Discrete-space partial dynamic equations on time scales and applications to stochastic processes

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Abstract

We consider a general class of discrete-space linear partial dynamic equations. The basic properties of solutions are provided (existence and uniqueness, sign preservation, maximum principle). Above all, we derive the following main results: first, we prove that the solutions depend continuously on the choice of the time scale. Second, we show that, under certain conditions, the solutions describe probability distributions of nonhomogeneous Markov processes, and that their time integrals remain the same for all underlying regular time scales.

Keywords: time scales, partial dynamic equations, stochastic process, nonhomogeneous Markov process, continuous dependence

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1 Introduction

Time scales calculus was created in order to study phenomena in discrete and continuous dynamical systems under one roof [6, 2]. It has proved to be useful not only in various theoretical considerations (see e.g. [14, 15]), but also in the study of applied problems in areas where discrete and continuous-time models naturally coexist, such as economics [1, 20] or control theory [7]. On the other hand, one mathematical field where continuous and discrete approaches are in balance has been almost absent – probability theory and stochastic processes. Only recently, stochastic dynamic equations have been studied in [3]. Additionally, our recent papers [19, 17, 18] were devoted to discrete-space partial dynamic equations and their relation to discrete-state Markov processes. In particular, Poisson-Bernoulli processes are related to solutions of the discrete-space dynamic transport equation, and random walks correspond to discrete-space dynamic diffusion equations. Consequently, the investigation of partial dynamic equations enables us to study the properties of these processes.

In this paper, we explore discrete-space partial dynamic equations and their connection to discrete-state stochastic processes in a more general context. We focus on the linear partial dynamic equation

$$u^{\Delta_t}(x,t) = \sum_{i=-m}^{m} a_i u(x+i,t), \quad x \in \mathbb{Z}, \quad t \in \mathbb{T}, \tag{1.1}$$

where $m \in \mathbb{N}$, $a_{-m}, \dots, a_m \in \mathbb{R}$, and $\mathbb{T}$ is a time scale (a closed subset of $\mathbb{R}$). The symbol u^{Δ_t} denotes the partial Δ-derivative with respect to t, which coincides with the standard partial derivative u_t when $\mathbb{T} = \mathbb{R}$, or with the forward partial difference $\Delta_t u$ when $\mathbb{T} = \mathbb{Z}$. Since the differences with respect to x are not used, we omit the lower index t in u^{Δ_t} and write u^{Δ} only. Throughout the paper, we assume that $\mathbb{T}$ is a closed subset of $[0, \infty)$ and $0 \in \mathbb{T}$. The time scale intervals are denoted by $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$.

Equations related to (1.1) have been studied in two directions. First, lattice dynamical systems with continuous or discrete time were discussed by several authors, see e.g. [4, 21]. Second, partial dynamic equations on general time and space structures have been considered in [8, 9].

In Section 2, we extend the results from [18] and discuss some basic properties of solutions to (1.1), such as existence and uniqueness, sum and sign preservation, and maximum principle. In Section 3, we prove a general theorem concerning the continuous dependence of solutions to (1.1) on initial values, coefficients on the right-hand side, as well as the choice of time scale. In Section 4, we show that under certain conditions, solutions of (1.1) correspond to probability distributions of discrete-state Markov processes. Using this probabilistic interpretation, we prove that the time integrals $\int_0^\infty u(x,t)\Delta t$, which give the expected value of the total time spent by the process in state x, do not depend on the choice of the time scale. Finally, we present examples illustrating the relation between (1.1) and Markov processes.

2 Basic Results

Throughout the paper, we consider the initial value problem

$$\begin{cases} u^{\Delta}(x,t) = \sum\limits_{i=-m}^{m} a_i u(x+i,t), \quad x \in \mathbb{Z},\ t \in \mathbb{T}, \\ u(x,0) = u_x^0, \quad x \in \mathbb{Z}. \end{cases} \tag{2.1}$$

In this section, we summarize some basic properties of solutions to (2.1). The statements presented here are generalizations of the results from our paper [17], which was concerned with the case $m = 1$; the proofs for a general $m \in \mathbb{N}$ are straightforward modifications of the original proofs, and we omit them.

In general, the forward solutions of (2.1) need not be unique. However, if the initial condition is bounded, there exists a unique solution of (2.1) which is bounded on every finite time interval. The proof of this fact is similar to the proof of [17, Theorem 3.5].

Theorem 2.1. *If $u^0 \in \ell^\infty(\mathbb{Z})$, there exists a unique solution $u : \mathbb{Z} \times [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ of (2.1) which is bounded on every interval $[0,T]_{\mathbb{T}}$, where $T \in [0,\infty)_{\mathbb{T}}$.*

According to the next theorem, bounded solutions of (2.1) preserve space sums whenever the coefficients a_i add up to zero. The proof is the same as the proof of [17, Theorem 4.1] with obvious modifications.

Theorem 2.2. *Assume that*

$$\sum_{i=-m}^{m} a_i = 0. \tag{2.2}$$

If $u : \mathbb{Z} \times [0,T]_{\mathbb{T}} \to \mathbb{R}$ is the unique bounded solution of (2.1) and the sum $\sum_{x\in\mathbb{Z}} |u(x,0)|$ is finite, then

$$\sum_{x\in\mathbb{Z}} u(x,t) = \sum_{x\in\mathbb{Z}} u(x,0), \quad t \in [0,T]_{\mathbb{T}}.$$

For time scales with a sufficiently fine graininess, the solutions of (2.1) preserve the sign of the initial condition; this is a straightforward generalization of [17, Lemma 4.3 and Corollary 4.4].

Theorem 2.3. *Assume that*

$$a_0 \le 0, \qquad a_i \ge 0 \ \text{ for } \ i \ne 0, \tag{2.3}$$

$$\mu(t) \le \frac{1}{|a_0|}, \quad t \in [0,T)_{\mathbb{T}}. \tag{2.4}$$

If $u_x^0 \ge 0$ for every $x \in \mathbb{Z}$ and $u : \mathbb{Z} \times [0,T]_{\mathbb{T}} \to \mathbb{R}$ is the unique bounded solution of (2.1), then $u(x,t) \ge 0$ for all $t \in [0,T]_{\mathbb{T}}$, $x \in \mathbb{Z}$.

Finally, we have the following maximum and minimum principles; see the proof of [17, Theorem 4.7].

Theorem 2.4. *Assume that* (2.2), (2.3) *and* (2.4) *hold. If* $u : \mathbb{Z} \times [0,T]_{\mathbb{T}} \to \mathbb{R}$ *is the unique bounded solution of* (2.1), *then*

$$\inf_{y\in\mathbb{Z}} u(y,0) \le u(x,t) \le \sup_{y\in\mathbb{Z}} u(y,0), \quad x \in \mathbb{Z}, \quad t \in [0,T]_{\mathbb{T}}.$$

3 Continuous Dependence

In this section, we consider sequences of time scales $\{\mathbb{T}_n\}_{n=1}^{\infty}$ such that $\mathbb{T}_n \to \mathbb{T}_0$ in the sense described below. If $u_n : \mathbb{Z} \times \mathbb{T}_n \to \mathbb{R}$ are the corresponding solutions of (1.1), we prove that $u_n \to u_0$ (for ordinary dynamic equations, the problem of continuous dependence with respect to time scale has been studied in several papers; see e.g. [5, 11]). In fact, our result applies in the more general situation in which both the coefficients $a_{-m}, \dots, a_m$ as well as the initial conditions can depend on n.

Theorem 3.1. *Assume that* $[0,T] \subset \mathbb{R}$, $\{\mathbb{T}_n\}_{n=0}^{\infty}$ *is a sequence of time scales such that* $0 \in \mathbb{T}_n$, $\sup \mathbb{T}_n \ge T$. *For every* $n \in \mathbb{N}_0$ *and* $t \in [0,T]$, *let* $g_n(t) = \inf\{s \in [0,T]_{\mathbb{T}_n};\ s \ge t\}$, *and suppose that* $\{g_n\}_{n=1}^{\infty}$ *is uniformly convergent to* g_0. *Also, assume that* $\{u^n\}_{n=1}^{\infty}$ *is a sequence in* $\ell^{\infty}(\mathbb{Z})$ *which is convergent to* $u^0 \in \ell^{\infty}(\mathbb{Z})$, $a_i^n \to a_i^0$ *for* $n \to \infty$ *and every* $i \in \{-m, \dots, m\}$, *and* $u_n : \mathbb{Z} \times \mathbb{T}_n \to \mathbb{R}$ *satisfy*

$$\begin{cases} u_n^{\Delta}(x,t) = \sum\limits_{i=-m}^{m} a_i^n u_n(x+i,t), & x \in \mathbb{Z}, \quad t \in \mathbb{T}_n, \quad n \in \mathbb{N}_0, \\ u_n(x,0) = u_x^n, & x \in \mathbb{Z}, \quad n \in \mathbb{N}_0. \end{cases} \tag{3.1}$$

Then, for every $\varepsilon > 0$, *there exists an* $n_0 \in \mathbb{N}$ *such that* $|u_n(x,t) - u_0(x,t)| < \varepsilon$ *for all* $n \ge n_0$, $x \in \mathbb{Z}$, $t \in [0,T]_{\mathbb{T}_n} \cap [0,T]_{\mathbb{T}_0}$.

Proof. For every $n \in \mathbb{N}_0$ and $t \in \mathbb{T}_n$, let $U_n(t) = \{u_n(x,t)\}_{x\in\mathbb{Z}}$. As explained in [17], U_n is the unique solution of the initial-value problem

$$U_n^{\Delta}(t) = A_n U(t), \quad t \in \mathbb{T}_n, \quad U_n(0) = u^n,$$

where $A_n : \ell^{\infty}(\mathbb{Z}) \to \ell^{\infty}(\mathbb{Z})$ is given by $A_n(\{u_x\}_{x\in\mathbb{Z}}) = \left\{\sum_{i=-m}^{m} a_i^n u_{x+i}\right\}_{x\in\mathbb{Z}}$. Let

$$U_n^*(t) = U_n(g_n(t)), \quad n \in \mathbb{N}_0, \quad t \in [0,T],$$

$$B_n(t) = \int_0^t A_n \,\mathrm{d}g_n(s) = (g_n(t) - g_n(0))A_n, \quad n \in \mathbb{N}_0, \quad t \in [0,T].$$

Using the relationship between dynamic equations on time scales and generalized ordinary differential equations (see [16]), we conclude that for every $n \in \mathbb{N}_0$, the function U_n^* is the unique solution of the generalized linear differential equation

$$U_n^*(t) = u^n + \int_0^t \mathrm{d}[B_n(s)]U_n^*(s), \quad t \in [0,T],$$

where the integral on the right-hand side is the Kurzweil-Stieltjes integral.

Clearly, $A_n \to A_0$ in the operator norm corresponding to $\ell^{\infty}(\mathbb{Z})$. Also, since $\{g_n\}_{n=1}^{\infty}$ is uniformly convergent to g_0 on $[0,T]$, it follows that $\{B_n\}_{n=1}^{\infty}$ is uniformly convergent to B_0, and $\operatorname{var} B_n \le (\sup_{k\in\mathbb{N}_0} \|A_k\|)(\sup_{k\in\mathbb{N}_0}(g_k(T) - g_k(0)))$. Hence, by the continuous dependence theorem for abstract linear generalized differential equations [12, Theorem 3.4], $\{U_n^*\}_{n=1}^{\infty}$ is uniformly convergent to U_0^*. The proof is finished by observing that $U_n^*(t) = U_n(t)$ for all $t \in [0,T]_{\mathbb{T}_n}$, $n \in \mathbb{N}_0$. □

Remark 3.2. Let us mention two typical situations where the previous theorem is applicable. First, if $\mathbb{T}_0 = \mathbb{T}_1 = \mathbb{T}_2 = \cdots$, then $\{g_n\}_{n=1}^{\infty}$ is uniformly convergent to g_0, and the previous theorem reduces to a continuous dependence theorem with respect to initial values and coefficients on the right-hand side. Second, if $\mathbb{T}_0 = \mathbb{R}$ and $\{\mathbb{T}_n\}_{n=1}^{\infty}$ are such that $\sup_{t\in[0,T]_{\mathbb{T}_n}} \mu(t) \to 0$ for $n \to \infty$, then $|g_n(t) - g_0(t)| = g_n(t) - t \le \sup_{t\in[0,T]_{\mathbb{T}_n}} \mu(t)$, i.e., $\{g_n\}_{n=1}^{\infty}$ converges uniformly to g_0. Hence, the theorem says that solutions corresponding to time scales whose graininess approaches uniformly zero are uniformly convergent to the solution of the semidiscrete equation; this answers one of the open problems stated in the conclusion of [17].

4 Relationship to Stochastic Processes

In this section, we focus on the relation between (2.1) and discrete-state Markov processes with continuous, discrete or mixed time. We assume that the initial condition u^0 satisfies

$$u_x^0 \ge 0, \quad x \in \mathbb{Z}, \quad \text{and} \sum_{x\in\mathbb{Z}} u_x^0 = 1, \tag{4.1}$$

i.e., u^0 is a discrete probability distribution. The following statement is an immediate consequence of Theorems 2.2 and 2.3.

Theorem 4.1. *Consider the problem* (2.1) *and assume that* (2.2)*,* (2.3)*,* (2.4) *and* (4.1) *hold. Then $u(\cdot,t)$ is a probability distribution for every fixed $t \in [0,\infty)_{\mathbb{T}}$.*

We say that $\mathbb{T}$ is a regular time scale if $\inf \mathbb{T} = 0$, $\sup \mathbb{T} = \infty$, and each bounded interval in $\mathbb{T}$ contains only finitely many right-scattered points (i.e., finitely many gaps). For a regular time scale, let us consider a nonhomogeneous Markov process $(X_t)_{t\in\mathbb{T}}$ whose state space is $\mathbb{Z}$ and the transitions are governed by a_i in the following way: if t is right-continuous, the intensity of transitions from state $x+i$ to state x equals a_i for all $i \in \{\pm1,\dots,\pm m\}$; if t is right-scattered, the probability of transition from state $x+i$ to state x at time t is $a_i\mu(t)$ for all $i \in \{\pm1,\dots,\pm m\}$. Under the assumptions (2.2), (2.3), (2.4) and (4.1), the solution $u(x,t)$ of (2.1) equals the probability that the process is in state x at time t. Hence, (2.1) provides a unified description of Markov processes with arbitrary regular time.

Our final result is concerned with the time integrals $\int_0^\infty u(x,t)\Delta t$, which give the expected value of the total time spent by the process in state x. In [18], we found explicit solutions of (2.1) for $m = 1$ and $\mathbb{T} = [0,\infty)$ or $\mathbb{T} = \mathbb{N}_0$. After lengthy calculations, we also obtained the values of their time integrals and observed that they are the same for both time scales. The next theorem, whose proof is based on the probabilistic interpretation of solutions, extends this result to more general time scales and arbitrary values of $m \in \mathbb{N}$.

Theorem 4.2. *Assume that* (2.2)*,* (2.3)*,* (2.4) *and* (4.1) *hold. If $\mathbb{T}_1$, $\mathbb{T}_2$ are regular time scales and $u_1 : \mathbb{Z}\times\mathbb{T}_1 \to \mathbb{R}$, $u_2 : \mathbb{Z}\times\mathbb{T}_2 \to \mathbb{R}$ are the corresponding solutions of* (2.1)*, then*

$$\int_{\mathbb{T}_1} u_1(x,t)\Delta t = \int_{\mathbb{T}_2} u_2(x,t)\Delta t.$$

Proof. Let $\mathbb{T}$ be a regular time scale and $(X_t)_{t\in\mathbb{T}}$ the Markov process described above. We define $T_0 := 0$, $T_i := \inf\{t \in (T_{i-1},\infty)_{\mathbb{T}};\, X_t \ne X_{T_{i-1}}\}$, $i \in \mathbb{N}$, and consider the embedded Markov chain $(Y_i)_{i\in\mathbb{N}_0}$ given by $Y_i = X_{T_i}$, $i \in \mathbb{N}_0$. In this chain, the transition probability from any state x to state $x+i$ is $p_{x,x+i} = a_{-i}/\sum_{j\ne0} a_j$, $i \in \{\pm1,\dots,\pm m\}$, and does not depend on $\mathbb{T}$. Indeed, the time scale affects only the time spent in a given state. The transition from an arbitrary state occurs with intensity $\sum_{i\ne0} a_i = -a_0$ if t is a right-continuous point, and with probability $\sum_{i\ne0} a_i\mu(t) = -a_0\mu(t)$ if t is right-scattered.

Now, let us study the expected time spent in a state x. Assuming that x has been entered at time t_0, let $T > t_0$ be the time of the next transition. Take an arbitrary strictly increasing sequence $\{t_i\}_{i=0}^{\infty}$ in $\mathbb{T}$ which contains all left- and right-scattered points of $[t_0,\infty)_{\mathbb{T}}$, and such that $\lim_{i\to\infty} t_i = \infty$. For all $t > t_i$, we have

$$P(T > t) = P(T > t \,|\, T > t_i) \prod_{j=1}^{i} P(T > t_j \,|\, T > t_{j-1}),$$

and therefore the expected time spent in x (i.e., the time before the next transition) equals

$$\mathrm{E}(T-t_0)=\int_{t_0}^{\infty}P(T>t)\Delta t=\sum_{i=0}^{\infty}\left(\prod_{j=1}^{i}P(T>t_j\,|\,T>t_{j-1})\right)\int_{t_i}^{t_{i+1}}P(T>t\,|\,T>t_i)\Delta t. \tag{4.2}$$

We define $h_i:=t_{i+1}-t_i$ and observe that if t_i is right-continuous, then

$$P(T>t_{i+1}\,|\,T>t_i)=\mathrm{e}^{a_0h_i},\qquad \int_{t_i}^{t_{i+1}}P(T>t\,|\,T>t_i)\Delta t=\int_0^{h_i}\mathrm{e}^{a_0s}\,\mathrm{d}s=\frac{\mathrm{e}^{a_0h_i}-1}{a_0}. \tag{4.3}$$

If t_i is right-scattered, then (note that $t_{i+1}=\sigma(t_i)$ and $h_i=\mu(t_i)$ in this case)

$$P(T>t_{i+1}\,|\,T>t_i)=1+a_0\mu(t_i),\qquad \int_{t_i}^{t_{i+1}}P(T>t\,|\,T>t_i)\Delta t=\int_{t_i}^{\sigma(t_i)}1\Delta t=\mu(t_i). \tag{4.4}$$

We see that the contribution of a discrete gap in $\mathbb{T}$ to (4.2) from both expressions in (4.4) is the same as if its place was occupied by a continuous interval of length $h_i^*=\frac{1}{a_0}\ln(1+a_0\mu(t_i))$. Thus, the integral (4.2) does not change when all gaps in $\mathbb{T}$ are replaced by adjusted continuous intervals (and the time scale is shifted accordingly), i.e., when $\mathbb{T}$ is replaced with $[0,\infty)$. Consequently, $\mathrm{E}(T-t_0)$ does not depend on the time scale.

The value of the integral $\int_{\mathbb{T}}u(x,t)\Delta t$ is the expected total time spent in x, which equals the product of the expected number of visits to x and the expected time spent in x in each visit (note that those two random variables are independent):

$$\int_{\mathbb{T}}u(x,t)\Delta t=\mathrm{E}\left(|\{i\in\mathbb{N}_0:Y_i=x\}|\right)\cdot\mathrm{E}\left(T-t_0\right).$$

The number of visits to x is the same as in the embedded Markov chain, which is independent of $\mathbb{T}$. Our previous discussion also implies that the expected time spent in x in each visit does not depend on $\mathbb{T}$. □

Remark 4.3. The previous proof shows that replacing gaps with continuous intervals does not change the time integral. However, the process itself and the solution of equation (2.1) changes. We note that the lengths of replacing continuous intervals satisfy $h_i^*>\mu(t_i)$ and $\frac{h_i^*}{\mu(t_i)}\to 1$ as $\mu(t_i)\to 0$.

Remark 4.4. For Markov processes with $\mathbb{T}=[0,\infty)$ or $\mathbb{T}=\mathbb{N}_0$, it is well known that the expected value $\int_{\mathbb{T}}u(x,t)\Delta t$ is infinite if and only if x is a recurrent state (see [13, Theorem 1.5.3 and Theorem 3.4.2]), i.e., if the probability of return to x is 1. Thus, Theorem 4.2 can be reformulated in the following way: if a state x is recurrent/transient on a regular time scale $\mathbb{T}$, then it is recurrent/transient on all regular time scales.

We conclude with three examples illustrating the relation between (2.1) and Markov processes.

Example 4.5 (Transport Equation – Counting processes). If $m=1$, $a_{-1}=r\in(0,\infty)$, $a_0=-r$, and $a_1=0$, then equation (2.1) reduces to the discrete-space transport equation studied in [19]. In the special case when $\mu(t)\in\left[0,\frac{1}{r}\right)$, the equation also describes a counting process. It is known that all states $x\in\mathbb{Z}$ are transient (see [19, Theorem 6.5]).

Example 4.6 (Diffusion Equation – Random walks). For $m=1$, equation (2.1) reduces to the discrete-space diffusion-type equation studied in [17, 18]. In the special case when $a_1=p\in[0,1]$, $a_{-1}=q\in[0,1]$, $a_0=-q-p$, and $\mu(t)\in\left[0,\frac{1}{p+q}\right)$, the equation describes a random walk where p, q are either the probabilities (if $\mathbb{T}=\mathbb{Z}$) or intensities (if $\mathbb{T}=\mathbb{R}$) of moving left and right. For both $\mathbb{T}=\mathbb{Z}$ and $\mathbb{T}=\mathbb{R}$, an arbitrary state $x\in\mathbb{Z}$ is recurrent if and only if the random walk is symmetric, i.e., if $p=q>0$ (see [18, Theorem 4.2 and Corollary 4.6]).

Example 4.7 (**Higher Order Equations – Insurance Classes**). Let $x \in \mathbb{Z}$ represent insurance classes. The insured moves yearly between them in the following way. If there is no claim then he/she is moved up to state $x+1$. In case of an insurance claim, the insured is moved down to $x-1$, $x-2$, $\ldots$, $x-m$ (depending on the type of the claim). This process can be represented by (2.1), where $a_m, \ldots, a_1$ are the probabilities of moving down, a_{-1} is the probability of moving up, and $a_0 = -a_m - \ldots - a_1 - a_{-1}$.

For $\mathbb{T} = \mathbb{Z}$, the Markov process is a special case of a random walk. Its states are recurrent if and only if the mean step $\sum_i ia_i$ is zero (see Theorem 8.2 and the subsequent paragraph in [10]). By Theorem 4.2, the same conclusion holds for all regular time scales.

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