An introduction to product integration

Antonín Slavík

slavik@karlin.mff.cuni.cz

Charles University, Prague

Motivation – differential equations

Consider the differential equation

$$\begin{aligned} y'(x) &= A(x)y(x) \\ y(a) &= y_0 \end{aligned}$$

where $x \in [a,b],\ y : [a,b] \to \mathbb{R}^n,\ y_0 \in \mathbb{R}^n,\ A : [a,b] \to \mathbb{R}^{n\times n}$.
Instead of it we can solve

$$\begin{aligned} Y'(x) &= A(x)Y(x) \\ Y(a) &= I, \end{aligned}$$

where $x \in [a,b],\ Y, A : [a,b] \to \mathbb{R}^{n\times n}$.

(The solution of the initial problem is then $y(x) = Y(x)y_0$.)

Standard solution

An equivalent formulation of our equation is

$$Y(x) = I + \int_a^x A(t)Y(t)\,dt.$$

Using the method of successive approximations we find the Peano-series solution

$$Y(x) = I + \int_a^x A(t_1)\,dt_1 + \int_a^x \int_a^{t_1} A(t_1)A(t_2)\,dt_2\,dt_1 +$$

$$+ \int_a^x \int_a^{t_1} \int_a^{t_2} A(t_1)A(t_2)A(t_3)\,dt_3\,dt_2\,dt_1 + \cdots$$

Another approach

Equation $Y'(x) = A(x)Y(x)$ implies that for small Δx

$$Y(x + \Delta x) \doteq Y(x) + Y'(x)\Delta x = (I + A(x)\Delta x)Y(x).$$

Therefore we expect (using the fact $Y(a) = I$) that

$$Y(x) = \lim_{\nu(D)\to 0} \prod_{i=m}^{1} (I + A(x_{i-1})\Delta x_i),$$

where D is a partition $a = x_0 \leq x_1 \leq \cdots \leq x_m = x$ of $[a, x]$ and $\Delta x_i = x_i - x_{i-1}$.

Product integral definition

For any function $A : [a,b] \to \mathbb{R}^{n\times n}$ denote

$$P(A,D,x) = \prod_{i=m}^{1} (I + A(\xi_i)\Delta x_i),$$

where $x \in [a,b]$, D is a partition $a = x_0 \le x_1 \le \cdots \le x_m = x$ of $[a,x]$ and $\xi_i \in [x_{i-1}, x_i]$, $\Delta x_i = x_i - x_{i-1}$.

If the limit $\lim_{\nu(D)\to 0} P(A,D,x)$ exists, we call it the product integral of A over $[a,x]$ and use the notation

$$\lim_{\nu(D)\to 0} P(A,D,x) = \prod_{a}^{x} (I + A(t)\,dt).$$

Existence of product integral

If $A : [a,b] \to \mathbb{R}^{n\times n}$ is a Riemann integrable function, then the product integral of A over $[a,b]$ exists and

$$\prod_a^b (I + A(t)\,dt) = I + \int_a^b A(t_1)\,dt_1 + \int_a^b \int_a^{t_1} A(t_1)A(t_2)\,dt_2\,dt_1 +$$

$$+ \int_a^b \int_a^{t_1} \int_a^{t_2} A(t_1)A(t_2)A(t_3)\,dt_3\,dt_2\,dt_1 + \cdots$$

Integration of continuous functions

Suppose $A : [a,b] \to \mathbb{R}^{n\times n}$ is continuous. Then the function

$$Y(x) = \prod_a^x (I + A(t)\,dt)$$

satisfies $Y'(x) = A(x)Y(x)$, $Y(a) = I$.

Conversely, if $Y : [a,b] \to \mathbb{R}^{n\times n}$ is such that $Y'(x) = A(x)Y(x)$ for every $x \in [a,b]$, then

$$\prod_a^b (I + A(t)\,dt) = Y(b)Y(a)^{-1}.$$

Multiplicative calculus

If $a \leq b \leq c$, then

$$\prod_a^c (I + A(t)\,dt) = \prod_b^c (I + A(t)\,dt) \prod_a^b (I + A(t)\,dt).$$

If $A(t) = A$ is a constant function on $[a, b]$, then

$$\prod_a^b (I + A(t)\,dt) = \exp((b - a)A).$$

Substitution theorem:

$$\prod_{\varphi(a)}^{\varphi(b)} (I + A(t)\,dt) = \prod_a^b (I + A(\varphi(t))\varphi'(t)\,dt)$$

History of product integration

- 1887 Vito Volterra – Riemann-type product integral
- 1931 Ludwig Schlesinger – Lebesgue product integral
- 1938 Bohuslav Hostinský – product integration of operator-valued functions
- 1947 Pesi Rustom Masani – Riemann production integration in Banach algebras

Recent development:

- Stieltjes product integral, integration of measures
- Henstock-Kurzweil and McShane product integral

Operator-valued functions

Let X be a Banach space. Denote $\mathcal{O}(X)$ the space of all operators on X and let $A : [a,b] \to \mathcal{O}(X)$. Put

$$P(A, D, x) = \prod_{i=m}^{1} (I + A(\xi_i)\Delta x_i),$$

where $x \in [a,b]$ and D is a partition of $[a,x]$.

If the limit $\lim_{\nu(D)\to 0} P(A,D,x)$ exists, we call it the product integral of A over $[a,x]$ and use the notation

$$\lim_{\nu(D)\to 0} P(A,D,x) = \prod_{a}^{x} (I + A(t)\,dt).$$

Example: Fluid flow (1)

Position of fluid particles at time $t \geq 0$ is specified using an operator $S(t) : \mathbb{R}^3 \to \mathbb{R}^3$.

$S(t)(x) \ldots$ position of particle which was at x at time t_0.

What is the velocity of the fluid at position x at time t?

$$V(t)(x) = \lim_{\Delta t \to 0} \frac{S(t + \Delta t)S(t)^{-1}(x) - x}{\Delta t}$$

Velocity is therefore an operator $V(t) : \mathbb{R}^3 \to \mathbb{R}^3$, where

$$V(t) = \lim_{\Delta t \to 0} \frac{S(t + \Delta t)S(t)^{-1} - I}{\Delta t}$$

Example: Fluid flow (2)

Given the velocity operator V, how to compute S?
For small Δt

$$S(t+\Delta t)S(t)^{-1} - I \doteq V(t)\Delta t$$

$$S(t+\Delta t) \doteq (I + V(t)\Delta t)S(t)$$

Therefore (using the fact that $S(t_0) = I$)

$$S(t) = \lim_{\nu(D)\to 0} \prod_{i=m}^{1} (I + V(t_{m-1})\Delta t_m) = \prod_{t_0}^{t} (I + V(u)\,du),$$

where D is a partition $t_0 \leq t_1 \leq \cdots \leq t_{m-1} \leq t_m = t$ of $[t_0, t]$

and $\Delta t_m = t_m - t_{m-1}$.

The end.