Henstock-Kurzweil and McShane product integration

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Motivation

Consider the differential equation

$$\begin{aligned} y'(x) &= A(x)y(x) \\ y(a) &= y_0 \end{aligned}$$

where $x \in [a,b]$, $y : [a,b] \to \mathbb{R}^n$, $y_0 \in \mathbb{R}^n$, $A : [a,b] \to \mathbb{R}^{n\times n}$.
Instead of it we can solve

$$\begin{aligned} Y'(x) &= A(x)Y(x) \\ Y(a) &= I, \end{aligned}$$

where $x \in [a,b]$, $Y, A : [a,b] \to \mathbb{R}^{n\times n}$.
(The solution of the original problem is then $y(x) = Y(x)y_0$.)

Peano series solution

An equivalent formulation of our equation is

$$Y(x) = I + \int_a^x A(t)Y(t)\,dt.$$

Using the method of successive approximations we find the Peano series solution

$$Y(x) = I + \int_a^x A(t_1)\,dt_1 + \int_a^x \int_a^{t_1} A(t_1)A(t_2)\,dt_2\,dt_1 +$$

$$+ \int_a^x \int_a^{t_1} \int_a^{t_2} A(t_1)A(t_2)A(t_3)\,dt_3\,dt_2\,dt_1 + \cdots$$

Another approach (Vito Volterra, 1887)

Equation $Y'(x) = A(x)Y(x)$ implies that for small Δx

$$Y(x+\Delta x) \doteq Y(x) + Y'(x)\Delta x = (I + A(x)\Delta x)Y(x).$$

Therefore we expect (using the fact $Y(a) = I$) that

$$Y(b) = \lim_{\nu(D)\to 0} \prod_{i=m}^{1} (I + A(x_{i-1})(x_i - x_{i-1})),$$

where D is a partition of $[a, b]$ with division points

$$a = x_0 \leq x_1 \leq \cdots \leq x_m = b.$$

Product integral definition

For any function $A : [a,b] \to \mathbb{R}^{n\times n}$ denote

$$P(A,D) = \prod_{i=m}^{1} (I + A(\xi_i)(x_i - x_{i-1})),$$

where D is a tagged partition of $[a,b]$ with division points $a = x_0 \le x_1 \le \cdots \le x_m = b$ and tags $\xi_i \in [x_{i-1}, x_i]$.

If the limit $\lim_{\nu(D)\to 0} P(A,D)$ exists, we call it the product integral of A over $[a,b]$ and use the notation

$$\lim_{\nu(D)\to 0} P(A,D) = \prod_a^b (I + A(x)\,dx).$$

Existence of product integral

If $A : [a,b] \to \mathbb{R}^{n\times n}$ is a Riemann integrable function, then the product integral of A over $[a,b]$ exists and

$$\prod_a^b (I + A(t)\,dt) = I + \int_a^b A(t_1)\,dt_1 + \int_a^b \int_a^{t_1} A(t_1)A(t_2)\,dt_2\,dt_1 +$$

$$+ \int_a^b \int_a^{t_1} \int_a^{t_2} A(t_1)A(t_2)A(t_3)\,dt_3\,dt_2\,dt_1 + \cdots$$

Indefinite product integral

If $A : [a, b] \to \mathbb{R}^{n \times n}$ is a Riemann integrable function, then

$$Y(x) = \prod_a^x (I + A(t)\,dt)$$

exists for $x \in [a, b]$ and

$$Y(x) = I + \int_a^x A(t)Y(t)\,dt.$$

If A is continuous, then $Y'(x) = A(x)Y(x)$ for every $x \in [a, b]$.

K-partitions and M-partitions

- A finite collection of point-interval pairs

$$D = \{(\xi_i, [x_{i-1}, x_i])\}_{i=1}^{m}$$

  is called an M-partition of interval $[a,b]$ if
  $a = x_0 \le x_1 \le \cdots \le x_m = b$ and $\xi_i \in [a,b]$, $i = 1, \ldots, m$.
- A K-partition is a M-partition which satisfies

$$\xi_i \in [x_{i-1}, x_i], \;\; i = 1, \ldots, m.$$

- Given a function $\Delta : [a,b] \to (0,\infty)$, the partition D is called Δ-fine if

$$[x_{i-1}, x_i] \subset (\xi_i - \Delta(\xi_i), \xi_i + \Delta(\xi_i)), \;\; i = 1, \ldots, m.$$

- For every $A:[a,b]\to\mathbb{R}^{n\times n}$ and for every M-partition $D=\{(\xi_i,[x_{i-1},x_i])\}_{i=1}^m$ of $[a,b]$ put

$$P(A,D)=\prod_{i=m}^{1}(I+A(\xi_i)(x_i-x_{i-1})).$$

- The function A is Henstock-Kurzweil product integrable, if there exists a matrix $P_A\in\mathbb{R}^{n\times n}$ such that for every $\varepsilon>0$ we can find $\Delta:[a,b]\to(0,\infty)$ so that

$$\|P(A,D)-P_A\|<\varepsilon$$

for every Δ-fine K-partition D of $[a,b]$.

Henstock-Kurzweil and McShane product integrals (2)

- The matrix P_A is called the Henstock-Kurzweil product integral of A over $[a,b]$.
- The definition of McShane product integral is obtained by replacing K-partitions by M-partitions.
- Notation:

$$(HK)\prod_a^b (I + A(t)\,dt), \quad (M)\prod_a^b (I + A(t)\,dt)$$

Indefinite Henstock-Kurzweil product integral

- J. Kurzweil, J. Jarník (1987):

Consider function $A : [a,b] \to \mathbb{R}^{n\times n}$ such that the integral $(HK)\prod_a^b (I + A(t)\,dt)$ exists and is invertible. Then the function

$$Y(x) = (HK)\prod_a^x (I + A(t)\,dt)$$

is well defined for every $x \in (a,b)$, it is ACG_* and satisfies $Y'(x) = A(x)Y(x)$ almost everywhere on $[a,b]$.

Indefinite McShane product integral

- Š. Schwabik, A. Slavík (2006):

Consider function $A:[a,b]\to\mathbb{R}^{n\times n}$ such that the integral $(M)\prod_a^b(I+A(t)\,dt)$ exists and is invertible. Then the function

$$Y(x)=(M)\prod_a^x(I+A(t)\,dt)$$

is well defined for every $x\in(a,b)$, it is absolutely continuous and satisfies $Y'(x)=A(x)Y(x)$ almost everywhere on $[a,b]$.

Corollary: The McShane product integral coincides with the Lebesgue/Bochner product integral.

History of product integration

- 1887 V. Volterra – Riemann-type product integral
- 1931 L. Schlesinger – Lebesgue product integral
- 1938 B. Hostinský – product integration of operator-valued functions
- 1947 P. R. Masani – Riemann product integration in Banach algebras
- 1987 J. Kurzweil, J. Jarník – H.-K. product integral
- 1994 Š. Schwabik – McShane product integral

In preparation:
Product integration, its history and applications.