Linear measure functional differential equations with infinite delay

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We use the theory of generalized linear ordinary differential equations in Banach spaces to study linear measure functional differential equations with infinite delay. We obtain new results concerning the existence, uniqueness, and continuous dependence of solutions. Even for equations with a finite delay, our results are stronger than the existing ones. Finally, we present an application to functional differential equations with impulses.

1 Introduction

In this paper, we deal with linear functional equations of the form

$$y(t)=y(a)+\int_a^t \ell(y_s,s)\,\mathrm{d}g(s)+\int_a^t p(s)\,\mathrm{d}g(s),\quad t\in[a,b], \tag{1.1}$$

where the functions y, ℓ, and p take values in $\mathbb{R}^n$, ℓ is linear in the first variable, and both integrals are the Kurzweil-Stieltjes integrals with respect to a nondecreasing function $g:[a,b]\to\mathbb{R}$. As is usual in the theory of functional differential equations, the symbol y_s stands for the function $y_s(\theta)=y(s+\theta)$, $\theta\in(-\infty,0]$.

Equation (1.1) represents a special case of the measure functional differential equation

$$y(t)=y(a)+\int_a^t f(y_s,s)\,\mathrm{d}g(s),\quad t\in[a,b] \tag{1.2}$$

introduced in [3] by M. Federson, J. G. Mesquita and A. Slavík for the case of finite delay; equations of this type with infinite delay were later studied by A. Slavík in [16].

For $g(s)=s$, equation (1.2) reduces to the classical functional differential equation studied by numerous authors (see e.g. [10]). Moreover, it was shown in [3] and [4] that impulsive functional differential equations as well as functional dynamic equations on time scales are special cases of the measure functional differential equation (1.2).

Our main tool in the study of equation (1.1) is the theory of generalized ordinary differential equations introduced by J. Kurzweil in [11]. The relation between functional differential equations and generalized ordinary differential equations in infinite-dimensional Banach spaces was first described by C. Imaz, F. Oliva and Z. Vorel in [9] and [14]. Later, a similar correspondence was established for impulsive

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functional differential equations by M. Federson and Š. Schwabik in [5], and for measure functional differential equations with finite and infinite delay in [3] and [16], respectively.

For equations with infinite delay, an important issue is the choice of the phase space; this topic is discussed in Section 2. In Section 3, we summarize the basic facts of the Kurzweil integration theory needed for our purposes and prove a new convergence theorem for the Kurzweil-Stieltjes integral. Section 4 describes the correspondence between linear measure functional differential equations and generalized linear ordinary differential equations. In Section 5, we prove a global existence-uniqueness theorem for linear measure functional differential equations. Section 6 contains the main results: a new continuous dependence theorem for generalized linear ordinary differential equations (inspired by the work of G. A. Monteiro and M. Tvrdý in [13]), and its counterpart for functional equations. Finally, in Section 7, we present an application of the previous results to impulsive functional differential equations.

Our paper confirms that the theory of generalized ordinary differential equations plays an important role in the study of functional differential equations. Moreover, by focusing on linear equations, we are able to obtain much stronger results than in the nonlinear case (even for equations with a finite delay).

2 Axiomatic description of the phase space

In contrast to classical functional differential equations, the solutions of measure functional differential equations are no longer continuous but merely regulated functions. Given an interval $[a,b]\subset\mathbb{R}$ and a Banach space X, recall that a function $f:[a,b]\to X$ is called regulated if the limits

$$\lim_{s\to t-} f(s)=f(t-)\in X,\quad t\in(a,b]\quad\text{and}\quad \lim_{s\to t+} f(s)=f(t+)\in X,\quad t\in[a,b)$$

exist. It is well known that every regulated function $f:[a,b]\to X$ is bounded; the symbol $\|f\|_\infty$ stands for the supremum norm of f.

Regulated functions on open or half-open intervals are defined in a similar way. Given an interval $I\subset\mathbb{R}$, we use the symbol $G(I,X)$ to denote the set of all regulated functions $f:I\to X$.

For equations with infinite delay, one of the crucial problems is the choice of a suitable phase space. In the axiomatic approach, we do not choose a fixed phase space, but instead deal with all spaces satisfying a given set of axioms. Consequently, there is no need to prove similar results repeatedly for different phase spaces. For classical functional differential equations with infinite delay, the axiomatic approach is well described in the paper [6] of J. K. Hale and J. Kato, as well as in the monograph [7] by S. Hino, S. Murakami, and T. Naito.

Our candidate for the phase space of a linear measure functional differential equation is a space $H_0\subset G((-\infty,0],\mathbb{R}^n)$ equipped with a norm denoted by $\|\cdot\|_\star$. We assume that H_0 satisfies the following conditions:

(HL1) H_0 is complete.

(HL2) If $y\in H_0$ and $t<0$, then $y_t\in H_0$.

(HL3) There exists a locally bounded function $\kappa_1:(-\infty,0]\to\mathbb{R}^+$ such that if $y\in H_0$ and $t\le 0$, then $\|y(t)\|\le\kappa_1(t)\|y\|_\star$.

(HL4) There exist functions $\kappa_2:[0,\infty)\to[1,\infty)$ and $\lambda:[0,\infty)\to\mathbb{R}^+$ such that if $u\le t\le 0$ and $y\in H_0$, then

$$\|y_t\|_\star\le\kappa_2(t-u)\sup_{s\in[u,t]}\|y(s)\|+\lambda(t-u)\|y_u\|_\star.$$

(HL5) There exists a locally bounded function $\kappa_3:(-\infty,0]\to\mathbb{R}^+$ such that if $y\in H_0$ and $t\le 0$, then

$$\|y_t\|_\star\le\kappa_3(t)\|y\|_\star.$$

Our conditions (HL1)–(HL5) are almost identical to conditions (H1)–(H5) in [16], except that (HL4) is stronger than (H4). Indeed, assume that $\sigma > 0$ and $y \in H_0$ is a function whose support is contained in $[-\sigma, 0]$. Using (HL4) with $t = 0$ and $u = -\sigma$, we obtain

$$\|y\|_\star \le \kappa_2(\sigma) \sup_{t\in[-\sigma,0]} \|y(t)\|,$$

which is precisely condition (H4). On the other hand, there is an additional condition (H6) in [16], which is however not strictly necessary (see [16, Remark 3.10]) and we omit it here.

Remark 2.1. One can replace (HL3) by the following condition, which is known from the axiomatic theory of classical functional differential equations with infinite delay (see [7]): *There exists a constant $\beta > 0$ such that if $y \in H_0$ and $t \le 0$, then $\|y(t)\| \le \beta\|y_t\|_\star$.* Indeed, combining this assumption with (HL5), we obtain $\|y(t)\| \le \beta\|y_t\|_\star \le \beta\kappa_3(t)\|y\|_\star$, i.e., (HL3) is satisfied with $\kappa_1 = \beta\kappa_3$.

The following example of a phase space is a simple modification of the space $C_\varphi((-\infty, 0], \mathbb{R}^n)$, which is well known from the classical theory of functional differential equations with infinite delay (see [7]).

Example 2.2. Consider the space

$$G_\varphi((-\infty, 0], \mathbb{R}^n) = \{y \in G((-\infty, 0], \mathbb{R}^n);\ y/\varphi \text{ is bounded}\},$$

where $\varphi : (-\infty, 0] \to \mathbb{R}$ is a fixed continuous positive function. The norm of a function $y \in G_\varphi((-\infty, 0], \mathbb{R}^n)$ is defined as

$$\|y\|_\varphi = \sup_{t\in(-\infty,0]} \frac{\|y(t)\|}{\varphi(t)}.$$

Assume that

$$\gamma_1(t) = \sup_{s\in(-\infty,-t]} \frac{\varphi(s+t)}{\varphi(s)} < \infty, \quad t \ge 0, \tag{2.1}$$

$$\gamma_2(t) = \sup_{s\in(-\infty,0]} \frac{\varphi(s+t)}{\varphi(s)} < \infty, \quad t \le 0, \tag{2.2}$$

and that γ_2 is a locally bounded function. (In the typical case when φ is nonincreasing, the first condition is satisfied automatically.) The following calculations show that under these hypotheses, the space $G_\varphi((-\infty, 0], \mathbb{R}^n)$ satisfies conditions (HL1)–(HL5).

- The mapping $y \mapsto y/\varphi$ is an isometric isomorphism between $G_\varphi((-\infty, 0], \mathbb{R}^n)$ and the space $BG((-\infty, 0], \mathbb{R}^n)$ of all bounded regulated functions on $(-\infty, 0]$, which is endowed with the supremum norm. The latter space is complete, and thus $G_\varphi((-\infty, 0], \mathbb{R}^n)$ is complete, too.

- For every $t < 0$ and $y \in G_\varphi((-\infty, 0], \mathbb{R}^n)$, we have

$$\sup_{s\in(-\infty,0]} \frac{\|y_t(s)\|}{\varphi(s)} \le \sup_{s\in(-\infty,0]} \frac{\|y(t+s)\|}{\varphi(t+s)} \sup_{s\in(-\infty,0]} \frac{\varphi(t+s)}{\varphi(s)} \le \|y\|_\varphi \gamma_2(t),$$

  which shows that (HL2) is true and (HL5) is satisfied with $\kappa_3(t) = \gamma_2(t)$.

- Since

$$\|y(t)\| \le \varphi(t)\frac{\|y(t)\|}{\varphi(t)} \le \varphi(t) \sup_{s\in(-\infty,0]} \frac{\|y(s)\|}{\varphi(s)},$$

  we see that (HL3) is satisfied with $\kappa_1(t) = \varphi(t)$.

- For an arbitrary $u \leq t \leq 0$, we have

$$\|y_t\|_\varphi = \sup_{s\in(-\infty,0]} \frac{\|y(t+s)\|}{\varphi(s)} = \sup_{s\in(-\infty,t]} \frac{\|y(s)\|}{\varphi(s-t)} \leq \sup_{s\in(-\infty,u]} \frac{\|y(s)\|}{\varphi(s-t)} + \sup_{s\in[u,t]} \frac{\|y(s)\|}{\varphi(s-t)}.$$

We estimate the first term as follows:

$$\sup_{s\in(-\infty,u]} \frac{\|y(s)\|}{\varphi(s-t)} \leq \sup_{s\in(-\infty,u]} \frac{\|y(s)\|}{\varphi(s-u)} \sup_{s\in(-\infty,u]} \frac{\varphi(s-u)}{\varphi(s-t)}$$

$$= \|y_u\|_\varphi \sup_{s\in(-\infty,u-t]} \frac{\varphi(s+t-u)}{\varphi(s)} \leq \|y_u\|_\varphi \gamma_1(t-u)$$

For the second term, we have the following estimate:

$$\sup_{s\in[u,t]} \frac{\|y(s)\|}{\varphi(s-t)} \leq \frac{\sup_{s\in[u,t]}\|y(s)\|}{\inf_{s\in[u,t]}\varphi(s-t)} = \frac{\sup_{s\in[u,t]}\|y(s)\|}{\inf_{s\in[u-t,0]}\varphi(s)}.$$

Thus,

$$\|y_t\|_\varphi \leq \|y_u\|_\varphi \gamma_1(t-u) + \frac{\sup_{s\in[u,t]}\|y(s)\|}{\inf_{s\in[u-t,0]}\varphi(s)},$$

which shows that (HL4) is satisfied with $\kappa_2(\sigma) = 1/(\inf_{s\in[-\sigma,0]}\varphi(s))$ and $\lambda(\sigma) = \gamma_1(\sigma)$, for $\sigma \in [0,\infty)$.

For example, when $\varphi(t) = 1$ for every $t \in (-\infty, 0]$, then $G_\varphi((-\infty,0],\mathbb{R}^n)$ coincides with the space $BG((-\infty,0],\mathbb{R}^n)$ of all bounded regulated functions on $(-\infty,0]$ and endowed with the supremum norm (see [16, Example 2.2]). Another important special case, which is a phase space commonly used for dealing with unbounded functions, is obtained by taking an arbitrary $\gamma \geq 0$ and letting $\varphi(t) = e^{-\gamma t}$ (see [16, Example 2.5]).

Besides the phase space H_0, we also need suitable spaces H_a of regulated functions defined on $(-\infty, a]$, where $a \in \mathbb{R}$. We obtain these spaces by shifting the functions from H_0. More precisely, for every $a \in \mathbb{R}$, denote $H_a = \{y \in G((-\infty,a],\mathbb{R}^n); y_a \in H_0\}$. Finally, define a norm $\|\cdot\|_\star$ on H_a by letting $\|y\|_\star = \|y_a\|_\star$ for every $y \in H_a$.

Example 2.3. Let H_0 be one of the phase spaces G_φ described in Example 2.2. Then H_a consists of all regulated functions $y : (-\infty, a] \to \mathbb{R}^n$ such that

$$\sup_{t\in(-\infty,a]} \frac{\|y(t)\|}{\varphi(t-a)} < \infty.$$

In this case, the value of the supremum equals $\|y\|_\star$.

The following lemma is a straightforward consequence of (HL1)–(HL5).

Lemma 2.4. *If $H_0 \subset G((-\infty,0],\mathbb{R}^n)$ is a space satisfying conditions* (HL1)–(HL5)*, then the following statements are true for every $a \in \mathbb{R}$:*

1. *H_a is complete.*
2. *If $y \in H_a$ and $t \leq a$, then $y_t \in H_0$.*
3. *If $t \leq a$ and $y \in H_a$, then $\|y(t)\| \leq \kappa_1(t-a)\|y\|_\star$.*
4. *If $y \in H_a$ and $u \leq t \leq a$, then*

$$\|y_t\|_\star \leq \kappa_2(t-u) \sup_{s\in[u,t]} \|y(s)\| + \lambda(t-u)\|y_u\|_\star.$$

5. *If $y \in H_a$ and $t \leq a$, then $\|y_t\|_\bigstar \leq \kappa_3(t-a)\|y\|_\bigstar$.*

The next lemma analyzes the particular case when a function $y \in H_b$ is defined as the prolongation of a function in H_0.

Lemma 2.5. *Let $\phi \in H_0$, $a, b \in \mathbb{R}$, with $a < b$, be given and consider a function $\tilde{x} \in H_b$ of the form*

$$\tilde{x}(\vartheta) = \begin{cases} \phi(\vartheta - a), & \vartheta \in (-\infty, a], \\ \phi(0), & \vartheta \in [a, b]. \end{cases}$$

Then, $\|\tilde{x}\|_\bigstar \leq (\kappa_2(b-a)\kappa_1(0) + \lambda(b-a))\|\phi\|_\bigstar$.

Proof. Using the definition of the norm in H_b, Lemma 2.4 and (HL3), we obtain

$$\begin{aligned} \|\tilde{x}\|_\bigstar = \|\tilde{x}_b\|_\bigstar &\leq \kappa_2(b-a) \sup_{s \in [a,b]} \|\tilde{x}(s)\| + \lambda(b-a)\|\tilde{x}_a\|_\bigstar \\ &= \kappa_2(b-a)\|\phi(0)\| + \lambda(b-a)\|\phi\|_\bigstar \leq (\kappa_2(b-a)\kappa_1(0) + \lambda(b-a))\|\phi\|_\bigstar. \quad \square \end{aligned}$$

3 Kurzweil integration

The integrals which occur in this paper represent special cases of the integral introduced by J. Kurzweil in [11] under the name "generalized Perron integral". For the reader's convenience, let us recall its definition.

Consider a Banach space X and let $\|\cdot\|_X$ denote its norm. As usual, the symbol $\mathcal{L}(X)$ denotes the space of all bounded linear operators on X.

A pair (D, ξ) is called a partition of the interval $[a, b]$, if $D : a = s_0 < s_1 < \cdots < s_m = b$ is a division of $[a, b]$, and $\tau_j \in [s_{j-1}, s_j]$ for every $j \in \{1, 2, \ldots, m\}$. Given a function $\delta : [a, b] \to \mathbb{R}^+$ (called a gauge on $[a, b]$), a partition is said to be δ-fine, if

$$[s_{j-1}, s_j] \subset (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)), \quad j \in \{1, 2, \ldots, m\}.$$

A function $U : [a, b] \times [a, b] \to X$ is Kurzweil integrable on $[a, b]$, if there exists a vector $I \in X$ such that for every $\varepsilon > 0$, there is a gauge $\delta : [a, b] \to \mathbb{R}^+$ such that

$$\left\| \sum_{j=1}^{m} [U(\tau_j, s_j) - U(\tau_j, s_{j-1})] - I \right\|_X < \varepsilon$$

for every δ-fine partition of $[a, b]$. In this case, we define the Kurzweil integral as $\int_a^b DU(\tau, s) = I$.

Basic properties of the Kurzweil integral, such as linearity, additivity with respect to adjacent intervals, as well as various convergence theorems, can be found in [17], [12].

When $U(\tau, s) = f(\tau)s$, where $f : [a, b] \to X$ is a given function, the definition above reduces to the definition of the well-known Henstock-Kurzweil integral, which generalizes the Lebesgue integral.

In this paper, we are particularly interested in Stieltjes-type integrals. The Kurzweil-Stieltjes integral $\int_a^b f \, \mathrm{d}g$ of a function $f : [a, b] \to \mathbb{R}^n$ with respect to a function $g : [a, b] \to \mathbb{R}$ is obtained by letting $U(\tau, s) = f(\tau)\, g(s)$. This is the integral which appears in the definition of a measure functional differential equation.

Secondly, we need the abstract Kurzweil-Stieltjes integral $\int_a^b \mathrm{d}[A]\, g$, where $A : [a, b] \to \mathcal{L}(X)$ and $g : [a, b] \to X$ (see [18]). This integral corresponds to the choice $U(\tau, s) = A(s)\, g(\tau)$, and it will appear in the definition of a generalized linear ordinary differential equation.

Let us recall that a function $f : [a, b] \to X$ has bounded variation on $[a, b]$, if

$$\operatorname{var}_{[a,b]} f = \sup \sum_{j=1}^{m} \|f(s_j) - f(s_{j-1})\|_X < \infty,$$

where the supremum is taken over all divisions $D : a = s_0 < s_1 < \ldots < s_m = b$ of the interval $[a, b]$. Let $BV([a, b], X)$ denote the set of all functions $f : [a, b] \to X$ with bounded variation. It is worth mentioning that $BV([a, b], X) \subset G([a, b], X)$.

The next theorem (see [18, Proposition 15]) provides a simple criterion for the existence of the abstract Kurzweil-Stieltjes integral.

Theorem 3.1. *If $A \in BV([a, b], \mathcal{L}(X))$ and $g : [a, b] \to X$ is a regulated function, then the integral $\int_a^b \mathrm{d}[A]g$ exists and we have*

$$\Big\| \int_a^b \mathrm{d}[A]g \Big\|_X \le (\operatorname{var}_{[a,b]} A)\|g\|_\infty.$$

The following property of the indefinite Kurzweil-Stieltjes integral (see [17, Theorem 1.16]) implies that solutions of measure functional differential equations are regulated functions.

Theorem 3.2. *Let $f : [a, b] \to \mathbb{R}^n$ and $g : [a, b] \to \mathbb{R}$ be such that g is regulated and the integral $\int_a^b f \,\mathrm{d}g$ exists. Then the function*

$$u(t) = \int_a^t f \,\mathrm{d}g, \quad t \in [a, b],$$

is regulated. Moreover, if g is left-continuous, then so is the function u.

The next convergence theorem for the Kurzweil-Stieltjes integral is inspired by a similar result from [13, Theorem 2.2]. Instead of requiring the uniform convergence of the sequence $\{A_k\}_{k=1}^\infty$ to A_0, we show that a weaker assumption is sufficient.

Theorem 3.3. *Let $A_k \in BV([a, b], \mathcal{L}(X))$, $g_k \in G([a, b], X)$ for $k \in \mathbb{N}_0$. Assume that the following conditions are satisfied:*

- $\lim_{k\to\infty} \|g_k - g_0\|_\infty = 0$.
- *There exists a constant $\gamma > 0$ such that $\operatorname{var}_{[a,b]} A_k \le \gamma$ for every $k \in \mathbb{N}$.*
- $\lim_{k\to\infty} \sup_{t\in[a,b]} \|[A_k(t) - A_0(t)]x\|_X = 0$ *for every $x \in X$.*

Then

$$\lim_{k\to\infty} \sup_{t\in[a,b]} \Big\| \int_a^t \mathrm{d}[A_k]g_k - \int_a^t \mathrm{d}[A_0]g_0 \Big\|_X = 0.$$

Proof. Let $\varepsilon > 0$ be given. Since g_0 is regulated, there exists a step function $g : [a, b] \to X$ such that $\|g_0 - g\|_\infty < \varepsilon$ (see [8, Theorem I.3.1]). Also, there exist a division $a = t_0 < t_1 < \cdots < t_m = b$ of $[a, b]$ and elements $c_1, \ldots, c_m \in X$ such that $g(t) = c_j$ for every $t \in (t_{j-1}, t_j)$.

Let $k_0 \in \mathbb{N}$ be such that

$$\|g_k - g_0\|_\infty < \varepsilon,$$

$$\|(A_k - A_0)(\tau)g(t_j)\|_X < \varepsilon/m, \quad j \in \{0, \ldots, m\},$$

$$\|(A_k - A_0)(\tau)c_j\|_X < \varepsilon/m, \quad j \in \{1, \ldots, m\}$$

for every $k \ge k_0$ and $\tau \in [a, b]$.

For an arbitrary $t \in [a, b]$, we have

$$\Big\| \int_a^t \mathrm{d}[A_k]g_k - \int_a^t \mathrm{d}[A_0]g_0 \Big\|_X \le \Big\| \int_a^t \mathrm{d}[A_k](g_k - g) \Big\|_X + \Big\| \int_a^t \mathrm{d}[A_k - A_0]g \Big\|_X + \Big\| \int_a^t \mathrm{d}[A_0](g - g_0) \Big\|_X.$$

For $k \geq k_0$, the first and third term on the right-hand side can be estimated using Theorem 3.1:

$$\left\| \int_a^t \mathrm{d}[A_k](g_k - g) \right\|_X \leq \gamma \|g_k - g\|_\infty \leq \gamma(\|g_k - g_0\|_\infty + \|g_0 - g\|_\infty) < 2\gamma\varepsilon,$$

$$\left\| \int_a^t \mathrm{d}[A_0](g - g_0) \right\|_X \leq (\operatorname{var}_{[a,b]} A_0)\|g - g_0\|_\infty < (\operatorname{var}_{[a,b]} A_0)\varepsilon.$$

Since g is a step function, we can calculate the integral $\int_a^t \mathrm{d}[A_k - A_0]g$ in the second term (see [18, Proposition 14]), and obtain the following estimate:

$$\begin{aligned} \left\| \int_a^t \mathrm{d}[A_k - A_0]g \right\|_X \leq & \sum_{j=1}^m \|[A_k - A_0](t_{j-1}+)g(t_{j-1}) - [A_k - A_0](t_{j-1})g(t_{j-1})\|_X \\ & + \sum_{j=1}^m \|[A_k - A_0](t_j-)c_j - [A_k - A_0](t_{j-1}+)c_j\|_X \\ & + \sum_{j=1}^m \|[A_k - A_0](t_j)g(t_j) - [A_k - A_0](t_j-)g(t_j)\|_X \leq 6\varepsilon. \end{aligned}$$

Consequently,

$$\sup_{t \in [a,b]} \left\| \int_a^t \mathrm{d}[A_k]g_k - \int_a^t \mathrm{d}[A_0]g_0 \right\|_X < \varepsilon(2\gamma + \operatorname{var}_{[a,b]} A_0 + 6)$$

for every $k \geq k_0$, which completes the proof. □

Remark 3.4. For the so-called interior integral, a result similar to Theorem 3.3 was proved by C. Hönig in [8, Theorem I.5.8].

4 Linear measure functional differential equations and generalized linear ordinary differential equations

A generalized linear ordinary differential equation is an integral equation of the form

$$x(t) = \tilde{x} + \int_a^t \mathrm{d}[A]\, x + h(t) - h(a), \qquad t \in [a,b], \tag{4.1}$$

where $A : [a,b] \to \mathcal{L}(X)$, $h \in G([a,b],X)$, and $\tilde{x} \in X$.

We say that a function $x : [a,b] \to X$ is a solution of (4.1) on the interval $[a,b]$, if the integral $\int_a^b \mathrm{d}[A]\, x$ exists and equality (4.1) holds for all $t \in [a,b]$.

Equations of the form (4.1), which represent a special case of generalized ordinary differential equations introduced by J. Kurzweil in [11], have been studied by numerous authors. The situation when X is a general Banach space was for the first time investigated by Š. Schwabik in [19], [20].

In this section, we clarify the relation between linear measure functional differential equations and generalized linear ordinary differential equations. Consider the functional equation

$$y(t) = y(a) + \int_a^t \ell(y_s, s)\,\mathrm{d}g(s) + \int_a^t p(s)\,\mathrm{d}g(s), \quad t \in [a,b],$$

where $g : [a,b] \to \mathbb{R}$ is nondecreasing, $\ell : H_0 \times [a,b] \to \mathbb{R}^n$ is linear in the first variable and $p : [a,b] \to \mathbb{R}^n$. We will show that, under certain assumptions, this functional equation is equivalent to the generalized equation (4.1), where $X = H_b$ and the functions A, h are defined as follows:

For every $t \in [a,b]$, $A(t) : H_b \to G((-\infty,b],\mathbb{R}^n)$ is the operator given by

$$(A(t)y)(\vartheta) = \begin{cases} 0, & -\infty < \vartheta \le a, \\ \int_a^{\vartheta} \ell(y_s,s)\,\mathrm{d}g(s), & a \le \vartheta \le t \le b, \\ \int_a^t \ell(y_s,s)\,\mathrm{d}g(s), & t \le \vartheta \le b, \end{cases} \tag{4.2}$$

and $h(t) \in G((-\infty,b],\mathbb{R}^n)$ is the function given by

$$h(t)(\vartheta) = \begin{cases} 0, & -\infty < \vartheta \le a, \\ \int_a^{\vartheta} p(s)\,\mathrm{d}g(s), & a \le \vartheta \le t \le b, \\ \int_a^t p(s)\,\mathrm{d}g(s), & t \le \vartheta \le b. \end{cases} \tag{4.3}$$

We introduce the following system of conditions, which will be useful later (in particular, conditions (A) and (E) guarantee that the integrals in (4.2) and (4.3) exist):

(A) The integral $\int_a^b \ell(y_t,t)\,\mathrm{d}g(t)$ exists for every $y \in H_b$.

(B) There exists a function $M : [a,b] \to \mathbb{R}^+$, which is Kurzweil-Stieltjes integrable with respect to g, such that

$$\left\| \int_u^v (\ell(y_t,t) - \ell(z_t,t))\,\mathrm{d}g(t) \right\| \le \int_u^v M(t)\|y_t - z_t\|_\star\,\mathrm{d}g(t)$$

whenever $y,z \in H_b$ and $[u,v] \subseteq [a,b]$.

(C) For every $y \in H_b$, $A(b)y$ is an element of H_b.

(D) H_b has the prolongation property for $t \ge a$, i.e., for every $y \in H_b$ and $t \in [a,b)$, the function $\bar{y} : (-\infty,b] \to \mathbb{R}^n$ given by

$$\bar{y}(s) = \begin{cases} y(s), & s \in (-\infty,t], \\ y(t), & s \in [t,b] \end{cases}$$

is an element of H_b.

(E) The integral $\int_a^b p(t)\,\mathrm{d}g(t)$ exists.

(F) There exists a function $N : [a,b] \to \mathbb{R}^+$, which is Kurzweil-Stieltjes integrable with respect to g, such that

$$\left\| \int_u^v p(t)\,\mathrm{d}g(t) \right\| \le \int_u^v N(t)\,\mathrm{d}g(t)$$

whenever $[u,v] \subseteq [a,b]$.

(G) $h(b)$ is an element of H_b.

We start with the following auxiliary statements.

Lemma 4.1. *Assume that $\ell : H_0 \times [a,b] \to \mathbb{R}^n$ is linear in the first variable and conditions* (A), (B) *are satisfied. Let $\tilde{\alpha} = \sup_{t\in[a,b]} \kappa_3(t-b)$. Then*

$$\left\| \int_u^v \ell(y_t,t)\,\mathrm{d}g(t) \right\| \le \|y\|_\star \int_u^v \tilde{\alpha} \cdot M(t)\,\mathrm{d}g(t)$$

whenever $y \in H_b$ and $[u,v] \subseteq [a,b]$.

Proof. For every $y \in H_b$ and $t \leq b$, we have

$$\|y_t\|_\bigstar \leq \kappa_3(t-b)\|y\|_\bigstar \leq \tilde{\alpha}\|y\|_\bigstar$$

(the first inequality follows from Lemma 2.4). To finish the proof, it is enough to apply (B) with $z \equiv 0$. □

Corollary 4.2. *Assume that $\ell : H_0 \times [a,b] \to \mathbb{R}^n$ is linear in the first variable and conditions* (A), (B) *are satisfied. For every bounded $O \subset H_b$, there exists a function $K : [a,b] \to \mathbb{R}^+$, which is Kurzweil-Stieltjes integrable with respect to g, such that*

$$\left\| \int_u^v \ell(y_t,t)\,\mathrm{d}g(t) \right\| \leq \int_u^v K(t)\,\mathrm{d}g(t)$$

whenever $y \in O$ and $[u,v] \subseteq [a,b]$.

The properties of the function A defined in (4.2), such as the fact that it has bounded variation on $[a,b]$, are described in the next lemma.

Lemma 4.3. *Assume that $\ell : H_0 \times [a,b] \to \mathbb{R}^n$ is linear in the first variable and conditions* (A)–(D) *are satisfied. For every $t \in [a,b]$, let $A(t) : H_b \to G((-\infty,b],\mathbb{R}^n)$ be given by* (4.2)*. Then, the function A takes values in $\mathcal{L}(H_b)$ and has bounded variation on $[a,b]$. Moreover,*

$$\operatorname{var}_{[a,b]} A \leq \kappa_2(b-a) \int_a^b \tilde{\alpha} \cdot M(s)\,\mathrm{d}g(s),$$

where $\tilde{\alpha} = \sup_{t\in[a,b]} \kappa_3(t-b)$.

Proof. It is clear that for every $t \in [a,b]$, $A(t)$ is a linear operator defined on H_b. Using (C), (D), and the definition of A, we see that $A(t)y \in H_b$ for every $y \in H_b$ and $t \in [a,b]$. Note that $(A(t)y)_a \equiv 0$ for every $y \in H_b$. Thus, by Lemma 2.4, we have

$$\|A(t)y\|_\bigstar = \|(A(t)y)_b\|_\bigstar \leq \kappa_2(b-a) \sup_{\vartheta\in[a,b]} \|(A(t)y)(\vartheta)\| = \kappa_2(b-a) \sup_{\vartheta\in[a,t]} \left\| \int_a^\vartheta \ell(y_s,s)\,\mathrm{d}g(s) \right\|.$$

By Lemma 4.1, $A(t)$ is a bounded linear operator on H_b and

$$\|A(t)\|_{\mathcal{L}(H_b)} \leq \kappa_2(b-a) \int_a^t \tilde{\alpha} \cdot M(s)\,\mathrm{d}g(s).$$

To show that $A : [a,b] \to \mathcal{L}(H_b)$ has bounded variation, consider $a \leq u < v \leq b$ and $y \in H_b$. By Lemmas 2.4 and 4.1,

$$\begin{aligned}
\|[A(v)-A(u)]y\|_\bigstar &\leq \kappa_2(b-a) \sup_{\vartheta\in[a,b]} \|([A(v)-A(u)]y)(\vartheta)\| \\
&= \kappa_2(b-a) \sup_{\vartheta\in[u,v]} \left\| \int_u^\vartheta \ell(y_s,s)\,\mathrm{d}g(s) \right\| \\
&\leq \kappa_2(b-a)\,\|y\|_\bigstar \int_u^v \tilde{\alpha} \cdot M(s)\,\mathrm{d}g(s).
\end{aligned}$$

Hence,

$$\|A(v)-A(u)\|_{\mathcal{L}(H_b)} \leq \kappa_2(b-a) \int_u^v \tilde{\alpha} \cdot M(s)\,\mathrm{d}g(s),$$

which concludes the proof. □

The following two theorems describe the relation between linear measure functional differential equations and generalized linear ordinary differential equations. Similar results for nonlinear equations were already obtained in [16]; therefore, it is sufficient to verify that the assumptions from [16] are satisfied.

Theorem 4.4. *Assume that $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, $\ell: H_0\times[a,b]\to\mathbb{R}^n$ is linear in the first variable, $\phi\in H_0$, and conditions* (A)–(G) *are satisfied. If $y\in H_b$ is a solution of the measure functional differential equation*

$$\begin{aligned} y(t) &= y(a)+\int_a^t \ell(y_s,s)\,\mathrm{d}g(s)+\int_a^t p(s)\,\mathrm{d}g(s), \quad t\in[a,b],\\ y_a &= \phi, \end{aligned}$$

then the function $x:[a,b]\to H_b$ given by

$$x(t)(\vartheta)=\begin{cases} y(\vartheta), & \vartheta\in(-\infty,t],\\ y(t), & \vartheta\in[t,b], \end{cases}$$

is a solution of the generalized ordinary differential equation

$$x(t)=x(a)+\int_a^t \mathrm{d}[A]x+h(t)-h(a), \quad t\in[a,b], \tag{4.4}$$

where A, h are given by (4.2), (4.3).

Proof. Consider the set $O=\{x(t);\,t\in[a,b]\}\subset H_b$. Clearly, O has the prolongation property for $t\geq a$. Observe that for every $t\in[a,b]$, the support of $x(t)-x(a)$ is contained in $[a,b]$. Thus, by Lemma 2.4, we have

$$\begin{aligned} \|x(t)\|_\star &\leq \|x(t)-x(a)\|_\star+\|x(a)\|_\star\leq \kappa_2(b-a)\sup_{\tau\in[a,b]}\|x(t)(\tau)-x(a)(\tau)\|+\|x(a)\|_\star\\ &\leq \kappa_2(b-a)\sup_{\tau\in[a,b]}\|x(b)(\tau)-x(a)(\tau)\|+\|x(a)\|_\star. \end{aligned}$$

The right-hand side does not depend on t, which means that the set O is bounded.

Let $f(y,t)=\ell(y,t)+p(t)$ for every $t\in[a,b]$ and $y\in H_0$. For every $y\in O$ and $[u,v]\subseteq[a,b]$, Corollary 4.2 and condition (F) lead to the estimate

$$\left\|\int_u^v f(y_s,s)\,\mathrm{d}g(s)\right\|\leq\left\|\int_u^v \ell(y_s,s)\,\mathrm{d}g(s)\right\|+\left\|\int_u^v p(s)\,\mathrm{d}g(s)\right\|\leq\int_u^v (K(s)+N(s))\,\mathrm{d}g(s).$$

The function F given by $F(y,t)=A(t)y+h(t)$ for every $t\in[a,b]$, $y\in H_b$ is well defined thanks to (A) and (E), and has values in H_b by (C), (D), and (G). This shows that all assumptions of Theorem 3.6 from [16] are satisfied. Consequently, x is a solution of the generalized ordinary differential equation whose right-hand side is F; however, this is equation coincides with (4.4). □

Theorem 4.5. *Assume that $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, $\ell: H_0\times[a,b]\to\mathbb{R}^n$ is linear in the first variable, $\phi\in H_0$, and conditions* (A)–(G) *are satisfied. Let A, h be given by* (4.2), (4.3). *If $x:[a,b]\to H_b$ is a solution of the generalized ordinary differential equation*

$$x(t)=x(a)+\int_a^t \mathrm{d}[A]x+h(t)-h(a), \quad t\in[a,b],$$

with the initial condition

$$x(a)(\vartheta)=\begin{cases} \phi(\vartheta-a), & \vartheta\in(-\infty,a],\\ \phi(0), & \vartheta\in[a,b], \end{cases}$$

then the function $y \in H_b$ defined by

$$y(\vartheta) = \begin{cases} x(a)(\vartheta), & \vartheta \in (-\infty, a], \\ x(\vartheta)(\vartheta), & \vartheta \in [a, b] \end{cases}$$

is a solution of the measure functional differential equation

$$\begin{aligned} y(t) &= y(a) + \int_a^t \ell(y_s, s)\,\mathrm{d}g(s) + \int_a^t p(s)\,\mathrm{d}g(s), \quad t \in [a, b], \\ y_a &= \phi. \end{aligned}$$

Proof. From the definition of A, it follows that the functions $x(t)$, where $t \in [a, b]$, coincide on $(-\infty, a]$. As in the proof of the previous theorem, the set $O = \{x(t),\, t \in [a, b]\}$ is bounded, has the prolongation property for $t \geq a$ (this follows from [16, Lemma 3.5]), and the functions f, F given by $f(y, t) = \ell(y, t) + p(t)$ and $F(y, t) = A(t)y + h(t)$ satisfy all assumptions of Theorem 3.7 from [16]; consequently, y is a solution of the given measure functional differential equation. □

5 Existence and uniqueness of solutions

A local existence and uniqueness theorem for nonlinear measure functional differential equations with a finite delay was obtained in [3, Theorem 5.3]. A generalized version for equations with infinite delay was proved in [16, Theorem 3.12]. For linear equations, it is possible to prove a much stronger global existence and uniqueness theorem; this is the content of the present section.

The following theorem guarantees the existence and uniqueness of a solution of the generalized linear ordinary differential equation, and corresponds to a special case of Proposition 2.8 from [19].

Theorem 5.1. *Consider a Banach space X and let $A \in BV([a, b], \mathcal{L}(X))$ be a left-continuous function. Then, for every $\tilde{x} \in X$ and every $h \in G([a, b], X)$, the equation*

$$x(t) = \tilde{x} + \int_a^t \mathrm{d}[A]\, x + h(t) - h(a), \qquad t \in [a, b]$$

has a unique solution x on $[a, b]$. Moreover, x is a regulated function.

With this in mind and using the relation established in the previous section, we derive the following result.

Theorem 5.2. *Assume that $g : [a, b] \to \mathbb{R}$ is a nondecreasing left-continuous function, $\ell : H_0 \times [a, b] \to \mathbb{R}^n$ is linear in the first variable, $\phi \in H_0$, conditions* (A)–(G) *are satisfied, and the function $x_0 : (-\infty, b] \to \mathbb{R}^n$ given by*

$$x_0(\vartheta) = \begin{cases} \phi(\vartheta - a), & \vartheta \in (-\infty, a], \\ \phi(0), & \vartheta \in [a, b], \end{cases}$$

is an element of H_b. Then, the measure functional differential equation

$$\begin{aligned} y(t) &= y(a) + \int_a^t \ell(y_s, s)\,\mathrm{d}g(s) + \int_a^t p(s)\,\mathrm{d}g(s), \quad t \in [a, b], \\ y_a &= \phi, \end{aligned}$$

has a unique solution on $[a, b]$.

Proof. Let A, h be given by (4.2), (4.3). It follows from Theorem 3.2 that A, h are regulated left-continuous functions. Moreover, by Lemma 4.3, A has bounded variation on $[a, b]$. Thus, Theorem 5.1 ensures the existence of a solution of the generalized linear ordinary differential equation

$$x(t) = x_0 + \int_a^t \mathrm{d}[A]x + h(t) - h(a), \quad t \in [a, b],$$

where x takes values in the Banach space $X = H_b$. By Theorem 4.5, there exists a corresponding solution of the given measure functional differential equation. If this equation had two different solutions, then, by Theorem 4.4, the corresponding generalized ordinary differential equation would have two different solutions, which is a contradiction. Thus, the solution has to be unique. □

6 Continuous dependence theorems

In this section, we use the theory of generalized linear ordinary differential equations (especially the results from [13]) to prove a new continuous dependence result for linear measure functional differential equations with infinite delay.

We remark that a continuous dependence theorem for nonlinear measure functional differential equations with a finite delay is available in [3, Theorem 6.3]. Although this theorem is also applicable to linear equations, our result is much stronger.

The following continuous dependence theorem (including its proof) is almost identical to Theorem 3.4 from [13]; however, our assumptions are weaker since we do not require that $\|A_k - A_0\|_\infty \to 0$. (On the other hand, [13, Theorem 3.4] does not require that A_k, $k \in \mathbb{N}_0$, are left-continuous functions.) Also, the result from the next theorem is more general than [1, Proposition A.3] when restricted to the linear case.

Theorem 6.1. *Let X be a Banach space. Consider $A_k \in BV([a,b],\mathcal{L}(X))$, $h_k \in G([a,b],X)$ and $\tilde{x}_k \in X$, for $k \in \mathbb{N}_0$. Assume that the following conditions are satisfied:*

- *For every $k \in \mathbb{N}_0$, A_k is a left-continuous function.*
- $\lim_{k\to\infty} \|\tilde{x}_k - \tilde{x}_0\|_X = 0$.
- $\lim_{k\to\infty} \sup_{t\in[a,b]} \|[A_k(t) - A_0(t)]x\|_X = 0$ *for every* $x \in X$.
- $\lim_{k\to\infty} \|h_k - h_0\|_\infty = 0$.
- *There exists a constant $\gamma > 0$ such that* $\operatorname{var}_{[a,b]} A_k \le \gamma$ *for every* $k \in \mathbb{N}$.

Then, for every $k \in \mathbb{N}_0$, the equation

$$x_k(t) = \tilde{x}_k + \int_a^t \mathrm{d}[A_k]\, x_k + h_k(t) - h_k(a), \quad t \in [a,b]$$

has a unique solution x_k on $[a,b]$, and $\lim_{k\to\infty} \|x_k - x_0\|_\infty = 0$.

Proof. Existence and uniqueness of solutions follow from Theorem 5.1. Next, observe that

$$\lim_{k\to\infty} \|x_k - x_0\|_\infty \le \lim_{k\to\infty} \|h_k - h_0\|_\infty + \lim_{k\to\infty} \|x_k - h_k - x_0 + h_0\|_\infty = \lim_{k\to\infty} \|w_k\|_\infty,$$

where

$$w_k = x_k - h_k - x_0 + h_0, \quad k \in \mathbb{N}.$$

Note that

$$w_k(a) = \tilde{x}_k - h_k(a) - \tilde{x}_0 + h_0(a), \quad k \in \mathbb{N},$$

and therefore $\lim_{k\to\infty} \|w_k(a)\|_X = 0$. A simple calculation reveals that

$$\begin{aligned} w_k(t) &= \tilde{x}_k + \int_a^t \mathrm{d}[A_k]x_k - h_k(a) - \tilde{x}_0 - \int_a^t \mathrm{d}[A_0]x_0 + h_0(a) \\ &= w_k(a) + \int_a^t \mathrm{d}[A_k]x_k - \int_a^t \mathrm{d}[A_0]x_0 \\ &= w_k(a) + \int_a^t \mathrm{d}[A_k]w_k + \int_a^t \mathrm{d}[A_k]h_k - \int_a^t \mathrm{d}[A_0]h_0 + \int_a^t \mathrm{d}[A_k - A_0](x_0 - h_0) \\ &= w_k(a) + \int_a^t \mathrm{d}[A_k]w_k + z_k(t) - z_k(a), \end{aligned}$$

where

$$z_k(t) = \int_a^t \mathrm{d}[A_k]h_k - \int_a^t \mathrm{d}[A_0]h_0 + \int_a^t \mathrm{d}[A_k - A_0](x_0 - h_0).$$

By Theorem 3.3,

$$\lim_{k\to\infty} \sup_{t\in[a,b]} \left\| \int_a^t \mathrm{d}[A_k]h_k - \int_a^t \mathrm{d}[A_0]h_0 \right\|_X = 0,$$

$$\lim_{k\to\infty} \sup_{t\in[a,b]} \left\| \int_a^t \mathrm{d}[A_k - A_0](x_0 - h_0) \right\|_X = 0,$$

and consequently $\lim_{k\to\infty} \|z_k\|_\infty = 0$. According to [13, Lemma 3.2], we have

$$\|w_k(t)\|_X \le (\|w_k(a)\|_X + \|z_k\|_\infty)e^\gamma, \quad t \in [a,b],$$

which implies that $\lim_{k\to\infty} \|w_k\|_\infty = 0$. □

For every $k \in \mathbb{N}_0$, we consider the linear measure functional differential equation

$$\begin{aligned} y_k(t) &= y_k(a) + \int_a^t \ell_k((y_k)_s, s)\,\mathrm{d}g_k(s) + \int_a^t p_k(s)\,\mathrm{d}g_k(s), \quad t \in [a,b], \\ (y_k)_a &= \phi_k, \end{aligned}$$

where $g_k : [a,b] \to \mathbb{R}$ is nondecreasing, $\ell_k : H_0 \times [a,b] \to \mathbb{R}^n$ is linear in the first variable, $p_k : [a,b] \to \mathbb{R}^n$ and $\phi_k \in H_0$. We define the corresponding operators $A_k(t) : H_b \to G((-\infty, b], \mathbb{R}^n)$ by

$$(A_k(t)y)(\vartheta) = \begin{cases} 0, & -\infty < \vartheta \le a, \\ \int_a^\vartheta \ell_k(y_s, s)\,\mathrm{d}g_k(s), & a \le \vartheta \le t \le b, \\ \int_a^t \ell_k(y_s, s)\,\mathrm{d}g_k(s), & t \le \vartheta \le b, \end{cases} \tag{6.1}$$

and functions $h_k(t) \in G((-\infty, b], \mathbb{R}^n)$ by

$$(h_k(t))(\vartheta) = \begin{cases} 0, & -\infty < \vartheta \le a, \\ \int_a^\vartheta p_k(s)\,\mathrm{d}g_k(s), & a \le \vartheta \le t \le b, \\ \int_a^t p_k(s)\,\mathrm{d}g_k(s), & t \le \vartheta \le b, \end{cases} \tag{6.2}$$

for every $k \in \mathbb{N}_0$ and $t \in [a,b]$.

To reflect the fact that we are dealing with sequences of functions, we modify conditions (A)–(G) from Section 4 as follows:

(A) The integral $\int_a^b \ell_k(y_t, t)\,\mathrm{d}g_k(t)$ exists for every $k \in \mathbb{N}_0$ and $y \in H_b$.

(B) For every $k \in \mathbb{N}_0$, there exists a function $M_k : [a,b] \to \mathbb{R}^+$, which is Kurzweil-Stieltjes integrable with respect to g_k, such that

$$\left\| \int_u^v (\ell_k(y_t, t) - \ell_k(z_t, t))\,\mathrm{d}g_k(t) \right\| \le \int_u^v M_k(t)\|y_t - z_t\|_\bigstar\,\mathrm{d}g_k(t)$$

whenever $y, z \in H_b$ and $[u,v] \subseteq [a,b]$.

(C) For every $k \in \mathbb{N}_0$ and $y \in H_b$, the function $A_k(b)y$ is an element of H_b.

(D) H_b has the prolongation property for $t \ge a$.

(E) For every $k \in \mathbb{N}_0$, the integral $\int_a^b p_k(t)\,\mathrm{d}g_k(t)$ exists.

(F) For every $k \in \mathbb{N}_0$, there exists a function $N_k : [a,b] \to \mathbb{R}^+$, which is Kurzweil-Stieltjes integrable with respect to g_k, such that

$$\left\|\int_u^v p_k(t)\,\mathrm{d}g_k(t)\right\| \le \int_u^v N_k(t)\,\mathrm{d}g_k(t)$$

whenever $[u,v] \subseteq [a,b]$.

(G) For every $k \in \mathbb{N}_0$, the function $h_k(b)$ is an element of H_b.

Now, using the correspondence established in Theorems 4.4 and 4.5, we derive the following result.

Theorem 6.2. *For every $k \in \mathbb{N}_0$, suppose that $g_k : [a,b] \to \mathbb{R}$ is a nondecreasing left-continuous function, and $\ell_k : H_0 \times [a,b] \to \mathbb{R}^n$ is a function linear with respect to the first variable. Assume that conditions* (A)–(G) *as well as the following conditions are satisfied:*

- *For every $y \in H_b$, $\lim_{k\to\infty} \sup_{t\in[a,b]} \left\| \int_a^t \ell_k(y_s,s)\mathrm{d}g_k(s) - \int_a^t \ell_0(y_s,s)\,\mathrm{d}g_0(s)\right\| = 0$.*
- *$\lim_{k\to\infty} \sup_{t\in[a,b]} \left\| \int_a^t p_k(s)\mathrm{d}g_k(s) - \int_a^t p_0(s)\,\mathrm{d}g_0(s)\right\| = 0$.*
- *There exists a constant $\gamma > 0$ such that $\int_a^b M_k(s)\mathrm{d}g_k(s) \le \gamma$ for all $k \in \mathbb{N}$.*

Further, consider a sequence of functions $\phi_k \in H_0$, $k \in \mathbb{N}_0$, such that

$$\lim_{k\to\infty} \|\phi_k - \phi_0\|_\bigstar = 0,$$

and such that for every $k \in \mathbb{N}_0$, the function

$$\tilde{x}_k(\vartheta) = \begin{cases} \phi_k(\vartheta - a), & \vartheta \in (-\infty, a], \\ \phi_k(0), & \vartheta \in [a,b], \end{cases}$$

is an element of H_b.

Then, for each $k \in \mathbb{N}_0$, there exists a solution $y_k : (-\infty, b] \to \mathbb{R}^n$ of the measure functional differential equation

$$\left.\begin{aligned} y_k(t) &= y_k(a) + \int_a^t \ell_k((y_k)_s, s)\,\mathrm{d}g_k(s) + \int_a^t p_k(s)\,\mathrm{d}g_k(s), \quad t \in [a,b], \\ (y_k)_a &= \phi_k, \end{aligned}\right\} \tag{6.3}$$

and the sequence $\{y_k\}_{k=1}^\infty$ converges uniformly to y_0 on $[a,b]$.

Proof. Consider A_k, h_k, for $k \in \mathbb{N}_0$, given by (6.1), (6.2). By Theorem 3.2, h_k and A_k are regulated left-continuous functions. In addition, by Lemma 4.3, $A_k \in BV([a,b], \mathcal{L}(H_b))$ with

$$\operatorname{var}_{[a,b]} A_k \le \kappa_2(b-a) \int_a^b \tilde{\alpha} \cdot M_k(s)\,\mathrm{d}g_k(s) \le \kappa_2(b-a) \cdot \tilde{\alpha} \cdot \gamma, \quad k \in \mathbb{N}_0, \tag{6.4}$$

where $\tilde{\alpha} = \sup_{t\in[a,b]} \kappa_3(t-a)$. For every $k \in \mathbb{N}_0$, Theorem 5.1 ensures the existence of a unique solution $x_k : [a,b] \to H_b$ of the generalized ordinary differential equation

$$x_k(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k]\, x_k + h_k(t) - h_k(a), \quad t \in [a,b].$$

Given $y \in H_b$, the definition of A_k, $k \in \mathbb{N}_0$, together with Lemma 2.4, implies

$$\|A_k(t)y - A_0(t)y\|_\bigstar \le \kappa_2(b-a) \sup_{\vartheta\in[a,b]} \|[A_k(t)y - A_0(t)y](\vartheta)\|$$
$$= \kappa_2(b-a) \sup_{\vartheta\in[a,b]} \left\| \int_a^\vartheta \ell_k(y_s,s)\mathrm{d}g_k(s) - \int_a^\vartheta \ell_0(y_s,s)\,\mathrm{d}g_0(s) \right\|$$

for every $t \in [a,b]$. Consequently,

$$\sup_{t\in[a,b]} \|[A_k(t) - A_0(t)]\,y\|_\bigstar \le \kappa_2(b-a) \sup_{\vartheta\in[a,b]} \left\| \int_a^\vartheta \ell_k(y_s,s)\mathrm{d}g_k(s) - \int_a^\vartheta \ell_0(y_s,s)\,\mathrm{d}g_0(s) \right\|,$$

and we conclude that

$$\lim_{k\to\infty} \sup_{t\in[a,b]} \|[A_k(t) - A_0(t)]\,y\|_\bigstar = 0, \quad y \in H_b.$$

Analogously, we have

$$\|h_k(t) - h_0(t)\|_\bigstar \le \kappa_2(b-a) \sup_{\vartheta\in[a,b]} \left\| \int_a^\vartheta p_k(s)\mathrm{d}g_k(s) - \int_a^\vartheta p_0(s)\,\mathrm{d}g_0(s) \right\|, \quad t \in [a,b],$$

and thus $\lim_{k\to\infty} \|h_k - h_0\|_\infty = 0$ holds.

To show that $\lim_{k\to\infty} \|\tilde{x}_k - \tilde{x}_0\|_\bigstar = 0$, it is enough to notice that, by Lemma 2.5, we have

$$\|\tilde{x}_k - \tilde{x}_0\|_\bigstar \le (\kappa_2(b-a)\kappa_1(0) + \lambda(b-a))\|\phi_k - \phi_0\|_\bigstar.$$

In summary, all hypotheses of Theorem 6.1 are satisfied, which proves that $\lim_{k\to\infty} \|x_k - x_0\|_\infty = 0$.

By Theorems 4.4 and 4.5, for each $k \in \mathbb{N}_0$, the function

$$y_k(\vartheta) = \begin{cases} x_k(a)(\vartheta), & \vartheta \in (-\infty, a], \\ x_k(\vartheta)(\vartheta), & \vartheta \in [a,b] \end{cases}$$

is the unique solution of Eq. (6.3). For $t \in [a,b]$, we can use Lemma 2.4 to see that

$$\|y_k(t) - y_0(t)\| = \|x_k(t)(t) - x_0(t)(t)\| \le \kappa_1(t-b)\|x_k(t) - x_0(t)\|_\bigstar \le \tilde{\beta} \sup_{\tau\in[a,b]} \|x_k(\tau) - x_0(\tau)\|_\bigstar,$$

where $\tilde{\beta} = \sup_{\sigma\in[a,b]} \kappa_1(\sigma - a)$. Thus, the sequence $\{y_k\}_{k=1}^\infty$ is uniformly convergent to y_0. □

Remark 6.3. By the fourth part of Lemma 2.4,

$$\|y_k - y_0\|_\bigstar = \|(y_k - y_0)_b\|_\bigstar \le \kappa_2(b-a) \sup_{s\in[a,b]} \|y_k(s) - y_0(s)\| + \lambda(b-a)\|\phi_k - \phi_0\|_\bigstar,$$

i.e., the sequence of solutions $\{y_k\}_{k=1}^\infty$ from Theorem 6.2 converges to y_0 also in the $\|\cdot\|_\bigstar$ norm.

7 Functional differential equations with impulses

As an example, we show how our results apply to functional differential equations with impulses. For simplicity, we restrict ourselves to the case when the phase space H_0 coincides with one of the spaces G_φ from Example 2.2. Recall that $G_\varphi((-\infty,0],\mathbb{R}^n) = \{y \in G((-\infty,0],\mathbb{R}^n);\ y/\varphi \text{ is bounded}\}$, and for $H_0 = G_\varphi$, the norm of a function $y \in H_a$ is given by

$$\|y\|_\bigstar = \sup_{t\in(-\infty,0]} \frac{\|y(t+a)\|}{\varphi(t)} = \sup_{t\in(-\infty,a]} \frac{\|y(t)\|}{\varphi(t-a)}.$$

We consider linear impulsive functional differential equations of the form

$$\begin{aligned} y'(t) &= \ell(y_t,t)+p(t), \quad \text{a.e. in } [a,b], \\ \Delta^+ y(t_i) &= A_i y(t_i)+b_i, \quad i\in\{1,\dots,k\}, \end{aligned} \tag{7.1}$$

where $\ell : H_0 \times [a,b] \to \mathbb{R}^n$ is linear in the first variable, $p : [a,b] \to \mathbb{R}^n$, $a \le t_1 < \cdots < t_k < b$, $A_1,\dots,A_k \in \mathbb{R}^{n\times n}$, $b_1,\dots,b_k \in \mathbb{R}^n$, and, as usual, $\Delta^+ y(s) = y(s+) - y(s)$, $s \in [a,b)$. In addition, assume the following conditions are satisfied:

(1) The Lebesgue integral $\int_a^b \ell(y_t,t)\,\mathrm{d}t$ exists for every $y \in H_b$.

(2) There exists a Lebesgue integrable function $M : [a,b] \to \mathbb{R}^+$ such that

$$\left\| \int_u^v (\ell(y_t,t)-\ell(z_t,t))\,\mathrm{d}t \right\| \le \int_u^v M(t)\|y_t-z_t\|_\star\,\mathrm{d}t$$

whenever $y,z \in H_b$ and $[u,v] \subseteq [a,b]$.

(3) The Lebesgue integral $\int_a^b p(t)\,\mathrm{d}t$ exists.

(4) There exists a Lebesgue integrable function $N : [a,b] \to \mathbb{R}^+$ such that

$$\left\| \int_u^v p(t)\,\mathrm{d}t \right\| \le \int_u^v N(t)\,\mathrm{d}t$$

whenever $[u,v] \subseteq [a,b]$.

The solutions of Eq. (7.1) are assumed to be left-continuous on $[a,b]$, and absolutely continuous on $[a,t_1]$, $(t_1,t_2],\dots,(t_k,b]$. The equivalent integral form of Eq. (7.1) is

$$y(t) = y(a) + \int_a^t (\ell(y_s,s)+p(s))\,\mathrm{d}s + \sum_{i;\, t_i<t} (A_i y(t_i)+b_i). \tag{7.2}$$

We need the following statement, which is a consequence of [4, Lemma 2.4].

Lemma 7.1. *Let $k \in \mathbb{N}$, $a \le t_1 < t_2 < \cdots < t_k < b$, and*

$$g(s) = s + \sum_{i=1}^k \chi_{(t_i,\infty)}(s), \quad s \in [a,b]$$

(the symbol χ_A denotes the characteristic function of a set $A \subset \mathbb{R}$). Consider an arbitrary function $f : [a,b] \to \mathbb{R}$ and let $\tilde f : [a,b] \to \mathbb{R}$ be such that $\tilde f(s) = f(s)$ for every $s \in [a,b]\backslash\{t_1,\dots,t_k\}$.

Then the integral $\int_a^b \tilde f(s)\,\mathrm{d}g(s)$ exists if and only if the integral $\int_a^b f(s)\,\mathrm{d}s$ exists; in that case, we have

$$\int_a^t \tilde f(s)\,\mathrm{d}g(s) = \int_a^t f(s)\,\mathrm{d}s + \sum_{\substack{i\in\{1,\dots,k\},\\ t_i<t}} \tilde f(t_i), \quad t \in [a,b].$$

Using the previous lemma, Eq. (7.2) can be rewritten as

$$y(t) = y(a) + \int_a^t \tilde\ell(y_s,s)\,\mathrm{d}g(s) + \int_a^t \tilde p(s)\,\mathrm{d}g(s), \tag{7.3}$$

where

$$\tilde\ell(z,t) = \begin{cases} \ell(z,t) & \text{if } t \in [a,b]\backslash\{t_1,\dots,t_k\}, \\ A_i z(0) & \text{if } t = t_i \text{ for some } i \in \{1,\dots,k\} \end{cases}$$

$$\tilde{p}(t)=\begin{cases}p(t) & \text{if } t\in[a,b]\backslash\{t_1,\dots,t_k\},\\ b_i & \text{if } t=t_i \text{ for some } i\in\{1,\dots,k\}\end{cases}$$

for every $t\in[a,b]$ and $z\in H_0$. Thus, the original linear impulsive functional differential equation (7.1) is equivalent to the linear measure functional differential equation (7.3).

It follows from Lemma 7.1 that assumptions (A), (E) from Section 4, where ℓ, p are replaced by $\tilde{\ell}$, $\tilde{p}$, are satisfied. For $[u,v]\subset[a,b]$, we obtain

$$\left\|\int_u^v \tilde{p}(t)\,\mathrm{d}g(t)\right\|=\left\|\int_u^v p(t)\,\mathrm{d}t+\sum_{i;\ t_i\in[u,v)} b_i\right\|\le\int_u^v N(t)\,\mathrm{d}t+\sum_{i;\ t_i\in[u,v)}\|b_i\|=\int_u^v \tilde{N}(t)\,\mathrm{d}g(t),$$

where

$$\tilde{N}(t)=\begin{cases}N(t) & \text{if } t\in[a,b]\backslash\{t_1,\dots,t_k\},\\ \|b_i\| & \text{if } t=t_i \text{ for some } i\in\{1,\dots,k\}.\end{cases}$$

This verifies assumption (F). Similarly, for every $y,z\in H_b$, we get

$$\left\|\int_u^v (\tilde{\ell}(y_t,t)-\tilde{\ell}(z_t,t))\,\mathrm{d}g(t)\right\|=\left\|\int_u^v (\ell(y_t,t)-\ell(z_t,t))\,\mathrm{d}t+\sum_{i;\ t_i\in[u,v)} A_i(y(t_i)-z(t_i))\right\|$$

$$\le\int_u^v M(t)\|y_t-z_t\|_\star\,\mathrm{d}t+\sum_{i;\ t_i\in[u,v)}\|A_i\|\kappa_1(0)\|y_{t_i}-z_{t_i}\|_\star=\int_u^v \tilde{M}(t)\|y_t-z_t\|_\star\,\mathrm{d}g(t),$$

where

$$\tilde{M}(t)=\begin{cases}M(t) & \text{if } t\in[a,b]\backslash\{t_1,\dots,t_k\},\\ \|A_i\|\kappa_1(0) & \text{if } t=t_i \text{ for some } i\in\{1,\dots,k\}.\end{cases}$$

Hence, assumption (B) from Section 4 is satisfied. The remaining assumptions (C), (D), and (G) are fulfilled, too: (C) and (G) follow from the fact that $A(b)y$ and $h(b)$ are regulated functions with compact support contained in $[a,b]$, and hence elements of H_b. Also, it is clear that our space H_b has the prolongation property for $t\ge a$.

Finally, let $\phi\in H_0$. We claim that the function $y:(-\infty,b]\to\mathbb{R}^n$ given by

$$y(\vartheta)=\begin{cases}\phi(\vartheta-a), & \vartheta\in(-\infty,a],\\ \phi(0), & \vartheta\in[a,b]\end{cases}$$

is an element of H_b. (Recall that assumptions of this type appear both in the existence-uniqueness theorem and in the continuous dependence theorem.) The claim follows from the estimate

$$\sup_{t\in(-\infty,b]}\frac{\|y(t)\|}{\varphi(t-b)}\le\sup_{t\in(-\infty,a]}\frac{\|y(t)\|}{\varphi(t-b)}+\sup_{t\in[a,b]}\frac{\|y(t)\|}{\varphi(t-b)},$$

since the second supremum is finite, and

$$\sup_{t\in(-\infty,a]}\frac{\|y(t)\|}{\varphi(t-b)}\le\left(\sup_{t\in(-\infty,a]}\frac{\|y(t)\|}{\varphi(t-a)}\right)\left(\sup_{t\in(-\infty,a]}\frac{\varphi(t-a)}{\varphi(t-b)}\right)=\|\phi\|_\star\left(\sup_{t\in(-\infty,a]}\frac{\varphi(t-a)}{\varphi(t-b)}\right)$$

is finite, too (we need (2.1) here).

Our calculations lead to the following existence-uniqueness theorem, which is an immediate consequence of Theorem 5.2. (For a local existence-uniqueness theorem for nonlinear impulsive functional differential equations with a finite delay, see [5, Theorem 5.1].)

Theorem 7.2. *Assume that $H_0 = G_\varphi$ and conditions (1)–(4) are satisfied. Then for every $\phi \in G_\varphi$, the impulsive functional differential equation*

$$\begin{aligned} y'(t) &= \ell(y_t, t) + p(t), \quad \text{a.e. in } [a,b], \\ \Delta^+ y(t_i) &= A_i y(t_i) + b_i, \quad i \in \{1,\dots,k\}, \\ y_a &= \phi \end{aligned}$$

has a unique solution on $[a,b]$.

We now proceed to continuous dependence and consider a sequence of impulsive functional differential equations of the form

$$\begin{aligned} y_j'(t) &= \ell_j((y_j)_t, t) + p_j(t), \quad \text{a.e. in } [a,b], \\ \Delta^+ y_j(t_i) &= A_j^i y_j(t_i) + b_j^i, \quad i \in \{1,\dots,k\}, \end{aligned}$$

where $\ell_j : H_0 \times [a,b] \to \mathbb{R}^n$ is linear in the first variable, $p_j : [a,b] \to \mathbb{R}^n$, $A_j^1, \dots, A_j^k \in \mathbb{R}^{n\times n}$, and $b_j^1, \dots, b_j^k \in \mathbb{R}^n$ for every $j \in \mathbb{N}_0$.

For linear equations, the following result is much stronger than the continuous dependence result stated in [5, Theorem 4.1].

Theorem 7.3. *Assume that $H_0 = G_\varphi$ and the following conditions are satisfied:*

- *The Lebesgue integral $\int_a^b \ell_j(y_t, t)\,\mathrm{d}t$ exists for every $j \in \mathbb{N}_0$ and $y \in H_b$.*
- *For every $j \in \mathbb{N}_0$, there exists a Lebesgue integrable function $M_j : [a,b] \to \mathbb{R}^+$ such that*

  $$\left\| \int_u^v (\ell_j(y_t, t) - \ell_j(z_t, t))\,\mathrm{d}t \right\| \le \int_u^v M_j(t) \|y_t - z_t\|_\star \,\mathrm{d}t$$

  whenever $y, z \in H_b$ and $[u,v] \subseteq [a,b]$.
- *For every $j \in \mathbb{N}_0$, the Lebesgue integral $\int_a^b p_j(t)\,\mathrm{d}t$ exists.*
- *For every $j \in \mathbb{N}_0$, there exists a Lebesgue integrable function $N_j : [a,b] \to \mathbb{R}^+$ such that*

  $$\left\| \int_u^v p_j(t)\,\mathrm{d}t \right\| \le \int_u^v N_j(t)\,\mathrm{d}t$$

  whenever $[u,v] \subseteq [a,b]$.
- *For every $y \in H_b$, $\lim_{j\to\infty} \sup_{t\in[a,b]} \left\| \int_a^t \ell_j(y_s, s)\,\mathrm{d}s - \int_a^t \ell_0(y_s, s)\,\mathrm{d}s \right\| = 0$.*
- $\lim_{j\to\infty} \sup_{t\in[a,b]} \left\| \int_a^t p_j(s)\,\mathrm{d}s - \int_a^t p_0(s)\,\mathrm{d}s \right\| = 0$.
- *There exists a constant $\gamma > 0$ such that $\int_a^b M_j(s)\,\mathrm{d}s \le \gamma$ for all $j \in \mathbb{N}$.*
- *For every $i \in \{1,\dots,k\}$, $\lim_{j\to\infty} A_j^i = A_0^i$ and $\lim_{j\to\infty} b_j^i = b_0^i$.*

Further, consider a sequence of functions $\phi_j \in H_0$, $j \in \mathbb{N}_0$, such that $\lim_{j\to\infty} \|\phi_j - \phi_0\|_\star = 0$.

Then, for each $j \in \mathbb{N}_0$, there exists a solution $y_j : (-\infty, b] \to \mathbb{R}^n$ of the impulsive functional differential equation

$$\begin{aligned} y_j'(t) &= \ell_j((y_j)_t, t) + p_j(t), \quad \text{a.e. in } [a,b], \\ \Delta^+ y_j(t_i) &= A_j^i y_j(t_i) + b_j^i, \quad i \in \{1,\dots,k\}, \\ (y_j)_a &= \phi_j, \end{aligned}$$

and the sequence $\{y_j\}_{j=1}^\infty$ converges uniformly to y_0 on $[a,b]$.

Proof. The theorem is a consequence of Theorem 6.2. Indeed, our sequence of impulsive equations is equivalent to the sequence of measure functional differential equations

$$\begin{aligned} y_j(t) &= y_j(a)+\int_a^t \tilde{\ell}_j((y_j)_s,s)\,\mathrm{d}g(s)+\int_a^t \tilde{p}_j(s)\,\mathrm{d}g(s), \quad t\in[a,b],\\ (y_j)_a &= \phi_j, \end{aligned}$$

where the functions $\tilde{\ell}_j$, $\tilde{p}_j$, and g are defined as in the beginning of the present section. Our previous calculations show that assumptions (A)–(G) from Section 5 (with ℓ_j, p_j replaced by $\tilde{\ell}_j$, $\tilde{p}_j$) are satisfied. In particular, we have the estimate

$$\left\|\int_u^v (\tilde{\ell}_j(y_t,t)-\tilde{\ell}_j(z_t,t))\,\mathrm{d}g(t)\right\| \le \int_u^v \tilde{M}_j(t)\|y_t-z_t\|_\star\,\mathrm{d}g(t),$$

where $\tilde{M}_j(t)=M_j(t)$ if $t\in[a,b]\backslash\{t_1,\dots,t_k\}$, and $\tilde{M}_j(t)=\|A_j^i\|\kappa_1(0)$ if $t=t_i$ for some $i\in\{1,\dots,k\}$. Hence,

$$\int_a^b \tilde{M}_j(s)\,\mathrm{d}g(s)=\int_a^b M_j(s)\,\mathrm{d}s+\sum_{i=1}^k\|A_j^i\|\kappa_1(0)\le\gamma+\kappa_1(0)\sum_{i=1}^k\sup_{j\in\mathbb{N}}\|A_j^i\|$$

for every $j\in\mathbb{N}$. By Lemma 7.1, we have

$$\left\|\int_a^t (\tilde{p}_j(s)-\tilde{p}_0(s))\,\mathrm{d}g(s)\right\| \le \left\|\int_a^t (p_j(s)-p_0(s))\,\mathrm{d}s\right\| + \left\|\sum_{i;\ t_i<t}(b_i^j-b_i^0)\right\|,$$

and it follows that

$$\lim_{j\to\infty}\sup_{t\in[a,b]}\left\|\int_a^t \tilde{p}_j(s)\,\mathrm{d}g(s)-\int_a^t \tilde{p}_0(s)\,\mathrm{d}g(s)\right\|=0.$$

Similarly, for every $y\in H_b$, we have

$$\left\|\int_a^t (\tilde{\ell}_j(y_s,s)-\tilde{\ell}_0(y_s,s))\,\mathrm{d}g(s)\right\| \le \left\|\int_a^t (\ell_j(y_s,s)-\ell_0(y_s,s))\,\mathrm{d}s\right\| + \left\|\sum_{i;\ t_i<t}(A_i^j-A_i^0)y(t_i)\right\|.$$

Since the last term can be majorized by $\sum_{i=1}^k \|A_i^j-A_i^0\|\cdot\|y(t_i)\|$, we obtain

$$\lim_{j\to\infty}\sup_{t\in[a,b]}\left\|\int_a^t \tilde{\ell}_j(y_s,s)\,\mathrm{d}g(s)-\int_a^t \tilde{\ell}_0(y_s,s)\,\mathrm{d}g(s)\right\|=0.$$

Thus, we have verified that all assumptions of Theorem 6.2 are satisfied. □

8 Conclusion

Finally, let us mention that the results of this paper are also applicable to linear functional dynamic equations on time scales. As shown in [3], functional dynamic equations represent another special case of measure functional differential equations.

Let $\mathbb{T}$ be a time scale, i.e., a nonempty closed subset of $\mathbb{R}$. Given $a,b\in\mathbb{T}$, $a<b$, let $[a,b]_{\mathbb{T}}=[a,b]\cap\mathbb{T}$. Using the notation from [15], let $t^*=\inf\{s\in\mathbb{T};\ s\ge t\}$ for every $t\in(-\infty,b]$.

Recall that in the time scale calculus, the usual derivative f' is replaced by the Δ-derivative f^Δ (see [2] for the basic definitions). Therefore, we consider the linear functional dynamic equation

$$y^\Delta(t)=\ell(y_t^*,t)+q(t),\quad t\in[a,b]_{\mathbb{T}}, \tag{8.1}$$

where $\ell : H_0 \times [a,b] \to \mathbb{R}^n$ is linear in the first variable and $q : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$. Here, the symbol y_t^* should be interpreted as $(y^*)_t$; as explained in [3], the reason for working with y_t^* instead of just y_t is that the first argument of ℓ has to be a function defined on the whole interval $(-\infty, 0]$.

For example, a linear delay dynamic equation of the form

$$y^{\Delta}(t) = \sum_{i=1}^{k} p_i(t)\, y(\tau_i(t)) + q(t),$$

where $\tau_i : \mathbb{T} \to \mathbb{T}$, $p_i : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ and $\tau_i(t) \le t$ for every $t \in [a,b]_{\mathbb{T}}$, $i \in \{1, \dots, k\}$, is a special case of Eq. (8.1) corresponding to the choice

$$\ell(y,t) = \sum_{i=1}^{k} p_i(t)\, y(\tau_i(t) - t).$$

Now, under certain assumptions on ℓ and q, the linear functional dynamic equation (8.1) is equivalent to the measure functional differential equation

$$z(t) = z(a) + \int_a^t \ell(z_s, s^*)\, \mathrm{d}g(s) + \int_a^t q(s^*)\, \mathrm{d}g(s), \quad t \in [a,b],$$

where $g(s) = s^*$, $s \in [a,b]$. More precisely, a function $z : [a,b] \to \mathbb{R}^n$ is a solution of the last equation if and only if $z(t) = y(t^*)$, $t \in [a,b]$, where $y : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of (8.1).

Therefore, the existence-uniqueness theorem as well as the continuous dependence theorem for linear functional dynamic equations of the form (8.1) are simple consequences of our results for measure functional differential equations. Moreover, the same approach also works for functional dynamic equations with impulses (see [4]). Since the whole procedure would be very similar to the one described in Section 7, we omit the details.

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