$\mathrm{Hom}(H_q(X, A, R), R) \simeq \mathrm{Hom}(F_q, R) \oplus \mathrm{Hom}(T_q, R) \simeq F_q$$

and by the Corollary 105

$$\mathrm{Ext}(H_q(X, A, R), R) \simeq \mathrm{Ext}(F_q, R) \oplus \mathrm{Ext}(T_q, R) \simeq T_q.$$

The result then follows from the previous Theorem.

Let $C$ and $G$ be finitely generated. Then we shall explore the relation of $H^*(C, G)$ with $H^*(C, R)$. Let $\mu : \mathrm{Hom}(A, G) \otimes G' \to \mathrm{Hom}(A, G \otimes G')$ be the functorial homomorphism defined by $[\mu(f \otimes g')](a) = f(a) \otimes g'$ for $f \in \mathrm{Hom}(A, G)$, $g' \in G'$ and $a \in A$.

**110 Lemma.** *If $A$ is a free $R$-module, $G'$ is a finitely generated $R$-module, then for any module $G$, $\mu$ is an isomorphism.*

*Proof.* The claim is evidently true for $G' = R$ since both Hom and $\otimes$ are over $R$. Since both the tensor product and the Hom functor commute with finite direct sums, the claim is true for finitely generated free module $G'$. Assume now $G'$ is finitely generated $R$-module. Then there exists a SES (a free presentation of $G'$) $0 \to \bar{\bar{G}} \to \bar{G} \to G' \to 0$ where both $\bar{\bar{G}}$ and $\bar{G}$ are finitely generated free $R$-modules. Consider now the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Hom}(A, G) \otimes \bar{\bar{G}} \arrow[r] \arrow[d, "\bar{\bar{\mu}}"] & \mathrm{Hom}(A, G) \otimes \bar{G} \arrow[r] \arrow[d, "\bar{\mu}"] & \mathrm{Hom}(A, G) \otimes G' \arrow[r] \arrow[d, "\mu'"] & 0 \\
\mathrm{Hom}(A, G \otimes \bar{\bar{G}}) \arrow[r] & \mathrm{Hom}(A, G \otimes \bar{G}) \arrow[r] & \mathrm{Hom}(A, G \otimes G') \arrow[r] & 0
\end{tikzcd}$$

By the Lemma 41, the first row is exact. Since $A$ is a free module over $R$, the functor $\mathrm{Hom}(A,-)$ observe exactness. Therefore, the exactness of the second row follows from the exactness of the free presentation of $G'$ and the Lemma 41. By the first part of the proof both $\overline{\overline{\mu}}$ and $\bar{\mu}$ are isomorphism. Hence, the Five isomorphism lemma implies $\mu : \mathrm{Hom}(A,G)\otimes G' \simeq \mathrm{Hom}(A, G\otimes G')$.

Let us now take in account another $R$-mod $B$, and redefine $\mu$ to be functorial homomorphism $\mu : \mathrm{Hom}(A,G)\otimes\mathrm{Hom}(B,G') \to \mathrm{Hom}(A\otimes B, G\otimes G')$. defined by $[\mu(f\otimes f')](a\otimes b) = f(a)\otimes f'(b)$ for $f\in\mathrm{Hom}(A,G)$, $f'\in\mathrm{Hom}(B,G')$, $a\in A$, and $b\in B$. Evidently, the case $B=R$ corresponds to the previous situation of the above Lemma.

**111 Lemma.** *Let $G'=R$. Then for arbitrary $R$-modules $A$ and $G$, and $B$ a finitely generated free module, $\mu$ is an isomorphism, i.e.*

$$\mu : \mathrm{Hom}(A,G)\otimes\mathrm{Hom}(B,R) \simeq \mathrm{Hom}(A\otimes B, G).$$

*Proof.* The claim is evidently true for $B=R$. Assume now $B$ is finitely generated free $R$-module, i.e. $B\simeq R^{\oplus}$. The claim then follows from the commutativity of the direct sum with tensor product and the functor Hom.

**112 Lemma.** *Let $A$ and $B$ are free modules. If either $A$ and $B$ or $B$ and $G'$ are finitely generated, then $\mu$ is an isomorphism, i.e.*

$$\mu : \mathrm{Hom}(A,G)\otimes\mathrm{Hom}(B,G') \simeq \mathrm{Hom}(A\otimes B, G\otimes G').$$

*Proof.* Since $A$ and $B$ are free, the module $A\otimes B$ is also a free $R$-mod. Assume $A$ and $B$ are finitely generated, so is $A\otimes B$. Consider now the following commutative diagram.

$$\begin{array}{ccc}
\begin{array}{c}[\mathrm{Hom}(R,G)\otimes\mathrm{Hom}(A,R)]\otimes \\ \otimes[\mathrm{Hom}(R,G')\otimes\mathrm{Hom}(B,R)]\end{array} & \xrightarrow{\bar{\mu}} & \mathrm{Hom}(R,G\otimes G')\otimes\mathrm{Hom}(A\otimes B,R) \\
\Big\downarrow{\scriptstyle \mu\otimes\mu} & & \Big\downarrow{\scriptstyle \mu} \\
\mathrm{Hom}(A,G)\otimes\mathrm{Hom}(B,G') & \xrightarrow{\quad\mu\quad} & \mathrm{Hom}(A\otimes B, G\otimes G')
\end{array}$$

where $\mu((f_1\otimes f_2)\otimes(f_3\otimes f_4) = \mu(f_1\otimes f_3)\otimes\mu(f_2\otimes f_4)$. From the previous Lemma, we can conclude that top horizontal map $\bar{\mu}$ and both vertical maps $\mu\otimes\mu$ and $\mu$ are isomorphisms. Then the commutativity of the diagram forces the bottom map $\mu$ to be an isomorphism.

Consider now the other case and assume $B$ and $G'$ are finitely generated. There is again a commutative diagram.

$$\begin{array}{ccc}
\mathrm{Hom}(A,G)\otimes\mathrm{Hom}(B,R)\otimes G' & \xrightarrow{\mathrm{Id}\otimes\mu} & \mathrm{Hom}(A\otimes G)\otimes\mathrm{Hom}(B\otimes G') \\
\Big\downarrow{\scriptstyle \mu\otimes\mathrm{Id}} & & \Big\downarrow{\scriptstyle \mu} \\
\mathrm{Hom}(A\otimes B,G)\otimes G' & \xrightarrow{\quad\mu\quad} & \mathrm{Hom}(A\otimes B, G\otimes G')
\end{array}$$

From the previous Lemma, we can conclude that the left-hand vertical map $\mu \otimes \mathrm{Id}$ is an isomorphism. The Lemma 110 implies that both horizontal maps $\mathrm{Id} \otimes \mu$ and $\mu$ are isomorphisms. Then the commutativity of the diagram forces the right-hand vertical map $\mu$ to be an isomorphism.

Let $G = B = R$ and $G' = A$. If $A$ is free and finitely generated, it follows immediately from the Lemma 110 that $\mu$ is an isomorphism, i.e $\mu : \mathrm{Hom}(A, R) \otimes A \simeq \mathrm{Hom}(A, A)$. The following Lemma is a partial converse of this result.

**113 Lemma.** *If $A$ is a $R$-module such that $\mu : \mathrm{Hom}(A, R) \otimes A \twoheadrightarrow \mathrm{Hom}(A, A)$ is an epimorphism, then $A$ is finitely generated.*

*Proof.* Let us consider $1_A \in \mathrm{Hom}(A, A)$. It follows that there exist $f_i \in \mathrm{Hom}(A, R)$ and $a_i \in A$ for $1 \leq i \leq n$ such that $\mu(\sum_i F_i \otimes a_i) = 1_A$. Then for any $a \in A$, we get

$$a = \mu(\sum_{i=1}^{n} f_i \otimes a_i)(a) = \sum_{i=1}^{n} f_i(a) \cdot a_i.$$

Hence, $A$ is finitely generated by $\{a_i\}_i \in A$.

**114 Definition.** *A graded $R$-mod $C \equiv \{C_q\}_q$ is of a* **finite type** *if $C_q$ is finitely generated for all $q$.*

Thus a graded $R$-mod $C$ is finitely generated iff $C_q = 0$ for all but finitely many $q \in Z$. The following Lemma gives suitable approximation to the chain complex with finitely generated homologies.

**115 Lemma.** *Let $C$ be a free chain complex such that $H(C)$ is of the finite type. Then there exists another free chain complex $C'$ of finite type that is chain equivalent to $C$.*

*Proof.* Let $F_q$ for all $q$ be a finitely generated submodule of $Z_q(C)$ such that $F_q$ maps onto $H_q(C)$ under the epimorphism $Z_q(C) \twoheadrightarrow H_q(C)$. Let $F'_q$ be the kernel of the epimorphism $F_q(C) \twoheadrightarrow H_q(C)$. Define a chain complex $C' \equiv \{C'_q, \partial'_q\}$ by $C'_q \stackrel{\text{def}}{=} F_q \oplus F'_{q-1}$ and $\partial'_q(c, c') = (c', 0)$ for $c \in F_q$ and $c' \in F'_{q-1}$. Then $C'$ is a free chain complex of finite type and $H_q(C') = F_q/F'_q \simeq H_q(C)$. To define a chain equivalence $\tau : C' \simeq C$, choose for all $q$ homomorphism $\varphi_q : F'_q \to C_{q+1}$ such that $\partial_{q+1}\varphi_q(c') = c'$ for $c' \in F'_q$. Then define $\tau$ by $\tau(c, c') = c + \varphi_{q-1}(c')$ for $c \in F_q$ and $c' \in F'_{q-1}$. One can easily verify that $\tau$ is a chain map. It induces an isomorphism $\tau_* : H(C') \simeq H(C)$. Because $C$ and $C'$ are free chain complexes, $H(C') \simeq H(C)$ implies via the Theorem (?) that $\tau$ is a chain equivalence.

**116 Theorem.** *Let $C$ be a free chain complex and $G$ be a $R$-module. If either $H(C)$ is of finite type or $G$ is finitely generated, then there exists a functorial SES*

$$0 \to H^q(C, R) \underset{R}{\otimes} G \xrightarrow{\mu} H^q(C, G) \to H^{q+1}(C, R) \star G \to 0$$

*and the sequence splits.*

*Proof.* Assume that $G$ is finitely generated. It follows from the Lemma 110

$$\mu : \mathrm{Hom}_R(C, R) \underset{R}{\otimes} G \simeq \mathrm{Hom}_R(C, G).$$

Because $\mathrm{Hom}_R(C, R)$ is a torsion free $R$-module, $\mathrm{Hom}_R(C, G) \star G = 0$. The claim then follows from the Theorem 85. Assume now that $H(C)$ is of finite type. By the previous Lemma, there exists a free chain complex $C'$ of finite type which is chain equivalent to C. Without loss of generality we can replace $C$ with $C'$. By the Lemma (?)

$$\mu : \mathrm{Hom}_R(C', R) \underset{R}{\otimes} G \simeq \mathrm{Hom}_R(C', G).$$

The claim then follows again from the Theorem 85.

**117 Theorem.** (Künneth formula for cohomology) *Let $C$ and $C'$ be free chain complexes and let $G$ and $G'$ be modules over PID, such that $G \star G' = 0$. Let either both $H(C)$ and $H(C')$ be of finite type or $H(C')$ be of finite type and $G'$ be finitely generated. Then there exists a functorial SES*

$$0 \to [H^*(C, G) \underset{R}{\otimes} H^*(C', G')]^q \to H^q(C \otimes C', G \otimes G') \to [H^*(C, G) \star H^*(C', G')]^{q+1} \to 0.$$

*and the sequence splits.*

*Proof.* If both $H(C)$ and $H(C')$ are of finite type, by the Lemma 115, we can replace $C, C'$ by free chain complexes of finite type. Hence we can assume that either both $C$ and $C'$ are of finite type or $C'$ is of finite type and $G'$ is finitely generated. By the Lemma 112, we get in both cases an isomorphism of $R-$modules .

$$\mu : \mathrm{Hom}(C, G) \underset{R}{\otimes} \mathrm{Hom}(C', G') \simeq \mathrm{Hom}(C \otimes C', G \otimes G')$$

The result will now follow from the Theorem (?) once we verify that

$$\mathrm{Hom}(C, G) \star \mathrm{Hom}(C', G')$$

is acyclic chain complex.

In the case both $C$ and $C'$ are of finite type, $\mathrm{Hom}(C_p, G)$ (for every $p$), resp. $\mathrm{Hom}(C'_q, G')$ (for every $q$), is isomorphic to the finite sum of copies of $G$, resp. $G'$. Because $G \star G' = 0$ by hypothesis, we get $\mathrm{Hom}(C_p, G) \star \mathrm{Hom}(C'_q, G') = 0$, and so $\mathrm{Hom}(C, G) \star \mathrm{Hom}(C', G')$ is acyclic chain complex.

In the case $C'$ is of finite type and $G'$ is finitely generated, $\mathrm{Hom}(C'_q, G')$ is isomorphic to a finite direct sum of $R$-modules $G'$. Therefore, it suffices to show that $\mathrm{Hom}(C, G) \star G' = 0$ for finitely generated $G'$. Let $0 \to \bar{\bar{G}} \to \bar{G} \to C' \to 0$ be a free resolution of $G'$ with finitely generated $\bar{\bar{G}}, \bar{G}$. Because $G \star G' = 0$, there exists a SES

$$0 \to G \underset{R}{\otimes} \bar{\bar{G}} \to G \underset{R}{\otimes} \bar{G} \to G \underset{R}{\otimes} C' \to 0$$

and a SES of cochain complexes

$$0 \to \mathrm{Hom}(C, G \underset{R}{\otimes} \bar{\bar{G}}) \to \mathrm{Hom}(C, G \underset{R}{\otimes} \bar{G}) \to \mathrm{Hom}(C, G \underset{R}{\otimes} C') \to 0.$$

Using the Lemma (?), we get a SES

$$0 \to \mathrm{Hom}(C, G) \underset{R}{\otimes} \bar{\bar{G}} \to \mathrm{Hom}(C, G) \underset{R}{\otimes} \bar{G} \to \mathrm{Hom}(C, G) \underset{R}{\otimes} C' \to 0.$$

Hence $\mathrm{Hom}(C, G) \star G' = 0$.

**118 Observation.** If $A$ is a free, finitely generated $R$-mod, then $A \simeq \mathrm{Hom}(\mathrm{Hom}(A, R), R)$. Since $\mathrm{Hom}(A, R)$ is also free and finitely generated it follows from the Lemma 112 that

$$A \underset{R}{\otimes} G \simeq \mathrm{Hom}(\mathrm{Hom}(A, R), R) \underset{R}{\otimes} \mathrm{Hom}(R, G) \xrightarrow{\mu} \mathrm{Hom}(\mathrm{Hom}(A, R), G).$$

**119 Theorem.** *Let $C$ be a free chain complex such that $H(C)$ is of finite type. Then for any $R$-module $G$, there exists a splitting (functorial) SES*

$$0 \to \mathrm{Ext}(H^{q+1}(C, R), G) \to H_q(C, G) \xrightarrow{h} \mathrm{Hom}_R(H^q(C, R), G) \to 0.$$

*Proof.* Using the Lemma 115 we can assume that $C$ is of finite type. Then $C \otimes G \simeq \mathrm{Hom}(\mathrm{Hom}(C, R), G)$, and the claim follows from the Theorem 108 taking $\mathrm{Hom}(C, R)$ instead of $C$ and changing the signs of degrees.

## 2.3 Cup and Cap Product

The cross product of cohomology classes from the tensor product of the cohomology of two spaces to the cohomology of their product space can be composed with the diagonal map of space into square. Thus it yields the ring structure in the cohomology module.

**120 Definition.** *If $\{X \times B, A \times Y\}$ is an excessive couple of topological space $X \times Y$, there exists a* **cohomology cross-product**

$$\mu' : H^p(X, A, G) \underset{R}{\otimes} H^q(Y, B, G') \to H^{p+q}((X, A) \times (Y, B), G \underset{R}{\otimes} G')$$

*induced by (functorial) homomorphism*

$$\begin{array}{c} \mathrm{Hom}_R(\Delta(X)/\Delta(A), G) \underset{R}{\otimes} \mathrm{Hom}_R(\Delta(Y)/\Delta(B), G') \\ \downarrow{\scriptstyle \mu} \\ \mathrm{Hom}_R(\Delta(X)/\Delta(A) \underset{R}{\otimes} \Delta(Y)/\Delta(B), G \underset{R}{\otimes} G') \end{array}$$

*followed by an Eilenberg-Zilber cochain equivalence*

$$\begin{array}{c}\operatorname{Hom}_R(\Delta(X)/\Delta(A) \underset{R}{\otimes} \Delta(Y)/\Delta(B), G \underset{R}{\otimes} G') \\ \Big\downarrow i \\ \operatorname{Hom}_R(\Delta(X \times Y)/\Delta(X \times B \cup A \times Y, G \underset{R}{\otimes} G').\end{array}$$

*If $u \in H^p(X, A, G)$ and $v \in H^q(Y, B, G')$, we define*

$$u \boxtimes v \overset{\text{def}}{=} \mu'(u \otimes v) \in H^{p+q}((X, A) \times (Y, B), G \otimes G').$$

From the Theorem (?) we obtain the Küneth formula for singular cohomology.

**121 Theorem.** *Let $\{X \times B, A \times Y\}$ be an excessive couple of topological space $X \times Y$, and let $G$ and $G'$ be $R$-modules fulfilling $G \star G' = 0$. If both $H_*(X, A, R)$ and $H_*(Y, B, R)$ are of finite type or $H_*(Y, B, R)$ is of finite type and $G'$ is finitely generated, then there exists a functorial SES*

$$\begin{aligned} 0 \to [H^*(X, A, G) \otimes H^*(Y, B, G')]^q &\xrightarrow{\mu'} H^q((X, A) \times (Y, B), G \otimes G') \to \\ &\to [H^*(X, A, G) \star H^*(Y, B, G')]^{q+1} \to 0 \end{aligned}$$

*and the sequence splits.*

**122 Observation.** The cohomology cross product satisfies the following statements:

(1) Let $f : (X, A) \to (X', A')$ and $g : (Y, B) \to (Y', B')$ be homomorphisms of topological pairs. Then for $u' \in H^p(X', A', G)$ and $v' \in H^q(Y', B', G')$, we have

$$(f \times g)^*(u' \times v') = f^*u' \boxtimes g^*v' \in H^{p+q}((X, A) \times (Y, B), G \otimes G').$$

(2) For $u \in H^p(X, A, G)$, $v \in H^q(Y, B, G')$, and $w \in H^r(Z, C, G'')$, we have

$$u \boxtimes (v \boxtimes w) = (u \boxtimes v) \boxtimes w \in H^{p+q+r}((X, A) \times (Y, B) \times (Z, C), G \otimes G' \otimes G'').$$

(3) Let $T : (X, A) \times (Y, B) \to (Y, B) \times (X, A)$ and $\tau : G \otimes G' \to G' \otimes G$ interchange the factors. For $u \in H^p(X, A, G)$ and $v \in H^q(Y, B, G')$, we have

$$T^*(u \boxtimes v) = (-1)^{pq} \tau_*(u \boxtimes v) \in H^{p+q}((X, A) \times (Y, B), G' \otimes G).$$

(4) Let $\{(X_1, A_1), (X_2, A_2)\}$ be an excessive couple of topological pairs in $X$, and let $u \in H^p(X_1 \cap X_2, A_1 \cap A_2)$ and $v \in H^q(Y, B, G')$. For the connecting homomorphisms of appropriate Mayer-Vietoris sequence we have

$$\begin{aligned} \delta^*(u \boxtimes v) &= \delta^* u \boxtimes v \in H^{p+q+1}((X_1 \cup X_2, A_1 \cup A_2) \times (Y, B), G \otimes G'), \\ \delta^*(v \boxtimes u) &= (-1)^q v \boxtimes \delta^* u \in H^{p+q+1}((Y, B) \times (X_1 \cup X_2, A_1 \cup A_2), G' \otimes G), \end{aligned}$$

Consider now the functors $\Delta : X \to \Delta(X)$ and $\Delta \otimes \Delta : X \to \Delta(X) \otimes \Delta(X)$ in the category **Top**. Because $\Delta(X)$ is a free abelian group with models $\{\Delta^q\}_{q \geq ?}$ and $\Delta \otimes \Delta$ is an acyclic functor with models $\{\Delta^q\}_{q \geq 0}$ (i.e. the reduced complex of $\Delta(X) \otimes \Delta(X)$ is acyclic for all q), it follows from the Theorem of the acyclic models that then there exists a functorial chain map $\tau_* : \Delta(X) \to \Delta(X) \otimes \Delta(X)$ which is unique up to homotopy equivalence. This map is called **diagonal approximation**. This name comes from the following fact. If $\tau'_X : \Delta(X \times X) \to \Delta(X) \otimes \Delta(X)$ is homotopy chain equivalence (given by the Eilenberg-Zilber Theorem 81) and the mapping $\mathrm{diag} : X \hookrightarrow X \times X$ is a diagonal embedding, then

$$i_X : \Delta(X) \xrightarrow{\Delta(\mathrm{diag})} \Delta(X \times X) \xrightarrow{\tau'_X} \Delta(X) \otimes \Delta(X)$$

is the diagonal approximation.

We will now construct a particular diagonal approximation called the **Alexander-Whitney** diagonal approximation. Let $\sigma : \Delta^q \to X$ be a singular $q$-simplex. The front $i$-face ${}_i\sigma$ is defined for $0 \leq i \leq q$ to be equal to $\sigma \circ \lambda_{i,q}$, where $\lambda_{i,q} : \Delta^i \to \Delta^q$ is the simplicial map defined by $\lambda_{i,q}(p_j) = p_j$ for $0 \leq j \leq i$. Similarly, the back $i$-face $\sigma_i$ is defined for $0 \leq i \leq q$ to be equal to $\sigma \circ \lambda'_{i,q}$, where $\lambda'_{i,q} : \Delta^i \to \Delta^q$ is the simplicial map defined by $\lambda'_{i,q}(p_j) = p_{j+q-i}$ for $0 \leq j \leq i$. Then

$$\tau(\sigma) \stackrel{\mathrm{def}}{=} \sum_{i+j=\deg\sigma} {}_i\sigma \otimes \sigma_j$$

defines Alexander-Whitney diagonal approximation.

**123 Definition.** *Let $G$ and $G'$ be $R$-modules. A* **pairing** *of $G$, $G'$ to an $R$-mod $G''$ is defined to be a homomorphism $\varphi \in \mathrm{Hom}_R(G \otimes G', G'')$.*

Given a pairing $\varphi$, there exists a (functorial) cochain map

$$\bar{\tau}_X : \mathrm{Hom}(\Delta(X), G) \underset{R}{\otimes} \mathrm{Hom}(\Delta(X), G') \to \mathrm{Hom}(\Delta(X), G'')$$

defined by

$$\mathrm{Hom}(\Delta(X), G) \otimes \mathrm{Hom}(\Delta(X), G') \xrightarrow{\mu} \mathrm{Hom}(\Delta(X) \otimes \Delta(X), G \otimes G') \to$$
$$\xrightarrow{\mathrm{Hom}(\tau_X, \varphi)} \mathrm{Hom}(\Delta(X), G'')$$

Let $A \subset X$. Then for $f \in \mathrm{Hom}(\Delta(X), G)$ and $f' \in \mathrm{Hom}(\Delta(X), G')$, we have

$$\bar{\tau}_X(f \otimes f') \restriction \Delta(A) = \bar{\tau}_A(f \restriction \Delta(A) \otimes f' \restriction \Delta(A)).$$

Let $A_1, A_2 \subset X$. If $f \restriction \Delta(A_1) = 0$ and $f' \restriction \Delta(A_2) = 0$, then $\bar{\tau}_X(f \otimes f') \restriction \Delta(A_1) + \Delta(A_2) = 0$.

**124 Definition.** *Let $\{A_1, A_2\}$ be an excessive couple. The homomorphism*

$$H^p(X, A_1, G) \otimes H^q(X, A_2, G') \to H^{p+q}(X, A_1 \cup A_2, G'')$$

*induced by $\bar{\tau}_X$ is called* **cup product** *homomorphism. For given $u \in H^p(X, A_1, G)$ and $v \in H^q(X, A_2, G')$, we use the notation $u \Cup v \in H^{p+q}(X, A_1 \cup A_2, G'')$.*

This $R$-bilinear mapping $\Cup$ depends only on mapping $\varphi$, not on the diagonal approximation. In the case of Alexander-Whitney diagonal approximation, the cup product $f \Cup f'$ for $f \in \mathrm{Hom}(\Delta_p(X), G)$ and $f \in \mathrm{Hom}(\Delta_q(X), G')$ is defined by

$$(f \Cup f')(\sigma) \overset{\mathrm{def}}{=} \varphi(f({}_p\sigma) \otimes f'(\sigma_q))$$

on cochains. Then, on classes we set

$$\{f\} \Cup \{f'\} = \{f \Cup f'\} \in H^{p+q}(X, A_1 \cup A_2, G'').$$

Properties of cup product correspond to the properties of the cross product.

(1) Let $\{X \times A_2, X \times A_1\}$ is an excessive couple in $X \times X$ and let $\varphi G \otimes G' \to G''$ be a pairing. Then for $u \in H^p(X, A_1, G)$ and $v \in H^q(X, B_2, G')$, we have

$$(u \Cup v) = \varphi_*(\mathrm{diag}^*(u \boxtimes v)).$$

(2) Let $f : X \to Y$ be mapping of topological spaces such that $A_1 \subset X \to B_1 \subset Y$ and $A_2 \subset X \to B_2 \subset Y$. Let $f_1(X, A_1) \to (Y, B_1)$ and $f_2(X, A_2) \to (Y, B_2)$ be mappings induced by $f$. Then for $u \in H^p(Y, B_1, G)$ and $v \in H^q(Y, B_2, G')$, we have

$$\bar{f}^*(u \Cup v) = f_1^*(u) \Cup f_2^*(v) \in H^{p+q}(X, A_1 \cup A_2, G'').$$

(3) ???

(4) Let $\varphi, \varphi', \psi, \psi'$ be pairings in the following commutative diagram

$$\begin{tikzcd}
G_1 \otimes (G_2 \otimes G_3) \arrow[d, "\mathrm{Id}\otimes\varphi'"'] & \simeq & (G_1 \otimes G_2) \otimes G_3 \arrow[r, "\varphi\otimes\mathrm{Id}"] & G_{1,2} \otimes G_3 \arrow[d, "\psi"] \\
G_1 \otimes G_{2,3} \arrow[rrr, "\psi'"] & & & G_{1,2,3}
\end{tikzcd}$$

Then for $u \in H^p(X, A_1, G_1)$, $v \in H^q(X, A_2, G_2)$, and $w \in H^r(X, A_3, G_3)$, we have

$$u \Cup (v \Cup w) = (u \Cup v) \Cup w \in H^{p+q+r}(X, A_1 \cup A_2 \cup A_3, G_{1,2,3}).$$

(5) Let $\varphi, \varphi'$ be pairings in the following commutative diagram

$$\begin{tikzcd}
G \otimes G' \arrow[dr, "\varphi"'] & \simeq & G' \otimes G \arrow[dl, "\varphi'"] \\
& G'' &
\end{tikzcd}$$

Then for $u \in H^p(X, A_1, G)$ and $v \in H^q(X, A_2, G')$, we have

$$u \Cup v = (-1)^{pq}(v \Cup u) \in H^{p+q}(X, A_1 \cup A_2, G'').$$

**125 Observation.** The above properties define on the cohomology a natural structure (2) of unital (3) associative (4) graded commutative (5) ring over $R$.

**126 Example.** Let $p, q \in \mathbb{N}, p, q \leq 1$. Let $X$ be a wedge product (a one-point union) of $\mathbb{S}^p, S^q, \mathbb{S}^{p+q}$. Denote $i : \mathbb{S}^p \hookrightarrow X$, $j : \mathbb{S}^q \hookrightarrow X$, and $k : \mathbb{S}^{p+q} \hookrightarrow X$ the corresponding inclusions. Then as the $R$-module we get

$$\widetilde{H}_*(X) \simeq i_*\widetilde{H}_*(\mathbb{S}^p) \oplus j_*\widetilde{H}_*(\mathbb{S}^q) \oplus k_*\widetilde{H}_*(\mathbb{S}^{p+q}).$$

The Künneth formula implies $H_*(X) \simeq H_*(\mathbb{S}^p \times \mathbb{S}^q)$.

(1) By the Universal Coefficient Theorem, $H_*(X, G) \simeq H_*(\mathbb{S}^p \times \mathbb{S}^q)$ for every $R$-mod $G$. Since $H^*(X, G) \simeq H^*(\mathbb{S}^p \times \mathbb{S}^q)$, we get

$$k^* H^{p+q}(X, \mathbb{Z}) \simeq H^{p+q}(\mathbb{S}^{p+q}) \simeq \mathbb{Z},$$

and $k^*$ commutes with the cup product of integral cohomology classes of degree $p$ and $q$ (note $H^p(\mathbb{S}^{p+q}, \mathbb{Z}) \simeq H^q(\mathbb{S}^{p+q}, \mathbb{Z}) \simeq \emptyset$).

(2) However, there are nontrivial integral cohomology classes of $\mathbb{S}^p \times \mathbb{S}^q$. In particular, for $0 \neq u \in H^p(\mathbb{S}^p, \mathbb{Z})$ and $0 \neq u \in H^p(\mathbb{S}^p, \mathbb{Z})$ we get

$$0 \neq (p_1^* u) \Cup (p_2^* v) \in H^{p+q}(\mathbb{S}^{p+q}, \mathbb{Z}),$$

where $p_1 : \mathbb{S}^{p+q} \to \mathbb{S}^p$ and $p_2 : \mathbb{S}^{p+q} \to \mathbb{S}^q$ are corresponding projections.

In other words, this example illustrates the case of compact polyhedra with isomorphic (co)homology modules but non-isomorphic as graded rings. Therefore, $X$ and $\mathbb{S}^p \times \mathbb{S}^q$ are not homeomorphic (nor even of the same homotopy type).

Let $C$ and $C'$ be chain complexes and let $\varphi \in H(G \otimes G', G'')$. Then there exists a (functorial) homomorphism

$$h : \mathrm{Hom}(C', G) \otimes (C \otimes C' \otimes G') \to C \otimes G''$$

defined by

$$h(f \otimes (c \otimes c' \otimes g') = c \otimes \varphi(f(c') \otimes g').$$

One can directly check that for $f \in \mathrm{Hom}(C'_q, G)$ and $\bar{c} \in (C \otimes C')_n \otimes G'$

$$\partial h(f \otimes \bar{c}) = (-1)^{n-q} h(\delta f \otimes \bar{c}) + h(f \otimes \partial \bar{c}).$$

Let $X$ be a topological space and let $\tau : \Delta(X) \to \Delta(X) \otimes \Delta(X)$ be a diagonal approximation. Then there exists a (functorial) mapping

$$\overline{\overline{\tau}} : \mathrm{Hom}(\Delta(X), G) \otimes (\Delta(X) \otimes G') \to \Delta(X) \otimes G''$$

defined by

$$\overline{\overline{\tau}}(f \otimes c) \to ???$$

with the boundary map

$$\partial \overline{\overline{\tau}}(f \otimes c) = (-1)^{\deg c - \deg f}\, \overline{\overline{\tau}}(\delta f \otimes c) + \overline{\overline{\tau}}(f \otimes \partial c).$$

For given $A \subset X$ and $f \in \mathrm{Hom}(\Delta(X), G)$ such that $f \restriction \Delta(A) = 0$ and for any $c \in \Delta(A) \otimes G'$, we get $(\overline{\overline{f}} \otimes c) = 0$. It follows that for $A_1, A_2 \subset X$, a cocycle $f \in \mathrm{Hom}(\Delta(X)/\Delta(A), G)$ and a chain $c \in \Delta(X) \otimes G'$ fulfilling $\partial c \in [\Delta(A_1) + \Delta(A_2)] \otimes G'$, we get $\overline{\overline{\tau}}(f \otimes c)$ to be a chain in $\Delta(X) \otimes G''$ whose boundary is in $\Delta(A_2) \otimes G''$ (because $\partial \overline{\overline{\tau}}(f \otimes c) = \tau(\overline{\overline{f}} \otimes \partial c)$). If $f$ is the boundary of cochain vanishing on $\Delta(A_1)$ or if $c$ equals to a boundary modulo $[\Delta(A_1) + \Delta(A_2)] \otimes G'$, then $\overline{\overline{\tau}}(f \otimes c)$ is a boundary modulo $\Delta(A_2) \otimes G''$. Hence $\overline{\overline{\tau}}$ defines a mapping

$$H^q(X, A_1, G) \otimes H_n(\Delta(X)/[\Delta(A_1) + \Delta(A_2)], G') \to H_{n-q}(X, A_2, G'')$$

defined by

$$\{f\} \otimes \{c\} \to \{\overline{\overline{\tau}}(f \otimes c)\}.$$

**127 Definition.** *Let $\{A_1, A_2\}$ be an excessive couple in topological space $X$. Then the mapping*

$$H^q(X, A_1, G) \otimes H_n(X, A_1 \cup A_2, G') \to H_{n-q}(X, A_2, G'')$$

*is called* **cap product**. *For given $u \in H^q(X, A_1, G)$ and $v \in H_n(X, A_1 \cup A_2, G')$, we use the notation $u \Cap v \in H_{n-q}(X, A_2, G'')$.*

The same remarks as in the case of cup product apply. In particular, Alexander-Whitney diagonal approximation is defined by

$$f \Cap c = f \Cap (\sum_\sigma \sigma \otimes g'_\sigma) = \sum_\sigma {}_{n-q}\sigma \otimes \varphi(f(\sigma_q) \otimes g'_\sigma),$$

where $f \in \mathrm{Hom}(\Delta_q(X), G)$ and $c = \sum_\sigma \sigma \otimes g'_\sigma \in \Delta(X) \otimes G'$. Then we set $\{f\} \Cap \{c\} = \{f \Cap c\}$. Similarly, one can list symmetric properties to the ones of the cup product.

## 2.4 Cohomology Algebra

The cup product in cohomology equips the cohomology $R$-module with the structure of a graded $R$-algebra.

**128 Definition.** *A* **graded** $R$**-algebra** $(A, \mu)$ *consists of*

*(1) graded $R$-module $A \equiv \{A^q\}_q,\ q \in \mathbb{Z}$,*
*(2) homomorphism $\mu : A \underset{R}{\otimes} A \to A$ of degree $0$ where*

$$\mu : a \otimes a' \in A^p \otimes A^q \mapsto \mu(a, a') \in A^{p+q}, \text{ for all } p, q \in \mathbb{Z}.$$

*The product $\mu$ is*

*(1) associative iff $(aa')a'' = a(a'a'')$ for all $a, a', a'' \in A$,*
*(2) graded commutative iff $aa' = (-1)^{\deg a \deg a'} a'a$ for all $a, a' \in A$,*

**129 Observation.** If (X,A) is a topological pair, then $H^*(X, A, R)$ is a graded algebra whose product is the cup product (with resp. to the pairing $R \otimes R \to R$). From the properties above it follows $H^*(X, A, R)$ is an associative, (graded) commutative and unital $R$-algebra called **cohomology algebra** of $(X, A)$ over $R$.

# 3. Supplementary material

## 3.1 Analytic realizations of cohomology functors

Let $M$ be (analytic, $C^\infty, \dots$) manifold. Let $\Omega^0(M)$ denote algebra of $\mathbb{R}(\mathbb{C})$-valued (analytic, $C^\infty, \dots$) functions on $M$ and let $\Omega^*(M) = \bigoplus_{k=0}^{\dim(M)} \Omega^k(M)$ denote algebra of $\mathbb{R}(\mathbb{C})$-valued (analytic, $C^\infty, \dots$) differentiable forms on $M$. This last algebra is evidently $\mathbb{Z}/2$ (=graded) commutative, ie. $\omega_1 \wedge \omega_2 = (-1)^{|\omega_1||\omega_2|} \omega_2 \wedge \omega_1$ for every $\omega_1 \in \Omega^{|\omega_1|}(M), \omega_2 \in \Omega^{|\omega_2|}(M)$. The dimension of the algebra $\Omega^*(M)$ as a $\mathbb{R}(\mathbb{C})$-module is infinity.

The algebra $\Omega^*(M)$ can be also viewed as a module over $\Omega^0(M)$. For $M = \mathbb{R}^n$ the algebra $\Omega^*(M)$ is free $\Omega^0(M)$-mod of dimension $2^n$ with generators (base) $\{dx_J\}_{\#|J|=1}^{\dim M}$. For $M$ compact manifold, the algebra $\Omega^*(M)$ is finitely generated $\Omega^0(M)$-mod (see the partition of unity for finite (because of compactness) cover). So, it's possible to write every differentiable form $\omega \in \Omega^k(M)$ as a finite sum $\omega = \sum_j f_j dg_{j_1} \wedge \dots \wedge dg_{j_k}$ for $f_j, g_{j_i} \in \Omega^0(M)$. For $M$ non-compact manifold, there exists only local finite atlas and the partition of unity. In other words, it's possible to write every differentiable form $\omega \in \Omega^k(M)$ as a locally finite sum $\omega = \sum_\alpha \omega_\alpha$ where for every $m \in M$ there exists $U \subset M$ $(m \in U)$ such that $|\{\alpha : U \cap \operatorname{supp}(\omega_\alpha) \neq 0\}| < \infty$.

Denote $\chi(M)$ the set of all derivatives of functions from $\Omega^0(M)$, ie. set of all operations $D$ on $\Omega^0(M)$ such that $D(fg) = f(Dg) + (Df)g$ for all $f, g \in \Omega^0(M)$. Let $\omega_1 \in \Omega^{|\omega_1|}(M), \omega_2 \in \Omega^{|\omega_2|}(M)$ denote $k = |\omega_1|$ and $l = |\omega_2|$. For $X_1, \dots, X_{k+l} \in \chi(M)$ define

$$(\omega_1 \wedge \omega_2)(X_1, \dots, X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \cdot \omega_1(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \cdot \omega_2(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)}).$$

**130 Lemma.** *Let $\omega_1 \wedge \dots \wedge \omega_k \in \Omega^1(M)$ and let $X_1, \dots, X_k \in \chi(M)$. Then*

$$(\omega_1 \dots \omega_k)(X_1, \dots, X_k) = \det(\omega_i(X_j))_{i,j=1}^k.$$

*Proof.* By induction. For $k = 1, 2$, the equality holds from the definition above. Assume the equality is true for $k = n - 1$ and prove it for $k = n$.

$$(\omega_1 \wedge (\omega_2 \wedge \dots \wedge \omega_n))(X_1, \dots, X_k) = \frac{1}{(n-1)!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \omega_1(X_{\sigma(1)})$$

$$(\omega_2 \wedge \dots \wedge \omega_n)(X_{\sigma(2)}, \dots, X_{\sigma(n)} = \frac{1}{(n-1)!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) X_{\sigma(1)})$$

$$\sum_{\tau \in S_{n-1}(\sigma(2) \dots \sigma(n))} \operatorname{sgn}(\tau) \cdot \omega_2(X_{\tau\sigma(2)}) \cdots \omega_n(X_{\tau\sigma(n)}) =$$

$$= \frac{1}{(n-1)!} \sum_{\sigma \in S_n} \sum_{\substack{\sigma' \in S_n \\ \sigma'(1) = \sigma(1)}} \operatorname{sgn}(\sigma') \cdot \omega_1(X_{\sigma'(1)}) \cdot \omega_2(X_{\sigma'(2)}) \cdots \omega_n(X_{\sigma'(n)}) =$$

$$= \sum_{\sigma' \in S_n} \operatorname{sgn}(\sigma') \cdot \omega_1(X_{\sigma'(1)}) \cdot \omega_2(X_{\sigma'(2)}) \cdots \omega_n(X_{\sigma'(n)}) =$$

$$= \det(\omega_i(X_j))_{i,j=1}^n.$$

## Graded Derivative on $\Omega^*(M)$

**131 Definition.** *For $j \in \mathbb{N}$, the* **graded derivative** *of degree $j$ is a linear mapping $D : \Omega^*(M) \to \Omega^*(M)$ where $\Omega^k(M) = \{\emptyset\}$ for $k \notin \{1, \dots, \dim(M)\}$ such that*

*(1) $D(\Omega^k(M)) \subseteq \Omega^{k+j}(M)$ for $k \in \mathbb{N}$,*

*(2) $D(\omega_1 \wedge \omega_2) = D\omega_1 \wedge \omega_2 + (-1)^{j|\omega_1|}\omega_1 \wedge D\omega_2$.*

*The graded derivative of degree 1 is called* **de Rham differential.**

**132 Definition.** *Let $X \in \chi(M)$. We define a linear mapping $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ by*

$$(i_X(\omega))(X_1, \dots, X_{k-1}) \stackrel{\text{def}}{=} \omega(X, X_1, \dots, X_{k-1})$$

*for all $X_1, \ldots, X_{k-1} \in \chi(M)$ and $\omega \in \Omega^k(M)$. This linear mapping $i_X$ is called* **contraction** *along vector field $X$.*

**133 Observation.** Let $\omega$ be decomposable $k$-form, ie. $\omega = \omega_1 \wedge \cdots \wedge \omega_k \in \Omega^k(M)$. Then for every $X \in \chi(M)$

$$i_X(\omega) = \sum_{j=1}^{l} (-1)^j \omega_j(X) \cdot (\omega_1 \wedge \cdots \wedge \omega_{j-1} \wedge \hat{\omega}_j \wedge \omega_{j+1} \wedge \cdots \wedge \omega_k) \in \Omega^{k-1}(M).$$

*Proof.* The result follows from the Lemma 130 and the expansion of determinant by the first column.

**134 Theorem.** *Let $X_1 \in \chi(M)$. Then $i_{X_1}$ is a graded derivative of degree $-1$.*

*Proof.* The mapping $i_{X_1}$ is linear. Therefore, it suffices to show that for two decomposable forms

$$i_{X_1}(\omega \wedge \tau) = i_{X_1}(\omega) \wedge \tau + (-1)^{-|\omega|} \omega \wedge i_{X_1}(\tau).$$

Assume that $\omega = \omega_1 \wedge \cdots \wedge \omega_k \in \Omega^k(M)$ and $\tau = \omega_{k+1} \wedge \cdots \wedge \omega_{k+l} \in \Omega^l(M)$. Then

$$i_{X_1}(\omega \wedge \tau)(X_2, \ldots, X_{k+l}) = (\omega \wedge \tau)(X_1, X_2, \ldots, X_{k+l}) = \det(\omega_i(X_j))_{i,j=1}^{k+l}.$$

The result follows from the expansion of determinant by the first column.

**135 Theorem.** *Let $D_1$, resp. $D_2$, be graded derivative of degree $j_1$, resp. $j_2$. Then their commutator $[D_2, D_1] \stackrel{\text{def}}{=} D_2 D_1 - (-1)^{j_1 j_2} D_1 D_2$ is graded derivative of degree $j_1 + j_2$.*

*Proof.* Let $\omega_1 \in \Omega^k(M), \omega_2 \in \Omega^l(M)$. Then

$$[D_2, D_1](\omega_1 \wedge \omega_2) = D_2[D_1\omega_1 \wedge \omega_2 + (-1)^{j_1 k} \omega_1 \wedge D_1\omega_2] - (-1)^{j_1 j_2} D_1[D_2\omega_1 \wedge \omega_2 + (-1)^{j_2 k} \omega_1 \wedge D_2\omega_2] =$$

$$= D_2 D_1 \omega_1 \wedge \omega_2 + (-1)^{j_2(k+j_1)} D_1\omega_1 \wedge D_2\omega_2 + (-1)^{j_1 k} D_2\omega_1 \wedge D_1\omega_2 + (-1)^{j_1 k}(-1)^{j_2 k} \omega_1 \wedge D_2 D_1 \omega_2 +$$
$$-(-1)^{j_1 j_2}[(D_1 D_2 \omega_1 \wedge \omega_2 + (-1)^{j_1(k+j_2)} D_2\omega_1 \wedge D_1\omega_2 + (-1)^{j_2 k} D_1\omega_1 \wedge D_2\omega_2 + (-1)^{j_2 k}(-1)^{j_1 k} \omega_1 \wedge D1 D_2 \omega_2] =$$
$$= [D_2, D_1]\omega_1 \wedge \omega_2 + (-1)^{(j_1+j_2)k} \omega_1 \wedge [D_2, D_1]\omega_2.$$

**136 Definition.** *Let $X \in \chi(M)$. Then the* **Lie derivative** *$L_X : \Omega^*(M) \to \Omega^*(M)$ is defined as a graded derivative of degree 0 by $L_X \stackrel{\text{def}}{=} [d, i_X] = d i_X + i_X d$.*

**137 Observation.** For $f \in \Omega^0(M)$ we get $L_X(f) = i_X d(f) = df(X) = X(f) \in \Omega^0(M)$. For $\omega = df \in \Omega^1(M)$ we get $L_X(\omega) = d i_X(df) = d(df(X)) = d(X(f)) \in \Omega^1(M)$.

**138 Theorem.** *For $X, Y \in \chi(M)$ we have*

*(1)* $[L_X, i_Y] = i_{[X,Y]}$,

*(2)* $[L_X, d] = 0$,
*(3)* $[L_X, L_Y] = L_{[X,Y]}$
*(4)* $[i_X, i_Y] = 0$.

*Proof.* Because all terms above are constructed by graded commutators, the corresponding ones are graded derivatives. Because of the Leibnitz rule, we need to prove above relations only for elements of $\Omega^0(M)$ and for exact elements of $\Omega^1(M)$.

(1) For elements of $\Omega^0(M)$ holds the equality trivially. For $\omega = df \in \Omega^1(M)$, we have $[L_X, i_Y](df) = L_X(df(Y)) - i_Y(L_X(df)) = XY(f) - d(X(f))(Y) = XY(f) - YX(f) = [X,Y](f) = i_{[X,Y]}(df)$
(2) Evidently $L_X d = di_X d + i_Y d^2 = di_X d = di_X d + d^2 i_Y = dL_X$. Hence, $[L_X, d] = L_X d - dL_X = 0$.
(3) Because $[L_X, d] = 0$, we need to prove above the equality only for elements of $\Omega^0(M)$. But it holds trivially from the definition.
(4) This follows from the fact that every graded derivative of degree $-2$ is trivial. This is true because of Leibnitz rule and trivial action on elements of $\Omega^0(M)$ and exact elements of $\Omega^1(M)$.

The following results can be used as a tool for computing differential for particular differential form without using specific trivialization and local map.

**139 Theorem.** *Let $X, X_0, \ldots, X_k \in \chi(M)$ and let $\omega \in \Omega^{k+1}(M)$. Then*

*(1)* $(L_X\omega)(X_1, \ldots, X_k) = L_X(\omega(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \omega(X_1, \ldots, X_{i-1}, [X, X_i], X_{i+1}, \ldots, X_k)$
*(2)*

$$(d\omega)(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)) +$$

$$+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).$$

*Proof.*

(1) The equivalent form of the equality is

$$i_{X_k} \cdots i_{X_1} L_X = L_X i_{X_k} \cdots i_{X_1} - \sum_{i=1}^{k} i_{X_k} \cdots i_{X_{i+1}} i_{[X,X_i]} i_{X_{i-1}} \cdots i_{X_1}.$$

Clearly, this follows from the fact $[L_X, i_Y] = i_{[X,Y]}$ proved above.
(2) By induction with respect to the degree of the form $\omega$. For $\omega = f \in \Omega^0(M)$ it reduced to the equality $df(X) = X(f)$ which holds by definition. Assume it holds for forms od degree less then $k-1$. For $\omega \in \Omega^k(M)$ we have

$$(d\omega)(X_0, \ldots, X_k) = (i_{X_k} \cdots i_{X_0})(d\omega) = (i_{X_k} \cdots i_{X_1})(L_{X_0} - di_{X_0})(\omega) = L_{X_0} i_{X_k} \cdots i_{X_1} \omega$$

$$-\sum_{i=1}^{k} i_{X_k}\cdots i_{X_{i+1}} i_{[X,X_i]} i_{X_{i-1}}\cdots i_{X_1} - \sum_{i=1}^{k}(-1)^{i-1}X_i(\omega(X_0,X_1,\ldots,\hat{X}_i,\ldots,X_k))+$$

$$+\sum_{1\le i<j\le k}(-1)^{i+j}\omega(X_0,[X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k) = X_0[\omega(X_1,\ldots,X_k)]+$$

$$+\sum_{i=1}^{k}(-1)^{i}X_i(\omega(X_0,X_1,\ldots,\hat{X}_i,\ldots,X_k))+$$

$$+\sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_0,X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k) =$$

$$=\sum_{i=0}^{k}(-1)^{i}X_i(\omega(X_0,\ldots,\hat{X}_i,\ldots,X_k))+$$

$$+\sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k).$$

**Homotopy invariance, de Rham complex, de Rham cohomology group**

**140 Definition.** **de Rham chain complex** *is a contravariant functor* $F_d^{\mathrm{compl}}R$ *from the category of manifolds to the category of chain complexes which assignees to any manifold* $M$ *a chain complex*

$$0\to\mathbb{R}\xrightarrow{d}\Omega^0(M)\xrightarrow{d}\Omega^1(M)\xrightarrow{d}\ldots\xrightarrow{d}\Omega^{\dim(M)}(M)\to 0$$

*and to any map between manifolds* $f: M\to N$ *a cochain complex map*

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "d"] & \Omega^0(M) \arrow[r, "d"] & \Omega^1(M) \arrow[r, "d"] \arrow[d, "f^*"] & \cdots \arrow[r, "d"] & \Omega^{\dim(M)}(M) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r, "d"] & \Omega^0(M) \arrow[r, "d"] & \Omega^1(M) \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Omega^{\dim(M)}(M) \arrow[r] & 0
\end{tikzcd}$$

*where* $d:\Omega^*(M)\to\Omega^{*+1}(M)$ *is de Rham differential such that* $d\circ d=0$. *The* $k^{th}$ **de Rham group** *is an algebraic functor* $F_dR: M\to H^k_{dR}(M)$ *where*

$$H^k_{dR}(M)\stackrel{\mathrm{def}}{=}\{\omega\in\Omega^k(M): d\omega=0\}/\{\omega\in\Omega^k(M):\omega=d\omega' \text{ for some } \omega'\in\Omega^{k-1}(M)\}$$

*and the abelian group operation is inherited from* $\Omega^*(M)$.

de Rham cohomology groups describe global topological properties of differential manifolds. This fact result from their following properties, so called (the same principle as in the case of singular (co)homology).

**141 Definition.** *Let $M$ be manifold and $I$ the unit interval $< 0,1 >$. Consider the manifold $M \times I$ with an edge $\partial(M \times I) = M \cup (-M)$. Let $F \in \Omega^0(M \times I)$. Then we get mapping*

$$F(x,t) \to f(x) = \int_0^1 F(x,t)dt$$

*from $\Omega^0(M \times I)$ to $\Omega^0(M \times I)$.*

*For a canonic vector field $X = \frac{\partial}{\partial t}$ on $M \times I$ we define a linear operator $L$ on $\Omega^*(M \times I)$ by*

$$L : \omega \in \Omega^p(M \times I) \mapsto L\omega \in \Omega^{p-1}(M \times I)$$

*where*

$$L\omega(X_2, \ldots, X_p) = \int_0^1 \omega(\frac{\partial}{\partial t}, X_2, \ldots, X_p)dt.$$

**142 Observation.** Let $(x_1, \ldots, x_n)$ be the local coordinates on $U \subset M$. Then for

$$\omega = \sum_{|I|=p-1} \alpha_I(t,x)dt \wedge dx_{|I|} + \sum_{|J|=p} \beta_J(t,x)dx_{|J|}$$

we get

$$\omega(\frac{\partial}{\partial t}, X_2, \ldots, X_p) = \sum_I \alpha_I(t,x)dx_I(X_2, \ldots, X_p).$$

Hence

$$L\omega = \sum_I (\int_0^1 \alpha_I(t,x)dt)dx_I \in \Omega^{p-1}(U).$$

**143 Theorem.** *Consider inclusions $s_i : M \hookrightarrow I \times M,\ i \in \{0,1\}$ defined by $s_i : x \mapsto (x,i)$. Then for every $\omega \in \Omega^p(I \times M)$, we get $(Ld + dL)\omega = (s_1^* - s_0^*)(\omega)$.*

*Proof.* Operators $L, d, s_0^*, s_1^*$ are linear mappings. The partition of unity on $I \times M$ forcing locally finite partition of $\omega = \sum_\alpha \omega_\alpha$ implies also a locally finite partition for all operators, e.g. $L\omega = \sum_\alpha L\omega_\alpha$. Suppose $\omega$ has a support in $I \times U$ where $U \subset M$. In a local coordinates $(t, x_1, \ldots, x_n)$ on $I \times U$, we can get two cases:

(1) $\omega = \beta dx_J, \quad |J| = p,$
(2) $\omega = \alpha dt \wedge dx_I, \quad |I| = p - 1.$

In the first one, there is $L\omega = 0$ and

$$L(d\omega) = (\int_0^1 \frac{d\beta}{dt}dt) \wedge dx_J = s_1^*\omega - s_0^*\omega.$$

In the second one, there is $s_i^*\omega = 0$ for $i = 0, 1$ since $s_i^*(dt) = 0$ for $i = 0, 1$. By the definition, we have

$$d(L\omega) = d(\int_0^1 \alpha(t,x)dt) \wedge dx_I = \sum_i (\int_0^1 \frac{\partial \alpha}{\partial t} dt) \wedge dx_i \wedge dx_I.$$

On the other hand, we get

$$L(d\omega) = L(\sum_i \frac{\partial \alpha}{\partial x_i} dx_i \wedge dt \wedge dx_I) = -\sum_i (\int_0^1 \frac{\partial \alpha}{\partial x_i} dt) \wedge dx_i \wedge dx_I$$

which proves the claim.

**144 Theorem.** *Let $f : M \to N$ be smooth mapping. Then $f^* : \Omega^*(N) \to \Omega^*(M)$ induces mapping $f^* : H^k_{dR}(N) \to H^k_{dR}(M)$. Moreover, if $f, g : M \to N$ are two homotopic mappings (i.e. there exists smooth mapping $h : I \times M \to N$ such that $h(0, m) = f(m)$, $h(1, m) = g(m)$ for every $m \in M$), then*

$$f^* = g^* : H^*_{dR}(N) \to H^*_{dR}(M).$$

*Proof.* The induced mapping on $H^*(M)$, $H^*(N)$ is well defined since for any $\omega$ such that $d\omega = 0$, $(f^*)\omega$ is also closed (It follows from the commutativity of the translation $f^*$ with $d$.). If $\omega_2 = \omega_1 + d\nu$, then $f^*\omega_2 = f^*\omega_1 + df^*\nu$. Thus, $[f^*\omega_2] = [f^*\omega_1]$. If $h$ is a (smooth) homotopy between $f$ and $g$, then for closed form $\omega$, we get

$$g^*\omega - f * \omega = s_1^* h^* \omega - s_0^* h^* \omega = (s_1^* - s_0^*) h^* \omega = d(L h^* \omega).$$

Hence, $[g^*\omega] = [f^*\omega]$.

**145 Theorem.**

(1) *If $f : M \to N$ is a smooth mapping homotopic with constant mapping, then $f^* : H^k_{dR}(N) \to H^k_{dR}(M)$ is trivial for all $k \geq 1$.*

(2) *(Poincaré Lemma) Let $M$ be a manifold. Then for every $m \in M$ there exists $U \subset M, m \in U$ such that for every $\omega, d\omega = 0$, there exists $\tau$ such that $\omega = d\tau$. In other words,*

$$H^k_{dR}(U) = 0 \quad \textit{for all} \quad k = 1, \ldots, n.$$

*Proof.*

(1) If $g$ is a constant mapping, then from the definition of translation of $g^*$ follows, that for every $\omega, |omega| > 0$ we get $g^*\omega = 0$.

(2) It is a corollary to the fact that for every $m \in M$ there exists $U \subset M, m \in U$ and a map $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ such that $\varphi(U)$ is a convex subset of $\mathbb{R}^n$. In other words, the identity on $U$ is homotopic to the constant mapping. Thus, $\mathrm{Id}^*$ is trivial. On the other hand, $\mathrm{Id}^*$ is the identical mapping on $H^*_{dR}(M)$, so is on $H^*_{dR}(U)$.

# References

[Sp] E. H. Spanier, *Algebraic Topology*, McGraw-Hill Series in Higher Math., 1966.

# Index