Measure functional differential equations with infinite delay

Antonín Slavík
Charles University, Faculty of Mathematics and Physics,
Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail: slavik@karlin.mff.cuni.cz

Abstract

We introduce measure functional differential equations with infinite delay and an axiomatically described phase space. We show how to transform these equations into generalized ordinary differential equations whose solutions take values in a suitable infinite-dimensional Banach space. Even in the special case of functional equations with finite delay, our result improves the existing one by imposing weaker conditions on the right-hand side.

Keywords: Measure functional differential equations, generalized ordinary differential equations, differential equations in Banach spaces, regulated functions, Kurzweil-Stieltjes integral, phase space

MSC 2010 subject classification: 34K05, 34G20, 34K45, 34N05

1 Introduction

Measure functional differential equations with finite delay have the form

$$y(t)=y(t_0)+\int_{t_0}^{t} f(y_s,s)\,\mathrm{d}g(s), \tag{1.1}$$

where y is an unknown function with values in $\mathbb{R}^n$ and the symbol y_s denotes the function $y_s(\tau)=y(s+\tau)$ defined on $[-r,0]$, $r\geq 0$ being a fixed number corresponding to the length of the delay. The integral on the right-hand side of (1.1) is the Kurzweil-Stieltjes integral with respect to a nondecreasing function g (see the definition in Section 3; this integral includes the Lebesgue-Stieltjes integral with respect to the measure generated by g).

Measure functional differential equations have been introduced in the paper [3]. In the special case $g(s)=s$, equation (1.1) reduces to the classical functional differential equation

$$y(t)=y(t_0)+\int_{t_0}^{t} f(y_s,s)\,\mathrm{d}s, \tag{1.2}$$

which has been studied by many authors (see e.g. [9]). On the other hand, the general form (1.1) includes other familiar types of equations such as functional differential equations with impulses or functional dynamic equations on time scales (see [3], [4]). For example, consider the impulsive functional differential equation

$$\left.\begin{array}{rcll} y'(t) &=& f(y_t,t), & t\in[t_0,\infty)\backslash\{t_1,t_2,\ldots\},\\ \Delta^+ y(t_i) &=& I_i(y(t_i)), & i\in\mathbb{N}, \end{array}\right\} \tag{1.3}$$

where the impulses take place at preassigned times $t_1,t_2,\ldots\in[t_0,\infty)$, and their action is described by the operators $I_i:\mathbb{R}^n\to\mathbb{R}^n$, $i\in\mathbb{N}$; the solution is assumed to be left-continuous at every point t_i. The corresponding integral form is

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s)\,\mathrm{d}s + \sum_{i\in\mathbb{N},\, t_i<t} I_i(y(t_i)), \quad t \in [t_0, \infty),$$

which is equivalent (see [4, Lemma 2.4]) to the measure functional differential equation

$$y(t) = y(t_0) + \int_{t_0}^{t} \tilde{f}(y_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, \infty),$$

where $g(s) = s + \sum_{i=1}^{\infty} \chi_{(t_i,\infty)}(s)$ and

$$\tilde{f}(x, s) = \begin{cases} f(x, s), & s \in [t_0, \infty)\backslash\{t_1, t_2, \ldots\}, \\ I_i(x(0)), & s = t_i, \text{ for } i \in \mathbb{N}. \end{cases}$$

The aim of this paper is to discuss measure functional differential equations with infinite delay, i.e., equations of the form (1.1), where y_s now denotes the function $y_s(\tau) = y(s + \tau)$ defined on $(-\infty, 0]$. The case $g(s) = s$ corresponds to classical functional differential equations with infinite delay, which have been studied by numerous authors (see e.g. [2], [8], [10] and the references in these works). The general case when g is a nondecreasing function includes certain other types of functional equations, such as the impulsive functional differential equation (1.3) with infinite delay. One particular example is the impulsive Volterra integro-differential equation

$$\begin{aligned} y'(t) &= \int_0^t a(y(s), s)\,\mathrm{d}s, \quad t \in [0, \infty)\backslash\{t_1, t_2, \ldots\}, \\ \Delta^+ y(t_i) &= I_i(y(t_i)), \quad i \in \mathbb{N}, \end{aligned}$$

which has the form (1.3) with

$$f(x, t) = \int_{-t}^{0} a(x(\tau), t + \tau)\,\mathrm{d}\tau$$

for a function $x : (-\infty, 0] \to \mathbb{R}^n$.

When dealing with infinite delay, the crucial problem is the choice of the phase space, i.e., the domain of the first argument of f. In the classical case (1.2), the elements of this phase space are continuous functions. Such a phase space is no longer suitable for a general measure functional differential equation, whose solutions are discontinuous functions. The problem of the choice of phase space is discussed in Section 2. We do not restrict ourselves to a particular phase space; instead, we introduce a certain system of conditions and allow the phase space to be any space satisfying these conditions. A similar axiomatic approach was used by various authors (see e.g. [2], [8], [10] and the references there) to describe the phase space of classical or impulsive functional differential equations with infinite delay.

In Section 3, we show that under certain natural assumptions, a measure functional differential equation can be transformed to a generalized ordinary differential equation whose solutions take values in an infinite-dimensional Banach space. Consequently, one can use the existing theory of generalized ordinary differential equations (see e.g. [13], [15]) to obtain new results for measure functional differential equations with infinite delay. The idea of transforming a classical functional differential equation to a generalized ordinary differential equation first appeared in the papers [11], [14] by C. Imaz, F. Oliva, and Z. Vorel. Later, it was extended to impulsive functional differential equations in the paper [5] by M. Federson and Š. Schwabik, and to measure functional differential equations with finite delay in the paper [3] by M. Federson, J. G. Mesquita and A. Slavík. In [1], [6] and [7], the correspondence between functional differential equations and generalized ordinary equations was used to obtain various results on boundedness and stability of solutions.

2 Phase space description

In general, solutions of measure functional differential equations are not continuous, but merely regulated functions; recall that a function $f : [a, b] \to \mathbb{R}^n$ is called regulated, if the limits

$$\lim_{s \to t-} f(s) = f(t-) \in \mathbb{R}^n, \quad t \in (a, b] \quad \text{and} \quad \lim_{s \to t+} f(s) = f(t+) \in \mathbb{R}^n, \quad t \in [a, b)$$

exist. Regulated functions on open or half-open intervals are defined in a similar way. Given an interval $I \subset \mathbb{R}$ and a set $B \subset \mathbb{R}^n$, we use the symbol $G(I, B)$ to denote the set of all regulated functions $f : I \to B$, and the symbol $C(I, B)$ to denote the set of all continuous functions $f : I \to B$.

Our candidate for the phase space of a measure functional differential equation with infinite delay is a linear space $H_0 \subset G((-\infty, 0], \mathbb{R}^n)$ equipped with a norm denoted by $\| \cdot \|_\star$. We assume that this normed linear space H_0 satisfies the following conditions:

(H1) H_0 is complete.

(H2) If $y \in H_0$ and $t < 0$, then $y_t \in H_0$.

(H3) There exists a locally bounded function $\kappa_1 : (-\infty, 0] \to \mathbb{R}^+$ such that if $y \in H_0$ and $t \le 0$, then $\|y(t)\| \le \kappa_1(t)\|y\|_\star$.

(H4) There exists a function $\kappa_2 : (0, \infty) \to [1, \infty)$ such that if $\sigma > 0$ and $y \in H_0$ is a function whose support is contained in $[-\sigma, 0]$, then

$$\|y\|_\star \le \kappa_2(\sigma) \sup_{t \in [-\sigma, 0]} \|y(t)\|.$$

(H5) There exists a locally bounded function $\kappa_3 : (-\infty, 0] \to \mathbb{R}^+$ such that if $y \in H_0$ and $t \le 0$, then

$$\|y_t\|_\star \le \kappa_3(t)\|y\|_\star.$$

(H6) If $y \in H_0$, then the function $t \mapsto \|y_t\|_\star$ is regulated on $(-\infty, 0]$.

To establish the correspondence between measure functional differential equations and generalized ordinary differential equations, we also need a suitable space H_a of regulated functions defined on $(-\infty, a]$, where $a \in \mathbb{R}$. We obtain this space by shifting the functions from H_0; more formally, for every $\tau \in \mathbb{R}$, let S_τ be the shift operator defined as follows: if $y : I \to \mathbb{R}^n$ is an arbitrary function, let $I + \tau = \{t + \tau, t \in I\}$ and define $S_\tau y : I + \tau \to \mathbb{R}^n$ by $(S_\tau y)(t) = y(t - \tau)$. Now, for every $a \in \mathbb{R}$, let $H_a = \{S_a y,\ y \in H_0\}$. Finally, define a norm $\| \cdot \|_\star$ on H_a by letting $\|y\|_\star = \|S_{-a} y\|_\star$ for every $y \in H_a$.

Note that if $y \in H_a$, then $y = S_a z$ for a certain $z \in H_0$, and $y_t = (S_a z)_t = z_{t-a} \in H_0$ for every $t \le a$.

The next lemma shows that the spaces H_a inherit all important properties of H_0; the statements follow immediately from the corresponding definitions.

Lemma 2.1. *If $H_0 \subset G((-\infty, 0], \mathbb{R}^n)$ is a space satisfying conditions* (H1)–(H6)*, then the following statements are true for every $a \in \mathbb{R}$:*

1. *H_a is complete.*
2. *If $y \in H_a$ and $t \le a$, then $y_t \in H_0$.*
3. *If $t \le a$ and $y \in H_a$, then $\|y(t)\| \le \kappa_1(t - a)\|y\|_\star$.*
4. *If $\sigma > 0$ and $y \in H_{a+\sigma}$ is a function whose support is contained in $[a, a + \sigma]$, then*

$$\|y\|_\star \le \kappa_2(\sigma) \sup_{t \in [a, a+\sigma]} \|y(t)\|.$$

5. *If $y \in H_{a+\sigma}$ and $t \le a+\sigma$, then $\|y_t\|_\star \le \kappa_3(t-a-\sigma)\|y\|_\star$.*

6. *If $y \in H_{a+\sigma}$, then the function $t \mapsto \|y_t\|_\star$ is regulated on $(-\infty, a+\sigma]$.*

In the rest of this section, we present examples of spaces satisfying conditions (H1)–(H6), which might be used as the phase space H_0 for measure functional differential equations with infinite delay. These spaces are modified versions of examples given in [10], where we have replaced continuous functions by regulated functions; other examples presented in [10] can be modified in a similar way.

Example 2.2. Probably the simplest example of a normed linear space H_0 satisfying conditions (H1)–(H6) is the space $BG((-\infty,0],\mathbb{R}^n)$, which consists of all bounded regulated functions on $(-\infty,0]$ and is endowed with the supremum norm

$$\|y\|_\infty = \sup_{t\in(-\infty,0]} \|y(t)\|, \quad y \in BG((-\infty,0],\mathbb{R}^n).$$

It is straightforward to check that conditions (H1)–(H5) are satisfied; in particular, (H3)–(H5) are true with $\kappa_1(t) = \kappa_3(t) = 1$ for every $t \le 0$ and $\kappa_2(\sigma) = 1$ for every $\sigma > 0$. Finally, condition (H6) is a consequence of the following Lemma 2.3.

Note also that $H_a = BG((-\infty,a],\mathbb{R}^n)$, i.e., the space of all bounded regulated functions on $(-\infty,a]$ with the supremum norm.

Lemma 2.3. *If $y : (-\infty,0] \to \mathbb{R}^n$ is a regulated function, then $t \mapsto \|y_t\|_\infty$ is regulated on $(-\infty,0]$.*

Proof. Let us show that $\lim_{t\to t_0-}\|y_t\|_\infty$ exists for every $t_0 \in (-\infty,0]$. The function y is regulated, and therefore satisfies the Cauchy condition at t_0, i.e., given an arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|y(u) - y(v)\| < \varepsilon, \quad u, v \in (t_0-\delta, t_0). \tag{2.1}$$

Consider a pair of numbers t_1, t_2 such that $t_0 - \delta < t_1 \le t_2 < t_0$. Obviously,

$$\|y_{t_1}\|_\infty \le \|y_{t_2}\|_\infty < \|y_{t_2}\|_\infty + \varepsilon.$$

For every $t \in [t_1,t_2]$, it follows from (2.1) that

$$\|y(t)\| \le \|y(t_1)\| + \|y(t_2) - y(t_1)\| < \|y_{t_1}\|_\infty + \varepsilon.$$

Since $\|y(t)\| \le \|y_{t_1}\|_\infty < \|y_{t_1}\|_\infty + \varepsilon$ for every $t \in (-\infty,t_1]$, we conclude that

$$\|y_{t_2}\|_\infty \le \|y_{t_1}\|_\infty + \varepsilon,$$

and consequently

$$\big|\|y_{t_1}\|_\infty - \|y_{t_2}\|_\infty\big| < \varepsilon, \ \ t_1, t_2 \in (t_0-\delta, t_0),$$

i.e. the Cauchy condition for the existence of $\lim_{t\to t_0-}\|y_t\|_\infty$ is satisfied. The existence of $\lim_{t\to t_0+}\|y_t\|_\infty$ for $t_0 \in (-\infty,0)$ can be proved similarly. □

Remark 2.4. In connection with the previous lemma, note that if $y : (-\infty,0] \to \mathbb{R}^n$ is a regulated function, then $t \mapsto y_t$ need not be regulated on $(-\infty,0]$. For example, let $y(t) = \sin t^2$, $t \in (-\infty,0]$. Then for every $h > 0$, we have $\limsup_{t\to-\infty}|y(t+h) - y(t)| = 2$, and therefore $\|y_{t+h} - y_t\|_\infty = 2$ for arbitrarily small values of h.

The next example is a phase space that can be used when dealing with unbounded functions; it consists of functions that do not grow faster than a certain exponential function as $t \to -\infty$.

Example 2.5. For an arbitrary $\gamma \geq 0$, let $G_\gamma((-\infty,0],\mathbb{R}^n)$ be the space of all regulated functions $y:(-\infty,0]\to\mathbb{R}^n$ such that $\sup_{t\in(-\infty,0]}\|e^{\gamma t}y(t)\|$ is finite. This space is endowed with the norm

$$\|y\|_\gamma = \sup_{t\in(-\infty,0]}\|e^{\gamma t}y(t)\|, \quad y\in G_\gamma((-\infty,0],\mathbb{R}^n).$$

The operator $T: G_\gamma((-\infty,0],\mathbb{R}^n)\to BG((-\infty,0],\mathbb{R}^n)$ defined by

$$(Ty)(t) = e^{\gamma t}y(t)$$

is an isometric isomorphism between $G_\gamma((-\infty,0],\mathbb{R}^n)$ and $BG((-\infty,0],\mathbb{R}^n)$, and thus $G_\gamma((-\infty,0],\mathbb{R}^n)$ is a complete space.

Again, it is not difficult to check that conditions (H1)–(H5) are satisfied with $\kappa_1(t)=\kappa_3(t)=e^{-\gamma t}$ and $\kappa_2(\sigma)=1$. To verify condition (H6), let $y\in G_\gamma((-\infty,0],\mathbb{R}^n)$ and note that

$$\|y_t\|_\gamma = \sup_{s\in(-\infty,0]}\|e^{\gamma s}y(s+t)\| = e^{-\gamma t}\sup_{s\in(-\infty,0]}\|e^{\gamma(s+t)}y(s+t)\| = e^{-\gamma t}\|z_t\|_\infty,$$

where $z(s)=e^{\gamma s}y(s)$ for every $s\leq 0$. By Lemma 2.3, the function $t\mapsto\|z_t\|_\infty$ is regulated, and thus $t\mapsto e^{-\gamma t}\|z_t\|_\infty$ is also regulated.

Remark 2.6. Consider the measure functional differential equation

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s,s)\,\mathrm{d}g(s),$$

where $g(s)=s$. In this special case, the solutions are continuous and it might be more appropriate to choose a space H_0 that contains continuous functions only. In particular, one can modify the previous two examples to obtain the space $BC((-\infty,0],\mathbb{R}^n)$ consisting of all bounded continuous functions, or the space $C_\gamma((-\infty,0],\mathbb{R}^n)$ consisting of all continuous functions $y:(-\infty,0]\to\mathbb{R}^n$ such that $\lim_{t\to-\infty}e^{\gamma t}y(t)$ exists and is finite. More information about phase spaces consisting of continuous functions can be found in [10].

3 Transformation to generalized ordinary differential equations

In this section, we establish the correspondence between measure functional differential equations with infinite delay and generalized ordinary differential equations. The notion of a generalized ordinary differential equation was introduced by J. Kurzweil in [12], and is closely related to the Kurzweil integral. We summarize the basic definitions and notation, and refer the reader to [15] or [13] for more information.

A function $\delta:[a,b]\to\mathbb{R}^+$ is called a gauge on $[a,b]$. A tagged partition of the interval $[a,b]$ with division points $a=s_0\leq s_1\leq\cdots\leq s_k=b$ and tags $\tau_i\in[s_{i-1},s_i]$, $i\in\{1,\ldots,k\}$, is called δ-fine if

$$[s_{i-1},s_i]\subset(\tau_i-\delta(\tau_i),\tau_i+\delta(\tau_i)), \ \ i\in\{1,\ldots,k\}.$$

Let X be a Banach space. A function $U:[a,b]\times[a,b]\to X$ is called Kurzweil integrable on $[a,b]$, if there is an element $I\in X$ such that for every $\varepsilon>0$, there is a gauge δ on $[a,b]$ such that

$$\left\|\sum_{i=1}^{k}(U(\tau_i,s_i)-U(\tau_i,s_{i-1}))-I\right\|<\varepsilon$$

for every δ-fine tagged partition of $[a,b]$. In this case, we define $\int_a^b DU(\tau,t)=I$.

An important special case is the Kurzweil-Stieltjes integral of a function $f : [a, b] \to X$ with respect to a function $g : [a, b] \to \mathbb{R}$, which appears in the definition of a measure functional differential equation, and is obtained by letting $U(\tau, t) = f(\tau)g(t)$. This integral will be denoted by $\int_a^b f(t)\,\mathrm{d}g(t)$ or simply $\int_a^b f\,\mathrm{d}g$.

Consider a set $O \subset X$, an interval $[a, b] \subset \mathbb{R}$ and a function $F : O \times [a, b] \to X$. A function $x : [a, b] \to O$ is called a solution of the generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t)$$

on the interval $[a, b]$, if

$$x(d) - x(c) = \int_c^d DF(x(\tau), t)$$

whenever $[c, d] \subseteq [a, b]$.

The following conditions on the right-hand side F occur frequently in the the theory of generalized ordinary differential equations:

(F1) There exists a nondecreasing function $h : [a, b] \to \mathbb{R}$ such that $F : O \times [a, b] \to X$ satisfies

$$\|F(x, s_2) - F(x, s_1)\| \le |h(s_2) - h(s_1)|$$

for every $x \in O$ and $s_1, s_2 \in [a, b]$.

(F2) There exists a nondecreasing function $k : [a, b] \to \mathbb{R}$ such that $F : O \times [a, b] \to X$ satisfies

$$\|F(x, s_2) - F(x, s_1) - F(y, s_2) + F(y, s_1)\| \le |k(s_2) - k(s_1)| \cdot \|x - y\|$$

for every $x, y \in O$ and $s_1, s_2 \in [a, b]$.

According to the following lemma, solutions of a generalized ordinary differential equation whose right-hand side satisfies condition (F1) are regulated functions (the proof follows directly from the definition of the Kurzweil integral and can be found in [15], Lemma 3.12).

Lemma 3.1. *Let X be a Banach space. Consider a set $O \subset X$, an interval $[a, b] \subset \mathbb{R}$ and a function $F : O \times [a, b] \to X$, which satisfies condition* (F1). *If $x : [a, b] \to O$ is a solution of the generalized ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t),$$

then

$$\|x(s_2) - x(s_1)\| \le h(t_2) - h(t_1)$$

for each pair $s_1, s_2 \in [a, b]$, and x is a regulated function.

Let $H_0 \subset G((-\infty, 0], \mathbb{R}^n)$ be a Banach space satisfying conditions (H1)–(H6), $t_0 \in \mathbb{R}$, $\sigma > 0$, $O \subset H_{t_0+\sigma}$ and $P = \{y_t;\ y \in O,\ t \in [t_0, t_0 + \sigma]\} \subset H_0$. Consider a nondecreasing function $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ and a function $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$. We will show that under certain assumptions, a measure functional differential equation of the form

$$y(t) = y(t_0) + \int_{t_0}^t f(y_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma] \tag{3.1}$$

is equivalent to a generalized ordinary differential equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t), \quad t \in [t_0, t_0 + \sigma], \tag{3.2}$$

where x takes values in O, and $F : O \times [t_0, t_0+\sigma] \to G((-\infty, t_0+\sigma], \mathbb{R}^n)$ is given by

$$F(x,t)(\vartheta) = \begin{cases} 0, & -\infty < \vartheta \le t_0, \\ \int_{t_0}^{\vartheta} f(x_s, s)\,\mathrm{d}g(s), & t_0 \le \vartheta \le t \le t_0+\sigma, \\ \int_{t_0}^{t} f(x_s, s)\,\mathrm{d}g(s), & t \le \vartheta \le t_0+\sigma \end{cases} \tag{3.3}$$

for every $x \in O$ and $t \in [t_0, t_0+\sigma]$. It will turn out that the relation between the solution x of (3.2) and the solution y of (3.1) is described by

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & \vartheta \in (-\infty, t], \\ y(t), & \vartheta \in [t, t_0+\sigma], \end{cases}$$

where $t \in [t_0, t_0+\sigma]$.

Definition 3.2. Let $I \subset \mathbb{R}$ be an interval, $t_0 \in I$, and O a set whose elements are functions $f : I \to \mathbb{R}^n$. We say that O has the prolongation property for $t \ge t_0$, if for every $y \in O$ and every $t \in I \cap [t_0, \infty)$, the function $\bar{y} : I \to \mathbb{R}^n$ given by

$$\bar{y}(s) = \begin{cases} y(s), & s \in (-\infty, t] \cap I, \\ y(t), & s \in [t, \infty) \cap I \end{cases}$$

is an element of O.

For example, consider an arbitrary set $B \subset \mathbb{R}^n$, an interval $I \subset \mathbb{R}$, and $t_0 \in I$. Then both the set $G(I, B)$ of all regulated functions $f : I \to B$ and the set $C(I, B)$ of all continuous functions $f : I \to B$ have the prolongation property for $t \ge t_0$.

The following theorem, which is a special case of Theorem 1.16 in [15], confirms that the values of F defined by (3.3) are indeed regulated functions on $(-\infty, t_0+\sigma]$. We use the symbol $\Delta^+ y(t)$ to denote $y(t+) - y(t)$.

Theorem 3.3. *Let $f : [a,b] \to \mathbb{R}^n$ and $g : [a,b] \to \mathbb{R}$ be a pair of functions such that g is regulated and $\int_a^b f\,\mathrm{d}g$ exists. Then the function*

$$u(t) = \int_a^t f(s)\,\mathrm{d}g(s), \quad t \in [a,b],$$

is regulated and satisfies $\Delta^+ u(t) = f(t)\Delta^+ g(t)$ for every $t \in [a,b)$.

To justify the relation between measure functional differential equations and generalized ordinary differential equations, we assume that the following three conditions are satisfied:

(A) The integral $\int_{t_0}^{t_0+\sigma} f(y_t, t)\,\mathrm{d}g(t)$ exists for every $y \in O$.

(B) There exists a function $M : [t_0, t_0+\sigma] \to \mathbb{R}^+$, which is Kurzweil-Stieltjes integrable with respect to g, such that

$$\left\| \int_a^b f(y_t, t)\,\mathrm{d}g(t) \right\| \le \int_a^b M(t)\,\mathrm{d}g(t)$$

whenever $y \in O$ and $[a,b] \subseteq [t_0, t_0+\sigma]$.

(C) There exists a function $L : [t_0, t_0 + \sigma] \to \mathbb{R}^+$, which is Kurzweil-Stieltjes integrable with respect to g, such that

$$\left\| \int_a^b (f(y_t, t) - f(z_t, t)) \, \mathrm{d}g(t) \right\| \leq \int_a^b L(t) \|y_t - z_t\|_\bigstar \, \mathrm{d}g(t)$$

whenever $y, z \in O$ and $[a, b] \subseteq [t_0, t_0 + \sigma]$ (we are assuming that the integral on the right-hand side exists).

Let us remark that the assumptions (B) and (C) are weaker than similar conditions used in [3] and [4]. In these papers, it was assumed that M and L are constants. In the special case $g(t) = t$, our conditions coincide with those given in [5].

The next lemma demonstrates the relation between conditions (A), (B), (C) and (F1), (F2). We point out that (F2) is not really needed to establish the relation between the two types of equations. However, we include it here for reader's convenience since it is often needed in proofs of various facts concerning generalized ordinary differential equations.

Lemma 3.4. *Let $O \subset H_{t_0+\sigma}$ and $P = \{y_t;\ y \in O,\ t \in [t_0, t_0 + \sigma]\}$. Assume that $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ is a nondecreasing function, $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies condition* (A)*, and $F : O \times [t_0, t_0 + \sigma] \to G((-\infty, t_0 + \sigma], \mathbb{R}^n)$ given by* (3.3) *has values in $H_{t_0+\sigma}$. Then the following statements are true:*

- *If f satisfies condition* (B)*, then F satisfies condition* (F1) *with*

$$h(t) = \kappa_2(\sigma) \int_{t_0}^t M(s) \, \mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma].$$

- *If f satisfies condition* (C)*, then F satisfies condition* (F2) *with*

$$k(t) = \kappa_2(\sigma) \left(\sup_{s \in [-\sigma, 0]} \kappa_3(s) \right) \int_{t_0}^t L(s) \, \mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma].$$

Proof. By condition (A), the integrals in the definition of F are guaranteed to exist. Assume that $t_0 \leq s_1 < s_2 \leq t_0 + \sigma$. We have

$$F(y, s_2)(\tau) - F(y, s_1)(\tau) = \begin{cases} 0, & -\infty < \tau \leq s_1, \\ \int_{s_1}^{\tau} f(y_s, s) \, \mathrm{d}g(s), & s_1 \leq \tau \leq s_2, \\ \int_{s_1}^{s_2} f(y_s, s) \, \mathrm{d}g(s), & s_2 \leq \tau \leq t_0 + \sigma \end{cases}$$

for every $y \in O$. Condition (B) and the fourth statement of Lemma 2.1 imply

$$\|F(y, s_2) - F(y, s_1)\|_\bigstar \leq \kappa_2(\sigma) \sup_{\tau \in [t_0, t_0+\sigma]} \|F(y, s_2)(\tau) - F(y, s_1)(\tau)\| =$$

$$= \kappa_2(\sigma) \sup_{\tau \in [s_1, s_2]} \left\| \int_{s_1}^{\tau} f(y_s, s) \, \mathrm{d}g(s) \right\| \leq \kappa_2(\sigma) \int_{s_1}^{s_2} M(s) \, \mathrm{d}g(s) = h(s_2) - h(s_1).$$

Similarly, condition (C) and the fourth statement of Lemma 2.1 imply that for every $y, z \in O$, we have

$$\|F(y, s_2) - F(y, s_1) - F(z, s_2) + F(z, s_1)\|_\bigstar$$

$$\leq \kappa_2(\sigma) \sup_{\tau \in [t_0, t_0+\sigma]} \|F(y, s_2)(\tau) - F(y, s_1)(\tau) - F(z, s_2)(\tau) + F(z, s_1)(\tau)\| =$$

$$= \kappa_2(\sigma) \sup_{\tau\in[s_1,s_2]} \left\| \int_{s_1}^{\tau} (f(y_s,s) - f(z_s,s))\,\mathrm{d}g(s) \right\| \le \kappa_2(\sigma) \int_{s_1}^{s_2} L(s) \|y_s - z_s\|_\bigstar \,\mathrm{d}g(s).$$

By the fifth statement of Lemma 2.1, the last expression is smaller than or equal to

$$\kappa_2(\sigma) \left(\sup_{s\in[t_0,t_0+\sigma]} \kappa_3(s - t_0 - \sigma) \right) \|y - z\|_\bigstar \left(\int_{s_1}^{s_2} L(s)\,\mathrm{d}g(s) \right) = (k(s_2) - k(s_1)) \cdot \|y - z\|_\bigstar. \qquad \square$$

The following statement is a slightly modified version of Lemma 3.7 from [3] (the original proof for equations with finite delay and $g(t) = t$ can be found in [5]).

Lemma 3.5. *Assume that O is a subset of $H_{t_0+\sigma}$, $P = \{y_t;\ y \in O,\ t \in [t_0, t_0 + \sigma]\}$, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ is a nondecreasing function, $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies condition* (A)*, and $F : O \times [t_0, t_0 + \sigma] \to G((-\infty, t_0 + \sigma], \mathbb{R}^n)$ given by* (3.3) *has values in $H_{t_0+\sigma}$. If $x : [t_0, t_0 + \sigma] \to O$ is a solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t)$$

and $x(t_0) \in H_{t_0+\sigma}$ is a function which is constant on $[t_0, t_0 + \sigma]$, then

$$x(v)(\vartheta) = x(v)(v), \quad t_0 \le v \le \vartheta \le t_0 + \sigma, \tag{3.4}$$

$$x(v)(\vartheta) = x(\vartheta)(\vartheta), \quad t_0 \le \vartheta \le v \le t_0 + \sigma. \tag{3.5}$$

Proof. Consider the case when $t_0 \le v \le \vartheta$. Since x is a solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t),$$

we have

$$x(v)(v) = x(t_0)(v) + \left(\int_{t_0}^{v} DF(x(\tau), t) \right)(v),$$

$$x(v)(\vartheta) = x(t_0)(\vartheta) + \left(\int_{t_0}^{v} DF(x(\tau), t) \right)(\vartheta).$$

Using the fact that $x(t_0)(\vartheta) = x(t_0)(v)$, we obtain

$$x(v)(\vartheta) - x(v)(v) = \left(\int_{t_0}^{v} DF(x(\tau), t) \right)(\vartheta) - \left(\int_{t_0}^{v} DF(x(\tau), t) \right)(v).$$

It follows from the existence of the integral $\int_{t_0}^{v} DF(x(\tau), t)$ that for every $\varepsilon > 0$, there is a tagged partition $\{(\tau_i, [s_{i-1}, s_i]),\ i = 1, \dots, k\}$ of $[t_0, v]$ such that

$$\left\| \sum_{i=1}^{k} (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})) - \int_{t_0}^{v} DF(x(\tau), t) \right\|_\bigstar < \varepsilon.$$

By the definition of F in (3.3), we have

$$F(x(\tau_i), s_i)(\vartheta) - F(x(\tau_i), s_{i-1})(\vartheta) = F(x(\tau_i), s_i)(v) - F(x(\tau_i), s_{i-1})(v)$$

for every $i \in \{1, \dots, k\}$, and consequently

$$\|x(v)(\vartheta) - x(v)(v)\| \le \left\| \left(\int_{t_0}^{v} DF(x(\tau), t) \right)(\vartheta) - \sum_{i=1}^{k} (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}))(\vartheta) \right\|$$

$$+\left\|\sum_{i=1}^{k}(F(x(\tau_i),s_i)-F(x(\tau_i),s_{i-1}))(\vartheta)-\sum_{i=1}^{k}(F(x(\tau_i),s_i)-F(x(\tau_i),s_{i-1}))(v)\right\|$$

$$+\left\|\sum_{i=1}^{k}(F(x(\tau_i),s_i)-F(x(\tau_i),s_{i-1}))(v)-\left(\int_{t_0}^{v}DF(x(\tau),t)\right)(v)\right\|<(\kappa_1(\vartheta-t_0-\sigma)+\kappa_1(v-t_0-\sigma))\varepsilon$$

(we have used the third statement of Lemma 2.1). This proves (3.4), since $\varepsilon>0$ can be arbitrarily small.

For $\vartheta\le v$, we have

$$x(v)(\vartheta)=x(t_0)(\vartheta)+\left(\int_{t_0}^{v}DF(x(\tau),t)\right)(\vartheta),$$

$$x(\vartheta)(\vartheta)=x(t_0)(\vartheta)+\left(\int_{t_0}^{\vartheta}DF(x(\tau),t)\right)(\vartheta),$$

$$x(v)(\vartheta)-x(\vartheta)(\vartheta)=\left(\int_{\vartheta}^{v}DF(x(\tau),t)\right)(\vartheta).$$

If $\{(\tau_i,[s_{i-1},s_i]),\ i=1,\dots,k\}$ is an arbitrary tagged partition of $[\vartheta,v]$, it follows from (3.3) that

$$F(x(\tau_i),s_i)(\vartheta)-F(x(\tau_i),s_{i-1})(\vartheta)=0,\quad i\in\{1,\dots,k\}.$$

Consequently, $\int_{\vartheta}^{v}DF(x(\tau),t)(\vartheta)=0$, and (3.5) is proved. □

We are now ready to prove two theorems describing the relationship between measure functional differential equations whose solutions take values in $\mathbb{R}^n$, and generalized ordinary differential equations whose solutions take values in $H_{t_0+\sigma}$. The basic ideas of both proofs are similar to the proofs given in [3] for equations with finite delay. However, as we have already noted, our conditions (B) and (C) are weaker, and thus the results presented here are more general even in the case of finite delay.

Theorem 3.6. *Assume that O is a subset of $H_{t_0+\sigma}$ having the prolongation property for $t\ge t_0$, $P=\{x_t;\ x\in O,\ t\in[t_0,t_0+\sigma]\}$, $\phi\in P$, $g:[t_0,t_0+\sigma]\to\mathbb{R}$ is a nondecreasing function, $f:P\times[t_0,t_0+\sigma]\to\mathbb{R}^n$ satisfies conditions* (A), (B), (C), *and $F:O\times[t_0,t_0+\sigma]\to G((-\infty,t_0+\sigma],\mathbb{R}^n)$ given by* (3.3) *has values in $H_{t_0+\sigma}$. If $y\in O$ is a solution of the measure functional differential equation*

$$\begin{aligned}y(t)&=y(t_0)+\int_{t_0}^{t}f(y_s,s)\,\mathrm{d}g(s),\quad t\in[t_0,t_0+\sigma],\\ y_{t_0}&=\phi,\end{aligned}$$

then the function $x:[t_0,t_0+\sigma]\to O$ given by

$$x(t)(\vartheta)=\begin{cases}y(\vartheta), & \vartheta\in(-\infty,t],\\ y(t), & \vartheta\in[t,t_0+\sigma]\end{cases}$$

is a solution of the generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}=DF(x,t),\quad t\in[t_0,t_0+\sigma].$$

Proof. We have to show that, for every $v\in[t_0,t_0+\sigma]$, the integral $\int_{t_0}^{v}DF(x(\tau),t)$ exists and

$$x(v)-x(t_0)=\int_{t_0}^{v}DF(x(\tau),t).$$

Let an arbitrary $\varepsilon > 0$ be given. Since the function

$$h(t) = \kappa_2(\sigma) \int_{t_0}^{t} M(s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma]$$

is nondecreasing, it can have only a finite number of points $t \in [t_0, v]$ such that $\Delta^+ h(t) \geq \varepsilon$; denote these points by $t_1, \ldots, t_m$. Consider a gauge $\delta : [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\delta(\tau) < \min\left\{ \frac{t_k - t_{k-1}}{2},\ k = 2, \ldots, m \right\}, \ \ \tau \in [t_0, t_0+\sigma],$$

$$\delta(\tau) < \min\left\{ |\tau - t_k|;\ k = 1, \ldots, m \right\}, \ \ \tau \in [t_0, t_0+\sigma].$$

These conditions imply that if a point-interval pair $(\tau, [c,d])$ satisfies $[c,d] \subset (\tau - \delta(\tau), \tau + \delta(\tau))$, then $[c,d]$ contains at most one of the points $t_1, \ldots, t_m$, and, moreover, $\tau = t_k$ whenever $t_k \in [c,d]$.

Since $y_{t_k} = x(t_k)_{t_k}$, it follows from Theorem 3.3 that

$$\lim_{s \to t_k+} \int_{t_k}^{s} L(s) \|y_s - x(t_k)_s\|_\star \,\mathrm{d}g(s) = L(t_k) \|y_{t_k} - x(t_k)_{t_k}\|_\star \Delta^+ g(t_k) = 0$$

for every $k \in \{1, \ldots, m\}$. Thus, the gauge δ might be chosen in such a way that

$$\int_{t_k}^{t_k+\delta(t_k)} L(s) \|y_s - x(t_k)_s\|_\star \,\mathrm{d}g(s) < \frac{\varepsilon}{2m+1}, \ \ k \in \{1, \ldots, m\}.$$

By condition (B), we have

$$\|y(\tau+t) - y(\tau)\| = \left\| \int_{\tau}^{\tau+t} f(y_s, s)\,\mathrm{d}g(s) \right\| \leq \int_{\tau}^{\tau+t} M(s)\,\mathrm{d}g(s) \leq h(\tau+t) - h(\tau)$$

(recall that $\kappa_2(\sigma) \geq 1$), and therefore

$$\|y(\tau+) - y(\tau)\| \leq \Delta^+ h(\tau) < \varepsilon, \quad \tau \in [t_0, t_0+\sigma] \backslash \{t_1, \ldots, t_m\}.$$

Thus, we can assume that the gauge δ is such that

$$\|y(\rho) - y(\tau)\| \leq \varepsilon$$

for every $\tau \in [t_0, t_0+\sigma] \backslash \{t_1, \ldots, t_m\}$ and $\rho \in [\tau, \tau+\delta(\tau))$.

Let $\{(\tau_i, [s_{i-1}, s_i]),\ i = 1, \ldots, l\}$ be a δ-fine tagged partition of $[t_0, v]$. Then

$$\big(x(s_i) - x(s_{i-1})\big)(\vartheta) = \begin{cases} 0, & \vartheta \in (-\infty, s_{i-1}], \\ \int_{s_{i-1}}^{\vartheta} f(y_s, s)\,\mathrm{d}g(s), & \vartheta \in [s_{i-1}, s_i], \\ \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s), & \vartheta \in [s_i, t_0+\sigma] \end{cases}$$

and

$$\big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(\vartheta) = \begin{cases} 0, & \vartheta \in (-\infty, s_{i-1}], \\ \int_{s_{i-1}}^{\vartheta} f(x(\tau_i)_s, s)\,\mathrm{d}g(s), & \vartheta \in [s_{i-1}, s_i], \\ \int_{s_{i-1}}^{s_i} f(x(\tau_i)_s, s)\,\mathrm{d}g(s), & \vartheta \in [s_i, t_0+\sigma]. \end{cases}$$

for every $i \in \{1, \ldots, l\}$. By combination of the these equalities, we obtain

$$\big(x(s_i) - x(s_{i-1})\big)(\vartheta) - \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(\vartheta) =$$

$$= \begin{cases} 0, & \vartheta \in (-\infty, s_{i-1}], \\ \int_{s_{i-1}}^{\vartheta} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\,\mathrm{d}g(s), & \vartheta \in [s_{i-1}, s_i], \\ \int_{s_{i-1}}^{s_i} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\,\mathrm{d}g(s), & \vartheta \in [s_i, t_0+\sigma]. \end{cases}$$

Consequently,

$$\|x(s_i) - x(s_{i-1}) - \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)\|_\star$$

$$\le \kappa_2(\sigma) \sup_{\vartheta\in[t_0,t_0+\sigma]} \|\big(x(s_i) - x(s_{i-1})\big)(\vartheta) - \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(\vartheta)\| =$$

$$= \kappa_2(\sigma) \sup_{\vartheta\in[s_{i-1},s_i]} \left\| \int_{s_{i-1}}^{\vartheta} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\,\mathrm{d}g(s) \right\| \tag{3.6}$$

(we have used the fourth statement of Lemma 2.1). By the definition of x, we see that $x(\tau_i)_s = y_s$ whenever $s \le \tau_i$. Thus,

$$\int_{s_{i-1}}^{\vartheta} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\,\mathrm{d}g(s) = \begin{cases} 0, & \vartheta \in [s_{i-1}, \tau_i], \\ \int_{\tau_i}^{\vartheta} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\,\mathrm{d}g(s), & \vartheta \in [\tau_i, s_i]. \end{cases}$$

We now use condition (C) to obtain the estimate

$$\left\| \int_{\tau_i}^{\vartheta} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\,\mathrm{d}g(s) \right\| \le \int_{\tau_i}^{\vartheta} L(s)\|y_s - x(\tau_i)_s\|_\star\,\mathrm{d}g(s) \le \int_{\tau_i}^{s_i} L(s)\|y_s - x(\tau_i)_s\|_\star\,\mathrm{d}g(s).$$

Given a particular point-interval pair $(\tau_i, [s_{i-1}, s_i])$, there are two possibilities:

(i) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \dots, t_m\}$ contains a single point $t_k = \tau_i$.

(ii) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \dots, t_m\}$ is empty.

In case (i), it follows from the definition of the gauge δ that

$$\int_{\tau_i}^{s_i} L(s)\|y_s - x(\tau_i)_s\|_\star\,\mathrm{d}g(s) \le \frac{\varepsilon}{2m+1},$$

and substitution back to Eq. (3.6) leads to

$$\|x(s_i) - x(s_{i-1}) - \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)\|_\star \le \frac{\kappa_2(\sigma)\varepsilon}{2m+1}.$$

In case (ii), if $s \in [\tau_i, s_i]$, then

$$\|y_s - x(\tau_i)_s\|_\star \le \kappa_2(\sigma) \sup_{\rho\in[-\sigma+s,s]} \|y(\rho) - x(\tau_i)(\rho)\| = \kappa_2(\sigma) \sup_{\rho\in[\tau_i,s]} \|y(\rho) - y(\tau_i)\| \le \kappa_2(\sigma)\varepsilon,$$

(we have used the definition of the gauge δ). Thus,

$$\int_{\tau_i}^{s_i} L(s)\|y_s - x(\tau_i)_s\|_\star\,\mathrm{d}g(s) \le \kappa_2(\sigma)\varepsilon \int_{\tau_i}^{s_i} L(s)\,\mathrm{d}g(s),$$

and substitution back to Eq. (3.6) gives

$$\|x(s_i) - x(s_{i-1}) - \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)\|_\star \le \kappa_2(\sigma)^2\varepsilon \int_{\tau_i}^{s_i} L(s)\,\mathrm{d}g(s).$$

Combining cases (i) and (ii) and using the fact that case (i) occurs at most $2m$ times, we obtain

$$\left\| x(v) - x(t_0) - \sum_{i=1}^{l} \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) \big) \right\|_{\star} \leq \varepsilon\kappa_2(\sigma) \left(\kappa_2(\sigma) \int_{t_0}^{t_0+\sigma} L(s)\,\mathrm{d}g(s) + \frac{2m}{2m+1} \right)$$

$$< \varepsilon\kappa_2(\sigma) \left(\kappa_2(\sigma) \int_{t_0}^{t_0+\sigma} L(s)\,\mathrm{d}g(s) + 1 \right),$$

which completes the proof. □

Theorem 3.7. *Assume that O is a subset of $H_{t_0+\sigma}$ having the prolongation property for $t \geq t_0$, $P = \{x_t;\ x \in O,\ t \in [t_0, t_0+\sigma]\}$, $\phi \in P$, $g : [t_0, t_0+\sigma] \to \mathbb{R}$ is a nondecreasing function, $f : P \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ satisfies conditions* (A), (B), (C), *and $F : O \times [t_0, t_0+\sigma] \to G((-\infty, t_0+\sigma], \mathbb{R}^n)$ given by* (3.3) *has values in $H_{t_0+\sigma}$. If $x : [t_0, t_0+\sigma] \to O$ is a solution of the generalized ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t), \quad t \in [t_0, t_0+\sigma]$$

with the initial condition

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & \vartheta \in (-\infty, t_0], \\ \phi(0), & \vartheta \in [t_0, t_0+\sigma], \end{cases}$$

then the function $y \in O$ defined by

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & \vartheta \in (-\infty, t_0], \\ x(\vartheta)(\vartheta), & \vartheta \in [t_0, t_0+\sigma] \end{cases}$$

is a solution of the measure functional differential equation

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ y_{t_0} &= \phi. \end{aligned}$$

Proof. The equality $y_{t_0} = \phi$ follows directly from the definitions of y and $x(t_0)$. It remains to prove that if $v \in [t_0, t_0+\sigma]$, then

$$y(v) - y(t_0) = \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s).$$

Using Lemma 3.5, we obtain

$$y(v) - y(t_0) = x(v)(v) - x(t_0)(t_0) = x(v)(v) - x(t_0)(v) = \left(\int_{t_0}^{v} DF(x(\tau), t) \right)(v).$$

Thus,

$$y(v) - y(t_0) - \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s) = \left(\int_{t_0}^{v} DF(x(\tau), t) \right)(v) - \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s). \tag{3.7}$$

Let an arbitrary $\varepsilon > 0$ be given. Since the function

$$h(t) = \kappa_2(\sigma) \int_{t_0}^{t} M(s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma]$$

is nondecreasing, it can have only a finite number of points $t \in [t_0, v]$ such that $\Delta^+ h(t) \geq \varepsilon$; denote these points by $t_1, \ldots, t_m$. As in the proof of Theorem 3.6, consider a gauge $\delta : [t_0, t_0 + \sigma] \to \mathbb{R}^+$ such that

$$\delta(\tau) < \min \left\{ \frac{t_k - t_{k-1}}{2},\ k = 2, \ldots, m \right\}, \quad \tau \in [t_0, t_0 + \sigma],$$

$$\delta(\tau) < \min \{ |\tau - t_k|;\ k = 1, \ldots, m \}, \quad \tau \in [t_0, t_0 + \sigma],$$

$$\int_{t_k}^{t_k + \delta(t_k)} L(s) \|y_s - x(t_k)_s\|_\star \, \mathrm{d}g(s) < \frac{\varepsilon}{2m+1}, \quad k \in \{1, \ldots, m\}.$$

Finally, assume that the gauge δ satisfies

$$\|h(\rho) - h(\tau)\| \leq \varepsilon, \quad \tau \in [t_0, t_0 + \sigma] \backslash \{t_1, \ldots, t_m\}, \quad \rho \in [\tau, \tau + \delta(\tau)).$$

Let $\{(\tau_i, [s_{i-1}, s_i]), i = 1, \ldots, l\}$ be a δ-fine tagged partition of $[t_0, v]$ such that

$$\left\| \int_{t_0}^{v} DF(x(\tau), t) - \sum_{i=1}^{l} \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) \big) \right\|_\star < \varepsilon$$

(the existence of such a partition follows from the definition of the Kurzweil integral). Using (3.7) and the third statement of Lemma 2.1, we obtain

$$\begin{aligned} \left\| y(v) - y(t_0) - \int_{t_0}^{v} f(y_s, s) \, \mathrm{d}g(s) \right\| &= \left\| \left(\int_{t_0}^{v} DF(x(\tau), t) \right)(v) - \int_{t_0}^{v} f(y_s, s) \, \mathrm{d}g(s) \right\| \\ &< \varepsilon \kappa_1 (v - t_0 - \sigma) + \left\| \sum_{i=1}^{l} \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) \big)(v) - \int_{t_0}^{v} f(y_s, s) \, \mathrm{d}g(s) \right\| \\ &\leq \varepsilon \kappa_1 (v - t_0 - \sigma) + \sum_{i=1}^{l} \left\| \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) \big)(v) - \int_{s_{i-1}}^{s_i} f(y_s, s) \, \mathrm{d}g(s) \right\|. \end{aligned} \tag{3.8}$$

The definition of F yields

$$\big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) \big)(v) = \int_{s_{i-1}}^{s_i} f(x(\tau_i)_s, s) \, \mathrm{d}g(s),$$

which implies

$$\left\| \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) \big)(v) - \int_{s_{i-1}}^{s_i} f(y_s, s) \, \mathrm{d}g(s) \right\| = \left\| \int_{s_{i-1}}^{s_i} \left(f(x(\tau_i)_s, s) - f(y_s, s) \right) \, \mathrm{d}g(s) \right\|.$$

By Lemma 3.5, for every $i \in \{1, \ldots, l\}$, we have $x(\tau_i)_s = x(s)_s = y_s$ for $s \in [s_{i-1}, \tau_i]$ and $y_s = x(s)_s = x(s_i)_s$ for $s \in [\tau_i, s_i]$. Therefore

$$\begin{aligned} \left\| \int_{s_{i-1}}^{s_i} \left(f(x(\tau_i)_s, s) - f(y_s, s) \right) \, \mathrm{d}g(s) \right\| &= \left\| \int_{\tau_i}^{s_i} \left(f(x(\tau_i)_s, s) - f(y_s, s) \right) \, \mathrm{d}g(s) \right\| = \\ &= \left\| \int_{\tau_i}^{s_i} \left(f(x(\tau_i)_s, s) - f(x(s_i)_s, s) \right) \, \mathrm{d}g(s) \right\| \leq \int_{\tau_i}^{s_i} L(s) \|x(\tau_i)_s - x(s_i)_s\|_\star \, \mathrm{d}g(s), \end{aligned}$$

where the last inequality follows from condition (C). Again, we distinguish two cases:

(i) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \dots, t_m\}$ contains a single point $t_k = \tau_i$.

(ii) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \dots, t_m\}$ is empty.

In case (i), it follows from the definition of the gauge δ that

$$\int_{\tau_i}^{s_i} L(s)\|y_s - x(\tau_i)_s\|_\bigstar \,\mathrm{d}g(s) \le \frac{\varepsilon}{2m+1},$$

and thus

$$\left\|\big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(v) - \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s)\right\| \le \frac{\varepsilon}{2m+1}.$$

In case (ii), we make use of the fact that $x(s_i) - x(\tau_i)$ is zero on $(-\infty, t_0]$ (this follows from the definition of F). Lemmas 3.1, 3.4 and 2.1 lead to the estimate

$$\|x(s_i)_s - x(\tau_i)_s\|_\bigstar \le \kappa_2(\sigma) \sup_{\rho\in[s-\sigma,s]} \|x(s_i)(\rho) - x(\tau_i)(\rho)\|$$

$$\le \kappa_2(\sigma)\left(\sup_{\rho\in[s-\sigma,s]} \kappa_1(\rho - t_0 - \sigma)\right)\|x(s_i) - x(\tau_i)\|_\bigstar \le \kappa_2(\sigma)\left(\sup_{\rho\in[-2\sigma,0]} \kappa_1(\rho)\right)(h(s_i) - h(\tau_i)) \le K\varepsilon$$

for every $s \in [\tau_i, s_i]$, where $K = \kappa_2(\sigma)\sup_{\rho\in[-2\sigma,0]} \kappa_1(\rho)$. Thus,

$$\left\|\big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(v) - \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s)\right\| \le K\varepsilon \int_{\tau_i}^{s_i} L(s)\,\mathrm{d}g(s).$$

Combining cases (i) and (ii) and using the fact that case (i) occurs at most $2m$ times, we obtain

$$\sum_{i=1}^{l}\left\|\big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(v) - \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s)\right\|$$

$$\le K\varepsilon \int_{t_0}^{t_0+\sigma} L(s)\,\mathrm{d}g(s) + \frac{2m\varepsilon}{2m+1} < \varepsilon\left(K\int_{t_0}^{t_0+\sigma} L(s)\,\mathrm{d}g(s) + 1\right).$$

Substitution back into (3.8) gives

$$\left\|y(v) - y(t_0) - \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s)\right\| < \varepsilon\left(\kappa_1(v - t_0 - \sigma) + K\int_{t_0}^{t_0+\sigma} L(s)\,\mathrm{d}g(s) + 1\right),$$

which completes the proof. □

Remark 3.8. It follows from Lemma 3.5 that the definition

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & \vartheta \in (-\infty, t_0], \\ x(\vartheta)(\vartheta), & \vartheta \in [t_0, t_0+\sigma] \end{cases}$$

from the previous theorem can be replaced by the single equality

$$y(\vartheta) = x(t_0+\sigma)(\vartheta), \quad \vartheta \in (-\infty, t_0+\sigma].$$

This also shows that y is indeed an element of O.

Remark 3.9. The statements of both Theorem 3.6 and Theorem 3.7 require that $F(x,t) \in H_{t_0+\sigma}$ for every $t \in [t_0, t_0+\sigma]$ and $x \in O$. Although this condition might seem difficult to verify, it is often satisfied automatically. Note that $F(x,t)$ is a regulated function whose support is contained in $[t_0, t_0+\sigma]$. Thus, for $H_0 = BG((-\infty,0],\mathbb{R}^n)$ or $H_0 = G_\gamma((-\infty,0],\mathbb{R}^n)$ (or when H_0 is any other space containing all regulated functions with a compact support), we always have $F(x,t) \in H_{t_0+\sigma}$.

In the classical case when $g(t) = t$, the function F is continuous, and $F(x,t) \in H_{t_0+\sigma}$ is always satisfied for both $H_0 = BC((-\infty,0],\mathbb{R}^n)$ and $H_0 = C_\gamma((-\infty,0],\mathbb{R}^n)$.

Remark 3.10. Throughout the paper, we have been assuming that our phase space H_0 satisfies conditions (H1)–(H6). However, assumptions (H5) and (H6) were never used in the proofs of Theorems 3.6 and 3.7, i.e., the two theorems remain valid if we assume (H1)–(H4) only. Assumption (H6) can be quite useful when dealing with condition (C). For the existence of the integral on the right-hand side of this condition, it is sufficient to prove that the function $t \mapsto L(t)\|y_t - z_t\|_\star$ is regulated. For example, if L is a regulated function, assumption (H6) guarantees the existence of the integral. Assumption (H5) was used to prove the second part of Lemma 3.4, which will be needed in the proof of Theorem 3.12.

Remark 3.11. An immediate consequence of the relation between the two types of equations is the following: The initial value problem

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(y_s,s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma],\\ y_{t_0} &= \phi \end{aligned}$$

has a unique solution if and only if the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t), \quad t \in [t_0, t_0+\sigma],$$

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & \vartheta \in (-\infty, t_0],\\ \phi(0), & \vartheta \in [t_0, t_0+\sigma], \end{cases}$$

has a unique solution. Indeed, if we assume that the first problem has more than one solution, then Theorem 3.6 implies that the second problem has more than one solution. Conversely, assume that the second problem has two different solutions x_1, x_2, which do not coincide at a point $t \in (t_0, t_0+\sigma]$, i.e., $x_1(t)(\tau) \neq x_2(t)(\tau)$ for a certain $\tau \in (-\infty, t_0+\sigma]$. By the definition of F, this is possible only for $\tau > t_0$. Moreover, by Lemma 3.5, we can assume that $\tau \le t$ (otherwise, t can be used instead of τ). According to Theorem 3.7, we obtain a pair of solutions y_1, y_2 of the first initial value problem, and

$$y_1(\tau) = x_1(\tau)(\tau) = x_1(t)(\tau) \neq x_2(t)(\tau) = x_2(\tau)(\tau) = y_2(\tau),$$

i.e., the first problem does not have a unique solution.

With Theorems 3.6 and 3.7 at our disposal, we can use existing theorems on generalized differential equations to obtain new results for measure functional differential equations with infinite delay. As an example, we present a theorem on the local existence and uniqueness of solutions for measure equations.

Theorem 3.12. *Assume that O is an open subset of $H_{t_0+\sigma}$ having the prolongation property for $t \ge t_0$, $P = \{x_t;\ x \in O,\ t \in [t_0, t_0+\sigma]\}$, $g : [t_0, t_0+\sigma] \to \mathbb{R}$ is a left-continuous nondecreasing function, $f : P \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ satisfies conditions* (A), (B), (C), *and $F : O \times [t_0, t_0+\sigma] \to G((-\infty, t_0+\sigma],\mathbb{R}^n)$ given by* (3.3) *has values in $H_{t_0+\sigma}$. If $\phi \in P$ is such that the functions*

$$x_0(t) = \begin{cases} \phi(t - t_0), & t \in (-\infty, t_0],\\ \phi(0), & t \in [t_0, t_0+\sigma], \end{cases}$$

and

$$x_1(t) = \begin{cases} \phi(t-t_0), & t \in (-\infty, t_0], \\ \phi(0) + f(\phi, t_0)\Delta^+ g(t_0), & t \in (t_0, t_0+\sigma] \end{cases}$$

are elements of O, then there exists a $\delta > 0$ and a function $y : (-\infty, t_0+\delta] \to \mathbb{R}^n$ which is a unique solution of the initial value problem

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^t f(y_s, s)\,\mathrm{d}g(s), \\ y_{t_0} &= \phi \end{aligned}$$

on $(-\infty, t_0+\delta]$.

Proof. By Theorems 3.6 and 3.7, the initial value problem is equivalent to

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t), \quad x(t_0) = x_0,$$

where F is given by (3.3). By Lemma 3.4, F satisfies conditions (F1) and (F2). Thus, by the existence and uniqueness theorem for generalized ordinary differential equations (see e.g. [5, Theorem 2.15] or [3, Theorem 5.1]), there exists a unique local solution of our initial value problem, provided that the function x_1 specified in the statement of the theorem is an element of O. □

Remark 3.13. It is slightly inconvenient that the above theorem refers to the function F, which represents the right-hand side of the corresponding generalized equation and plays an auxiliary role only. However, as we mentioned in Remark 3.9, the condition $F(x,t) \in H_{t_0+\sigma}$ is often satisfied automatically.

We conclude this section by presenting modified versions of Theorems 3.6 and 3.7 for equations whose solutions are defined on the unbounded interval $[t_0, \infty)$.

In this case, the space $H_{t_0+\sigma}$ has to be replaced by a space H consisting of regulated functions defined on the whole real line. The space H together with its associated norm $\|\cdot\|_H$ can be an arbitrary space having the following properties:

1. H is complete.
2. For every $a \in [t_0, \infty)$, the space H_a is isometrically embedded into H as follows: if $y \in H_a$, then the function $z : \mathbb{R} \to \mathbb{R}^n$ given by

$$z(s) = \begin{cases} y(s), & s \in (-\infty, a], \\ y(a), & s \in (a, \infty) \end{cases}$$

   is an element of H and $\|z\|_H = \|y\|_\star$.

The simplest example of a space H satisfying these conditions is the space $BG(\mathbb{R}, \mathbb{R}^n)$ of all bounded regulated functions with the supremum norm.

Alternatively, we can take an arbitrary $\gamma \geq 0$ and consider the space G_{γ,t_0} of all regulated functions $y : \mathbb{R} \to \mathbb{R}^n$ such that

$$\max\left(\sup_{t\in(-\infty,t_0]} \|e^{\gamma(t-t_0)}y(t)\|, \sup_{t\in[t_0,\infty)} \|e^{\gamma(t_0-t)}y(t)\|\right)$$

is a finite number; this number is then defined to be the norm of y.

For the unbounded interval $[t_0, \infty)$, conditions (A), (B), (C) have to be modified as follows:

(A') The integral $\int_{t_0}^{t_0+\sigma} f(y_t,t)\,\mathrm{d}g(t)$ exists for every $y \in O$ and every $\sigma > 0$.

(B') There exists a function $M : [t_0,\infty) \to \mathbb{R}^+$, which is locally Kurzweil-Stieltjes integrable with respect to g, such that

$$\left\| \int_a^b f(y_t,t)\,\mathrm{d}g(t) \right\| \le \int_a^b M(t)\,\mathrm{d}g(t)$$

whenever $y \in O$ and $[a,b] \subset [t_0,\infty)$.

(C') There exists a function $L : [t_0,\infty) \to \mathbb{R}^+$, which is locally Kurzweil-Stieltjes integrable with respect to g, such that

$$\left\| \int_a^b (f(y_t,t) - f(z_t,t))\,\mathrm{d}g(t) \right\| \le \int_a^b L(t)\|y_t - z_t\|_\bigstar\,\mathrm{d}g(t)$$

whenever $y,z \in O$ and $[a,b] \subset [t_0,\infty)$ (we are assuming that the integral on the right-hand side exists).

We can now present modified versions of Theorem 3.6 and 3.7 for the interval $[t_0,\infty)$.

Theorem 3.14. *Assume that O is a subset of H having the prolongation property for $t \ge t_0$, $P = \{x_t;\ x \in O,\ t \in [t_0,\infty)\}$, $\phi \in P$, $g : [t_0,\infty) \to \mathbb{R}$ is a nondecreasing function, $f : P \times [t_0,\infty) \to \mathbb{R}^n$ satisfies conditions* (A'), (B'), (C'), *and the function $F : O \times [t_0,\infty) \to G(\mathbb{R},\mathbb{R}^n)$ given by*

$$F(x,t)(\vartheta) = \begin{cases} 0, & -\infty < \vartheta \le t_0, \\ \int_{t_0}^{\vartheta} f(x_s,s)\,\mathrm{d}g(s), & t_0 \le \vartheta \le t < \infty, \\ \int_{t_0}^{t} f(x_s,s)\,\mathrm{d}g(s), & t_0 \le t \le \vartheta < \infty \end{cases} \tag{3.9}$$

has values in H. If $y \in O$ is a solution of the measure functional differential equation

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(y_s,s)\,\mathrm{d}g(s), \quad t \in [t_0,\infty), \\ y_{t_0} &= \phi, \end{aligned}$$

then the function $x : [t_0,\infty) \to O$ given by

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & \vartheta \in (-\infty,t], \\ y(t), & \vartheta \in [t,\infty). \end{cases}$$

is a solution of the generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t), \quad t \in [t_0,\infty).$$

Theorem 3.15. *Assume that O is a subset of H having the prolongation property for $t \ge t_0$, $P = \{x_t;\ x \in O,\ t \in [t_0,\infty)\}$, $\phi \in P$, $g : [t_0,\infty) \to \mathbb{R}$ is a nondecreasing function, $f : P \times [t_0,\infty) \to \mathbb{R}^n$ satisfies conditions* (A'), (B'), (C'), *and the function $F : O \times [t_0,\infty) \to G(\mathbb{R},\mathbb{R}^n)$ given by* (3.9) *has values in H. If $x : [t_0,\infty) \to O$ is a solution of the generalized ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t), \quad t \in [t_0,\infty)$$

with the initial condition

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & \vartheta \in (-\infty,t_0], \\ x(t_0)(t_0), & \vartheta \in [t_0,\infty), \end{cases}$$

then the function $y \in O$ defined by

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & \vartheta \in (-\infty, t_0], \\ x(\vartheta)(\vartheta), & \vartheta \in [t_0, \infty). \end{cases}$$

is a solution of the measure functional differential equation

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \\ y_{t_0} &= \phi. \end{aligned}$$

Let us explain why the previous two theorems are true. In both cases, the fact that $x : [t_0, \infty) \to O$ is a solution of the generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t)$$

is equivalent to saying that

$$x(t_0 + \sigma) - x(t_0) = \int_{t_0}^{t_0+\sigma} DF(x(\tau), t) \tag{3.10}$$

for every $\sigma \geq 0$, where the integral on the right hand side is taken in the space H with respect to the norm $\|\cdot\|_H$. For a fixed $\sigma \geq 0$, Eq. (3.10) is equivalent to

$$x(t_0 + \sigma)(\vartheta) - x(t_0)(\vartheta) = \left(\int_{t_0}^{t_0+\sigma} DF(x(\tau), t) \right)(\vartheta) \tag{3.11}$$

for every $\vartheta \in [t_0, t_0 + \sigma]$; this follows from the fact the both sides are constant functions on $[t_0 + \sigma, \infty)$. However, since $H_{t_0+\sigma}$ is isometrically embedded in H, Eq. (3.11) is equivalent to

$$\bar{x}(t_0 + \sigma) - \bar{x}(t_0) = \int_{t_0}^{t_0+\sigma} DF(x(\tau), t),$$

where $\bar{x}(t)$ denotes the restriction of $x(t)$ to $(-\infty, t_0 + \sigma]$ and the integral on the right hand side is taken in the space $H_{t_0+\sigma}$ with respect to the norm $\|\cdot\|_\bigstar$. The whole problem is now reduced to the finite interval $[t_0, t_0 + \sigma]$ (where $\sigma \geq 0$ is arbitrary but fixed), where Theorems 3.6 and 3.7 apply.

4 Conclusion

Using the correspondence between both types of equations described in the previous section, it is possible to utilize the existing theory of generalized ordinary differential equations to obtain results on measure functional differential equations with infinite delay. This approach was already demonstrated in [3] for equations with finite delay, and can be repeated for equations with infinite delay without any substantial changes.

As shown in [3], functional dynamic equations on time scales represent a special case of measure functional differential equations. Consequently, another possible application of our results is in the investigation of functional dynamic equations with infinite delay; again, one can follow the methods presented in [3] for equations with finite delay. As described in [4], the same approach also works for functional differential and dynamic equations with impulses.

Acknowledgment. The author is grateful to the anonymous referee for suggestions which helped to improve the paper.

References

[1] S. M. Afonso, E. M. Bonotto, M. Federson, L. P. Gimenes, *Boundedness of solutions of retarded functional differential equations with variable impulses via generalized ordinary differential equations*, Math. Nachr. 285 (2012), 545–561.

[2] T. Faria, M. C. Gadotti, José J. Oliveira, *Stability results for impulsive functional differential equations with infinite delay*, Nonlinear Analysis 75 (2012), 6570–6587.

[3] M. Federson, J. G. Mesquita, A. Slavík, *Measure functional differential equations and functional dynamic equations on time scales*, J. Differential Equations 252 (2012), 3816–3847.

[4] M. Federson, J. G. Mesquita, A. Slavík, *Basic results for functional differential and dynamic equations involving impulses*, Mathematische Nachrichten (to appear). Online version available at: `http://dx.doi.org/10.1002/mana.201200006`

[5] M. Federson, Š. Schwabik, *Generalized ODE approach to impulsive retarded functional differential equations*, Differential Integral Equations 19 (2006), no. 11, 1201–1234.

[6] M. Federson, Š. Schwabik, *A new approach to impulsive retarded differential equations: stability results*, Funct. Differ. Equ. 16 (2009), no. 4, 583–607.

[7] M. Federson, Š. Schwabik, *Stability for retarded functional differential equations*, Ukrainian Math. J. 60 (2008), no. 1, 121–140.

[8] J. K. Hale, J. Kato, *Phase space for retarded equations with infinite delay*, Funkcial. Ekvac. 21 (1978), 11–41.

[9] J. K. Hale, S. M. V. Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[10] Y. Hino, S. Murakami, T. Naito, *Functional Differential Equations with Infinite Delay*, Springer-Verlag, 1991.

[11] C. Imaz, Z. Vorel, *Generalized ordinary differential equations in Banach spaces and applications to functional equations*, Bol. Soc. Mat. Mexicana, 11 (1966), 47–59.

[12] J. Kurzweil, *Generalized ordinary differential equations and continuous dependence on a parameter*, Czechoslovak Math. J. 7 (82), 1957, 418–448.

[13] J. Kurzweil, *Generalized Ordinary Differential Equations. Not Absolutely Continuous Solutions*, World Scientific, 2012.

[14] F. Oliva, Z. Vorel, *Functional equations and generalized ordinary differential equations*, Bol. Soc. Mat. Mexicana, 11 (1966), 40–46.

[15] Š. Schwabik, *Generalized Ordinary Differential Equations*, World Scientific, Singapore, 1992.