Generalized differential equations: differentiability of solutions with respect to initial conditions and parameters

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Abstract

We show that under certain assumptions, solutions of generalized ordinary differential equations are differentiable with respect to initial conditions and parameters. This result unifies and extends several existing theorems for other types of equations, such as impulsive differential equations or dynamic equations on time scales.

Keywords: Generalized ordinary differential equations, differentiability of solutions, Kurzweil integral, differential equations with impulses, dynamic equations on time scales

1 Introduction

In the classical theory of ordinary differential equations, it is well known that under certain assumptions, solutions of the problem

$$\begin{aligned} x'(t) &= f(x(t),t), \quad t \in [a,b], \\ x(t_0) &= x_0, \end{aligned}$$

are differentiable with respect to the initial condition; that is, if $x(t,x_0)$ denotes the value of the solution at $t \in [a,b]$, then the function $x_0 \mapsto x(t,x_0)$ is differentiable. The key requirement is that the right-hand side f should be differentiable with respect to x. Moreover, the derivative as a function of t is known to satisfy the so-called variational equation, which might be helpful in determining the value of the derivative.

Similarly, under suitable assumptions, solutions of a differential equation whose right-hand side depends on a parameter are differentiable with respect to that parameter.

These two types of theorems concerning differentiability of solutions with respect to initial conditions and parameters can be found in many differential equations textbooks, see e.g. [3]. Theorems of a similar type are also available for other types of equations, such as differential equations with impulses (see [6]) or dynamic equations on time scales (see [2]).

In 1957, Jaroslav Kurzweil introduced a class of integral equations called generalized ordinary differential equations (see [4]). His original motivation was to use them in the study of continuous dependence of solutions with respect to parameters. However, it became clear that generalized equations encompass various other types of equations, including equations with impulses (see [8]), dynamic equations on time scales (see [11]), or measure differential equations (see [8]).

The aim of this paper is to obtain differentiability theorems for generalized ordinary differential equations. Despite the fact that solutions of generalized equations need not be differentiable or even continuous with respect to t, we show that differentiability of the right-hand side with respect to x (and possibly with respect to parameters) still guarantees that the solutions are differentiable with respect to initial conditions (and parameters, respectively). Consequently, our result unifies and extends existing theorems for other types of equations.

2 The Kurzweil integral and generalized differential equations

We start this section by presenting a short overview of the Kurzweil integral, which is fundamental for the study of generalized ordinary differential equations. For more information on Kurzweil integration, see [5, 8].

A function $\delta : [a, b] \to \mathbb{R}^+$ is called a gauge on $[a, b]$. A tagged partition of the interval $[a, b]$ with division points $a = t_0 \le t_1 \le \cdots \le t_k = b$ and tags $\tau_i \in [t_{i-1}, t_i]$, $i \in \{1, \dots, k\}$, is called δ-fine if

$$[t_{i-1}, t_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)), \quad i \in \{1, \dots, k\}.$$

A matrix-valued function $U : [a, b] \times [a, b] \to \mathbb{R}^{n\times m}$ is called Kurzweil integrable on $[a, b]$, if there is a matrix $K \in \mathbb{R}^{n\times m}$ such that for every $\varepsilon > 0$, there is a gauge δ on $[a, b]$ such that

$$\left\| \sum_{i=1}^{k} (U(\tau_i, t_i) - U(\tau_i, t_{i-1})) - K \right\| < \varepsilon$$

for every δ-fine tagged partition of $[a, b]$. In this case, we define $\int_a^b \mathrm{D}_t U(\tau, t) = K$.

Obviously, a matrix-valued function U is integrable if and only if all its components are integrable. Note also that we restrict ourselves to functions with values in finite-dimensional spaces; in this case, the Kurzweil integral coincides with the strong Kurzweil integral (see Chapter 19 in [5]).

An important special case is the Kurzweil-Stieltjes integral (also known as the Perron-Stieltjes, Henstock-Stieltjes or Henstock-Kurzweil-Stieltjes integral) of a function $f : [a, b] \to \mathbb{R}^n$ with respect to a function $g : [a, b] \to \mathbb{R}$, which corresponds to the choice $U(\tau, t) = f(\tau)g(t)$ and will be denoted by $\int_a^b f(t)\,\mathrm{d}g(t)$.

We are now ready to introduce generalized ordinary differential equations. Consider a set $B \subset \mathbb{R}^n$, an interval $[a, b] \subset \mathbb{R}$ and a function $F : B \times [a, b]^2 \to \mathbb{R}^n$. A generalized ordinary differential equation with the right-hand side F has the form

$$\frac{\mathrm{d}}{\mathrm{d}t} x = \mathrm{D}_t F(x, \tau, t), \tag{1}$$

which is a shorthand notation for the integral equation

$$x(s) = x(a) + \int_a^s \mathrm{D}_t F(x(\tau), \tau, t), \quad s \in [a, b]. \tag{2}$$

In other words, a function $x : [a, b] \to B$ is a solution of (1) if and only if (2) is satisfied. We emphasize that (1) is a symbolic notation only and does not mean that x has to be differentiable.

Equations of the form (1) have been introduced by J. Kurzweil in [5]. In many situations, it is sufficient to consider the less general type of equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x = \mathrm{D}_t F(x, t), \tag{3}$$

where the right-hand side does not depend on τ (in fact, most existing sources devoted to generalized equations focus on this less general type; see e.g. the pioneering paper [4] by J. Kurzweil, or the monograph [8] by Š. Schwabik). The corresponding integral equation has the form

$$x(s) = x(a) + \int_a^s \mathrm{D}_t F(x(\tau), t), \quad s \in [a, b].$$

Under certain conditions, an equation of the form (1) can be transformed to an equation of the form (3) (see Chapter 27 in [5]). However, as will be clear in Section 4, the more general type (1) is quite useful for our purposes.

The rest of this section summarizes some basic facts concerning the Kurzweil integral that will be needed later.

The following existence theorem can be found in [8, Corollary 1.34] or [5, Chapter 20].

Theorem 2.1. *If $f : [a,b] \to \mathbb{R}^n$ is a regulated function and $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, then the integral $\int_a^b f(s)\,\mathrm{d}g(s)$ exists.*

The next estimate follows directly from the definition of the Kurzweil integral.

Lemma 2.2. *Let $U : [a,b]^2 \to \mathbb{R}^{n\times n}$ be a Kurzweil integrable function. Assume there exists a pair of functions $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ such that f is regulated, g is nondecreasing, and*

$$\|U(\tau,t) - U(\tau,s)\| \le f(\tau)|g(t) - g(s)|, \quad \tau, t, s \in [a,b].$$

Then

$$\left\| \int_a^b \mathrm{D}_t U(\tau,t) \right\| \le \int_a^b f(\tau)\,\mathrm{d}g(\tau).$$

For the first part of the following statement, see [5, Corollary 14.18] or [8, Theorem 1.16]; the second part is a direct consequence of Lemma 2.2.

Theorem 2.3. *Assume that $U : [a,b]^2 \to \mathbb{R}^{n\times m}$ is Kurzweil integrable and $u : [a,b] \to \mathbb{R}^{n\times m}$ is its primitive, i.e.,*

$$u(s) = u(a) + \int_a^s \mathrm{D}_t U(\tau,t), \quad s \in [a,b].$$

If U is regulated in the second variable, then u is regulated and satisfies

$$\begin{aligned} u(\tau+) &= u(\tau) + U(\tau,\tau+) - U(\tau,\tau), \quad \tau \in [a,b) \\ u(\tau-) &= u(\tau) + U(\tau,\tau-) - U(\tau,\tau), \quad \tau \in (a,b]. \end{aligned}$$

Moreover, if there exists a nondecreasing function $h : [a,b] \to \mathbb{R}$ such that

$$\|U(\tau,t) - U(\tau,s)\| \le |h(t) - h(s)|, \quad \tau, t, s \in [a,b],$$

then

$$\|u(t) - u(s)\| \le |h(t) - h(s)|, \quad t, s \in [a,b].$$

The following theorem represents an analogue of Gronwall's inequality for the Kurzweil-Stieltjes integral; the proof can be found in [8, Corollary 1.43].

Theorem 2.4. *Let $h : [a,b] \to [0,\infty)$ be a nondecreasing left-continuous function, $k > 0$, $l \ge 0$. Assume that $\psi : [a,b] \to [0,\infty)$ is bounded and satisfies*

$$\psi(\xi) \le k + l \int_a^\xi \psi(\tau)\,\mathrm{d}h(\tau), \quad \xi \in [a,b].$$

Then $\psi(\xi) \le k e^{l(h(\xi)-h(a))}$ for every $\xi \in [a,b]$.

3 Linear equations

In this section, we consider the homogeneous linear generalized ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} z = \mathrm{D}_t[A(\tau,t)z], \tag{4}$$

where $A : [a,b]^2 \to \mathbb{R}^{n\times n}$ is a given matrix-valued function and the solution z takes values in $\mathbb{R}^n$. The integral form of this equation is

$$z(s) = z(a) + \int_a^s \mathrm{D}_t[A(\tau,t)z(\tau)], \quad s \in [a,b].$$

The case when A does not depend on τ has been studied in numerous works (see e.g. [8, 9]). However, to prove the main result of this paper, we need some basic facts concerning the more general type (4).

For convenience, let us introduce the following condition:

(A) There exists a nondecreasing function $h : [a,b] \to \mathbb{R}$ such that

$$\|A(\tau,t) - A(\tau,s)\| \le |h(t) - h(s)|, \quad \tau,t,s \in [a,b].$$

Note that if (A) is satisfied, then for every fixed $\tau \in [a,b]$, the function $t \mapsto A(\tau,t)$ is regulated (this follows from the Cauchy condition for the existence of one-sided limits). Also, if h is left-continuous, then $t \mapsto A(\tau,t)$ is left-continuous as well.

Lemma 3.1. *Assume that $A : [a,b]^2 \to \mathbb{R}^{n\times n}$ satisfies* (A). *Let $y, z : [a,b] \to \mathbb{R}^n$ be a pair of functions such that*

$$z(s) = z(a) + \int_a^s \mathrm{D}_t[A(\tau,t)y(\tau)], \quad s \in [a,b].$$

Then z is regulated on $[a,b]$.

Proof. Let $U(\tau,t) = A(\tau,t)y(\tau)$; note that

$$z(s) = z(a) + \int_a^s \mathrm{D}_t U(\tau,t), \quad s \in [a,b].$$

By condition (A), U is regulated in the second variable. By Theorem 2.3, z is regulated. □

A simple consequence of the previous lemma is that every solution of the linear generalized differential equation (4) is regulated.

Lemma 3.2. *Assume that $A : [a,b]^2 \to \mathbb{R}^{n\times n}$ satisfies* (A) *with a left-continuous function h. Then for every $z_0 \in \mathbb{R}^n$, the initial value problem*

$$\frac{\mathrm{d}}{\mathrm{d}t} z = \mathrm{D}_t[A(\tau,t)z], \quad z(a) = z_0$$

has at most one solution.

Proof. Consider a pair of functions $z_1, z_2 : [a,b] \to \mathbb{R}^n$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} z_i = \mathrm{D}_t[A(\tau,t)z_i], \quad \text{for } i \in \{1,2\}.$$

Then

$$\begin{aligned}\|z_1(s) - z_2(s)\| &\le \|z_1(a) - z_2(a)\| + \left\| \int_a^s \mathrm{D}_t[A(\tau,t)(z_1(\tau) - z_2(\tau))] \right\| \\ &\le \|z_1(a) - z_2(a)\| + \int_a^s \|z_1(\tau) - z_2(\tau)\| \,\mathrm{d}h(\tau), \quad s \in [a,b]\end{aligned}$$

(the last inequality follows from Lemma 2.2). By Gronwall's inequality from Theorem 2.4, we obtain

$$\|z_1(s) - z_2(s)\| \le \|z_1(a) - z_2(a)\| e^{h(b)-h(a)}, \quad s \in [a,b].$$

Thus, if $z_1(a) = z_2(a)$, then z_1, z_2 coincide on $[a,b]$. □

The proof of the following lemma is almost identical to the proof of Theorem 3.14 in [8]; however, since our assumptions are different, we repeat the proof here for reader's convenience.

Lemma 3.3. *Let $z_k : [a,b] \to \mathbb{R}^n$, $k \in \mathbb{N}$, be a uniformly bounded sequence of functions, which is pointwise convergent to a function $z : [a,b] \to \mathbb{R}^n$. Assume that $A : [a,b]^2 \to \mathbb{R}^{n\times n}$ satisfies* (A) *and the integral $\int_a^b \mathrm{D}_t[A(\tau,t)z_k(\tau)]$ exists for every $k \in \mathbb{N}$. Then $\int_a^b \mathrm{D}_t[A(\tau,t)z(\tau)]$ exists as well and equals $\lim_{k\to\infty} \int_a^b \mathrm{D}_t[A(\tau,t)z_k(\tau)]$.*

Proof. Let A^i denote the i-th row of A. Clearly, it is enough to prove that for every $i \in \{1, \ldots, n\}$, the integral $\int_a^b \mathrm{D}_t[A^i(\tau,t)z(\tau)]$ exists and equals $\lim_{k\to\infty} \int_a^b \mathrm{D}_t[A^i(\tau,t)z_k(\tau)]$.

Let $i \in \{1, \ldots, n\}$ be fixed, $U(\tau,t) = A^i(\tau,t)z(\tau)$ and $U_k(\tau,t) = A^i(\tau,t)z_k(\tau)$ for every $k \in \mathbb{N}$ (note that U and U_k are scalar functions). Consider an arbitrary fixed $\varepsilon > 0$. For every $\tau \in [a,b]$, there is a number $p(\tau) \in \mathbb{N}$ such that

$$\|z_k(\tau) - z(\tau)\| < \frac{\varepsilon}{h(b) - h(a) + 1}, \quad k \geq p(\tau).$$

Let $M > 0$ be such that $\|z_k(\tau)\| \leq M$ for every $k \in \mathbb{N}$, $\tau \in [a,b]$. The function

$$\mu(t) = \frac{\varepsilon h(t)}{h(b) - h(a) + 1}, \quad t \in [a,b],$$

is nondecreasing and $\mu(b) - \mu(a) < \varepsilon$. If $\tau, t_1, t_2 \in [a,b]$, $t_1 \leq t_2$, and $k \geq p(\tau)$, we have

$$|U_k(\tau,t_2) - U_k(\tau,t_1) - U(\tau,t_2) + U(\tau,t_1)| \leq \|A^i(\tau,t_2) - A^i(\tau,t_1)\| \cdot \|z_k(\tau) - z(\tau)\|$$

$$\leq \|A(\tau,t_2) - A(\tau,t_1)\| \cdot \|z_k(\tau) - z(\tau)\| \leq \frac{\varepsilon(h(t_2) - h(t_1))}{h(b) - h(a) + 1} = \mu(t_2) - \mu(t_1)$$

and

$$M(-h(t_2) + h(t_1)) \leq U_k(\tau,t_2) - U_k(\tau,t_1) \leq M(h(t_2) - h(t_1)).$$

The conclusion now follows from the dominated convergence theorem for the Kurzweil integral ([8, Corollary 1.31]). □

Lemma 3.4. *Assume that $A : [a,b]^2 \to \mathbb{R}^{n\times n}$ is Kurzweil integrable and satisfies* (A). *Then for every regulated function $y : [a,b] \to \mathbb{R}^n$, the integral $\int_a^b \mathrm{D}_t[A(\tau,t)y(\tau)]$ exists.*

Proof. Every regulated function is a uniform limit of step functions. Thus, in view of Lemma 3.3, it is sufficient to prove that the statement is true for every step function $y : [a,b] \to \mathbb{R}^n$.

Let $a = t_0 < t_1 < \cdots < t_k = b$ be a partition of $[a,b]$ such that y is constant on each interval (t_{i-1}, t_i). For $t_{i-1} < u < \sigma < v < t_i$, the integrability of A implies that the integrals $\int_u^\sigma \mathrm{D}_t[A(\tau,t)y(\tau)]$ and $\int_\sigma^v \mathrm{D}_t[A(\tau,t)y(\tau)]$ exist and are regulated functions of u, v (this follows from (A) and Lemma 3.1). According to Hake's theorems for the Kurzweil integral (see Theorems 14.20 and 14.22 in [5], or Theorem 1.14 and Remark 1.15 in [8]), the integrals $\int_{t_{i-1}}^\sigma \mathrm{D}_t[A(\tau,t)y(\tau)]$ and $\int_\sigma^{t_i} \mathrm{D}_t[A(\tau,t)y(\tau)]$ exist as well. Thus, $\int_{t_{i-1}}^{t_i} \mathrm{D}_t[A(\tau,t)y(\tau)]$ exists for every $i \in \{1, \ldots, k\}$, which proves the statement. □

Up to a small detail, the proof of the following theorem is the same as the proof of Theorem 23.4 in [5]. Therefore, we provide only a sketch of the proof and leave the details to the reader.

Theorem 3.5. *Assume that $A : [a,b]^2 \to \mathbb{R}^{n\times n}$ is Kurzweil integrable and satisfies* (A) *with a left-continuous function h. Then for every $z_0 \in \mathbb{R}^n$, the initial value problem*

$$\frac{\mathrm{d}}{\mathrm{d}t} z = \mathrm{D}_t[A(\tau,t)z], \quad z(a) = z_0 \tag{5}$$

has a unique solution $z : [a,b] \to \mathbb{R}^n$.

Proof. Uniqueness of solutions follows immediately from Lemma 3.2. To prove the existence of a solution of (5) on $[a,b]$, choose a partition $a = s_0 < s_1 < \cdots < s_k = b$ of $[a,b]$ such that $h(s_i) - h(s_{i-1}+) \leq \frac{1}{2}$ for every $i \in \{1, \ldots, k\}$. Now, it is sufficient to prove that for every $w_0 \in \mathbb{R}^n$ and $i \in \{1, \ldots, k\}$, the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t} z = \mathrm{D}_t[A(\tau,t)z], \quad z(s_{i-1}) = w_0$$

has a solution on $[s_{i-1}, s_i]$. This solution can be obtained by the method of successive approximations: Let

$$\begin{aligned} v_0(s) &= w_0, \quad s \in [s_{i-1}, s_i], \\ v_j(s) &= w_0 + \int_{s_{i-1}}^{s} \mathrm{D}_t[A(\tau, t)v_{j-1}(\tau)], \quad s \in [s_{i-1}, s_i], \ \ j \in \mathbb{N}. \end{aligned}$$

The existence of the integral on the right-hand side is guaranteed by Lemma 3.4 (this is the only difference against the proof of Theorem 23.4 in [5]). By Theorem 2.3, we have

$$v_j(s) = \widehat{w_0} + \int_{s_{i-1}+}^{s} \mathrm{D}_t[A(\tau, t)v_{j-1}(\tau)], \quad s \in [s_{i-1}, s_i], \ \ j \in \mathbb{N},$$

where $\widehat{w_0} = w_0 + A(s_{i-1}, s_{i-1}+)w_0 - A(s_{i-1}, s_{i-1})w_0$. Using Lemma 2.2, it is not difficult to see that

$$\|v_1(s) - v_0(s)\| \le \|w_0\|(h(s_i) - h(s_{i-1})), \quad s \in [s_{i-1}, s_i],$$

$$\|v_{j+1}(s) - v_j(s)\| \le \sup_{\tau \in [s_{i-1}, s_i]} \|v_j(\tau) - v_{j-1}(\tau)\|(h(s_i) - h(s_{i-1}+)), \quad s \in (s_{i-1}, s_i], \ \ j \in \mathbb{N}.$$

Since $h(s_i) - h(s_{i-1}+) \le \frac{1}{2}$, it follows by induction that

$$\|v_{j+1}(s) - v_j(s)\| \le \|w_0\| \left(\frac{1}{2}\right)^j (h(s_i) - h(s_{i-1})), \quad s \in (s_{i-1}, s_i], \ \ j \in \mathbb{N}.$$

This implies that $\{v_j\}_{j=1}^{\infty}$ is uniformly convergent to a function $v : [s_{i-1}, s_i] \to \mathbb{R}^n$, which satisfies

$$v(s) = \lim_{j \to \infty} v_j(s) = w_0 + \lim_{j \to \infty} \int_{s_{i-1}}^{s} \mathrm{D}_t[A(\tau, t)v_{j-1}(\tau)] = w_0 + \int_{s_{i-1}}^{s} \mathrm{D}_t[A(\tau, t)v(\tau)]$$

for every $s \in [s_{i-1}, s_i]$ (we have used Lemma 3.3). □

Throughout this section, we focused our attention on equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} z = \mathrm{D}_t[A(\tau, t)z], \quad z(a) = z_0,$$

where z takes values in $\mathbb{R}^n$. More generally, it is possible to consider equations where the unknown function has values in $\mathbb{R}^{n \times n}$. For example, if $Z_0 \in \mathbb{R}^{n \times n}$ is an arbitrary matrix, then

$$\frac{\mathrm{d}}{\mathrm{d}t} Z = \mathrm{D}_t[A(\tau, t)Z], \quad Z(a) = Z_0$$

is a shorthand notation for the integral equation

$$Z(s) = Z_0 + \int_a^s \mathrm{D}_t[A(\tau, t)Z(\tau)], \quad s \in [a, b]. \tag{6}$$

For an arbitrary matrix $X \in \mathbb{R}^{n \times n}$, let X^i be the i-th column of X. Then it is obvious that (6) holds if and only if

$$Z^i(s) = Z_0^i + \int_a^s \mathrm{D}_t[A(\tau, t)Z^i(\tau)], \quad s \in [a, b], \ \ i \in \{1, \dots, n\}.$$

We will encounter equations with matrix-valued solutions in the following section.

4 The main results

Consider a generalized ordinary differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \mathrm{D}_t F(x,\tau,t), \quad x(a) = x^0(\lambda), \tag{7}$$

where the solution x takes values in $\mathbb{R}^n$, and $x^0 : \mathbb{R}^l \to \mathbb{R}^n$ is a function which describes the dependence of the initial condition on a parameter $\lambda \in \mathbb{R}^l$.

Let $x(s,\lambda)$ be the value of the solution at $s \in [a,b]$. Our goal is to show that under certain conditions, $x(s,\lambda)$ is differentiable with respect to λ. Using the definition of the Kurzweil integral, we see that the value of $x(s,\lambda)$ can be approximated by

$$x^0(\lambda) + \sum_{j=1}^{k} \left(F(x(\tau_j,\lambda),\tau_j,t_j) - F(x(\tau_j,\lambda),\tau_j,t_{j-1})\right),$$

where $a = t_0 < t_1 < \cdots < t_k = s$ is a sufficiently fine partition of $[a,s]$ with tags $\tau_j \in [t_{j-1},t_j]$, $j \in \{1,\ldots,k\}$. Assuming that all expressions are differentiable with respect to λ at $\lambda_0 \in \mathbb{R}^l$, we see that the derivative $x_\lambda(s,\lambda_0)$, i.e., the matrix $\{\frac{\partial x_i}{\partial \lambda_j}(s,\lambda_0)\}_{i,j} \in \mathbb{R}^{n\times l}$, should be approximately equal to

$$x^0_\lambda(\lambda_0) + \sum_{j=1}^{k} \left(F_x(x(\tau_j,\lambda_0),\tau_j,t_j)x_\lambda(\tau_j,\lambda_0) - F_x(x(\tau_j,\lambda),\tau_j,t_{j-1})x_\lambda(\tau_j,\lambda_0)\right).$$

Now, the right-hand side is an approximation to

$$x^0_\lambda(\lambda_0) + \int_a^s \mathrm{D}_t[F_x(x(\tau,\lambda_0),\tau,t)x_\lambda(\tau,\lambda_0)].$$

Thus, it seems reasonable to expect that the derivative $Z(t) = x_\lambda(t,\lambda_0)$, $t \in [a,b]$, is a solution of the linear equation

$$\frac{\mathrm{d}}{\mathrm{d}t}z = \mathrm{D}_t G(z,\tau,t), \quad z(a) = x^0_\lambda(\lambda_0), \tag{8}$$

where $G(z,\tau,t) = F_x(x(\tau,\lambda_0),\tau,t)z$. This provides a motivation for the following theorem.

Note that even in the case when the right-hand side of Eq. (7) does not depend on τ and has the form $F(x,t)$, the right-hand side of Eq. (8) has the form $G(z,\tau,t) = F_x(x(\tau,\lambda_0),t)z$, i.e., still depends on τ. That is why we had to study the more general type of equations in the previous section.

The following proof is based on elementary estimates and Gronwall's inequality; it is inspired by a proof of Theorem 3.1 in the paper [2], which is concerned with dynamic equations on time scales.

Theorem 4.1. *Let $B \subset \mathbb{R}^n$ be an open set, $\lambda_0 \in \mathbb{R}^l$, $\rho > 0$, $\Lambda = \{\lambda \in \mathbb{R}^l; \|\lambda - \lambda_0\| < \rho\}$, $x^0 : \Lambda \to B$, $F : B \times [a,b]^2 \to \mathbb{R}^n$. Assume that F is regulated and left-continuous in the third variable, and that for every $\lambda \in \Lambda$, the initial value problem*

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \mathrm{D}_t F(x,\tau,t), \quad x(a) = x^0(\lambda) \tag{9}$$

has a solution defined on $[a,b]$; let $x(t,\lambda)$ be the value of that solution at $t \in [a,b]$. Moreover, let the following conditions be satisfied:

1. *For every fixed pair $(t,\tau) \in [a,b]^2$, the function $x \mapsto F(x,\tau,t)$ is continuously differentiable on B.*

2. *The function x^0 is differentiable at λ_0.*

3. *There exists a left-continuous nondecreasing function $h : [a,b] \to \mathbb{R}$ such that*

$$\|F_x(x,\tau,t) - F_x(x,\tau,s)\| \le |h(t) - h(s)|, \quad s,t,\tau \in [a,b], \quad x \in B.$$

4. There exists a continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ such that $\omega(0) = 0$ and

$$\|F_x(x,\tau,t)-F_x(x,\tau,s)-F_x(y,\tau,t)+F_x(y,\tau,s)\| \le \omega(\|x-y\|)\cdot|h(t)-h(s)|, \quad s,t,\tau \in [a,b], \quad x,y \in B.$$

5. There exists a left-continuous nondecreasing function $k : [a,b] \to \mathbb{R}$ such that

$$\|F(x,\tau,t) - F(x,\tau,s) - F(y,\tau,t) + F(y,\tau,s)\| \le \|x-y\| \cdot |k(t) - k(s)|, \quad s,t,\tau \in [a,b], \quad x,y \in B.$$

6. There exists a number $\eta > 0$ such that if $x \in \mathbb{R}^n$ satisfies $\|x - x(t,\lambda_0)\| < \eta$ for some $t \in [a,b]$, then $x \in B$ (i.e., the η-neighborhood of the solution $t \mapsto x(t,\lambda_0)$ is contained in B).

Then the function $\lambda \mapsto x(t,\lambda)$ is differentiable at λ_0, uniformly for all $t \in [a,b]$. Moreover, its derivative $Z(t) = x_\lambda(t,\lambda_0)$, $t \in [a,b]$, is the unique solution of the generalized differential equation

$$Z(s) = x^0_\lambda(\lambda_0) + \int_a^s \mathrm{D}_t[F_x(x(\tau,\lambda_0),\tau,t)Z(\tau)], \quad s \in [a,b]. \tag{10}$$

Proof. According to the assumptions, we have

$$x(s,\lambda) = x^0(\lambda) + \int_a^s \mathrm{D}_t F(x(\tau,\lambda),\tau,t), \quad \lambda \in \Lambda, \quad s \in [a,b].$$

By Theorem 2.3, every solution x is a regulated and left-continuous function on $[a,b]$. If $\Delta\lambda \in \mathbb{R}^l$ is such that $\|\Delta\lambda\| < \rho$, then

$$\|x(s,\lambda_0+\Delta\lambda) - x(s,\lambda_0)\| \le \|x^0(\lambda_0+\Delta\lambda) - x^0(\lambda_0)\| + \left\| \int_a^s \mathrm{D}_t V(\tau,t) \right\|,$$

where $V(\tau,t) = F(x(\tau,\lambda_0+\Delta\lambda),\tau,t) - F(x(\tau,\lambda_0),\tau,t)$. By assumption 5, we obtain

$$\|V(\tau,t_1) - V(\tau,t_2)\| \le \|x(\tau,\lambda_0+\Delta\lambda) - x(\tau,\lambda_0)\| \cdot |k(t_1) - k(t_2)|,$$

and consequently (using Lemma 2.2)

$$\|x(s,\lambda_0+\Delta\lambda) - x(s,\lambda_0)\| \le \|x^0(\lambda_0+\Delta\lambda) - x^0(\lambda_0)\| + \int_a^s \|x(\tau,\lambda_0+\Delta\lambda) - x(\tau,\lambda_0)\| \,\mathrm{d}k(\tau).$$

for every $s \in [a,b]$. Gronwall's inequality from Theorem 2.4 implies

$$\|x(s,\lambda_0+\Delta\lambda) - x(s,\lambda_0)\| \le \|x^0(\lambda_0+\Delta\lambda) - x^0(\lambda_0)\| e^{k(b)-k(a)}, \quad s \in [a,b].$$

Thus we see that for $\Delta\lambda \to 0$, $x(s,\lambda_0+\Delta\lambda)$ approaches $x(s,\lambda_0)$ uniformly for all $s \in [a,b]$.

By assumption 3, the function $A(\tau,t) = F_x(x(\tau,\lambda_0),\tau,t)$ satisfies condition (A). By Theorem 3.5, Eq. (10) has a unique solution $Z : [a,b] \to \mathbb{R}^{n\times n}$. By Lemma 3.1, the solution is regulated. Consequently, there exists a constant $M > 0$ such that $\|Z(t)\| \le M$ for every $t \in [a,b]$.

For every $\Delta\lambda \in \mathbb{R}^l$ such that $\|\Delta\lambda\| < \rho$, let

$$\xi(r,\Delta\lambda) = \frac{x(r,\lambda_0+\Delta\lambda) - x(r,\lambda_0) - Z(r)\Delta\lambda}{\|\Delta\lambda\|}, \quad r \in [a,b].$$

Our goal is to prove that if $\Delta\lambda \to 0$, then $\xi(r,\Delta\lambda) \to 0$ uniformly for $r \in [a,b]$.

Let $\varepsilon > 0$ be given. There exists a $\delta > 0$ such that if $\Delta\lambda \in \mathbb{R}^l$ and $\|\Delta\lambda\| < \delta$, then

$$\|x(t,\lambda_0+\Delta\lambda) - x(t,\lambda_0)\| < \min(\varepsilon,\eta), \quad t \in [a,b],$$

and

$$\frac{\|x^0(\lambda_0+\Delta\lambda) - x^0(\lambda_0) - x^0_\lambda(\lambda_0)\Delta\lambda\|}{\|\Delta\lambda\|} < \varepsilon.$$

Observe that

$$\xi(a,\Delta\lambda)=\frac{x^0(\lambda_0+\Delta\lambda)-x^0(\lambda_0)-x^0_\lambda(\lambda_0)\Delta\lambda}{\|\Delta\lambda\|},$$

$$\begin{aligned}\xi(r,\Delta\lambda)-\xi(a,\Delta\lambda)&=\frac{x(r,\lambda_0+\Delta\lambda)-x(a,\lambda_0+\Delta\lambda)}{\|\Delta\lambda\|}-\frac{x(r,\lambda_0)-x(a,\lambda_0)}{\|\Delta\lambda\|}-\frac{(Z(r)-Z(a))\Delta\lambda}{\|\Delta\lambda\|}\\&=\int_a^r \mathrm{D}_t U(\tau,t),\end{aligned}$$

where

$$U(\tau,t)=\frac{F(x(\tau,\lambda_0+\Delta\lambda),\tau,t)-F(x(\tau,\lambda_0),\tau,t)-F_x(x(\tau,\lambda_0),\tau,t)Z(\tau)\Delta\lambda}{\|\Delta\lambda\|}.$$

For every $\tau\in[a,b]$ and $u\in[0,1]$, we have

$$\|ux(\tau,\lambda_0+\Delta\lambda)+(1-u)x(\tau,\lambda_0)-x(\tau,\lambda_0)\|\le\|x(\tau,\lambda_0+\Delta\lambda)-x(\tau,\lambda_0)\|<\eta.$$

By assumption 6, the point $ux(\tau,\lambda_0+\Delta\lambda)+(1-u)x(\tau,\lambda_0)$ is an element of B. In other words, the segment connecting $x(\tau,\lambda_0+\Delta\lambda)$ and $x(\tau,\lambda_0)$ is contained in B. Thus we can use the mean-value theorem for vector-valued functions (see e.g. [3, Lemma 8.11]) to examine the following difference:

$$\begin{aligned}U(\tau,t)-U(\tau,s)&=\frac{F(x(\tau,\lambda_0+\Delta\lambda),\tau,t)-F(x(\tau,\lambda_0),\tau,t)-F(x(\tau,\lambda_0+\Delta\lambda),\tau,s)+F(x(\tau,\lambda_0),\tau,s)}{\|\Delta\lambda\|}\\&\quad-\frac{(F_x(x(\tau,\lambda_0),\tau,t)-F_x(x(\tau,\lambda_0),\tau,s))(x(\tau,\lambda_0+\Delta\lambda)-x(\tau,\lambda_0))}{\|\Delta\lambda\|}\\&\quad+\frac{(F_x(x(\tau,\lambda_0),\tau,t)-F_x(x(\tau,\lambda_0),\tau,s))(x(\tau,\lambda_0+\Delta\lambda)-x(\tau,\lambda_0)-Z(\tau)\Delta\lambda)}{\|\Delta\lambda\|}\\&=\frac{1}{\|\Delta\lambda\|}\left(\int_0^1(F_x(ux(\tau,\lambda_0+\Delta\lambda)+(1-u)x(\tau,\lambda_0),\tau,t)-F_x(ux(\tau,\lambda_0+\Delta\lambda)+(1-u)x(\tau,\lambda_0),\tau,s))\,\mathrm{d}u\right.\\&\quad\left.-\int_0^1(F_x(x(\tau,\lambda_0),\tau,t)-F_x(x(\tau,\lambda_0),\tau,s))\,\mathrm{d}u\right)\cdot(x(\tau,\lambda_0+\Delta\lambda)-x(\tau,\lambda_0))\\&\quad+(F_x(x(\tau,\lambda_0),\tau,t)-F_x(x(\tau,\lambda_0),\tau,s))\xi(\tau,\Delta\lambda)\end{aligned}$$

(In the second integral above, we are simply integrating a constant function.) If $\|\Delta\lambda\|<\delta$, then (by assumption 4)

$$\begin{aligned}&\|F_x(ux(\tau,\lambda_0+\Delta\lambda)+(1-u)x(\tau,\lambda_0),\tau,t)-F_x(ux(\tau,\lambda_0+\Delta\lambda)+(1-u)x(\tau,\lambda_0),\tau,s)-F_x(x(\tau,\lambda_0),\tau,t)\\&\quad+F_x(x(\tau,\lambda_0),\tau,s)\|\le\omega(\|ux(\tau,\lambda_0+\Delta\lambda)+(1-u)x(\tau,\lambda_0)-x(\tau,\lambda_0)\|)|h(t)-h(s)|\\&\quad=\omega(\|u(x(\tau,\lambda_0+\Delta\lambda)-x(\tau,\lambda_0))\|)|h(t)-h(s)|\le\omega(\varepsilon)|h(t)-h(s)|,\end{aligned}$$

and thus (using assumption 3)

$$\begin{aligned}\|U(\tau,t)-U(\tau,s)\|&\le\omega(\varepsilon)|h(t)-h(s)|\frac{\|x(\tau,\lambda_0+\Delta\lambda)-x(\tau,\lambda_0)\|}{\|\Delta\lambda\|}+|h(t)-h(s)|\cdot\|\xi(\tau,\Delta\lambda)\|\\&\le|h(t)-h(s)|\left(\omega(\varepsilon)\frac{\|x(\tau,\lambda_0+\Delta\lambda)-x(\tau,\lambda_0)-Z(\tau)\Delta\lambda\|+\|Z(\tau)\Delta\lambda\|}{\|\Delta\lambda\|}+\|\xi(\tau,\Delta\lambda)\|\right)\\&\le|h(t)-h(s)|\left(\omega(\varepsilon)(\|\xi(\tau,\Delta\lambda)\|+M)+\|\xi(\tau,\Delta\lambda)\|\right).\end{aligned}$$

Consequently, by Lemma 2.2,

$$\|\xi(r,\Delta\lambda)-\xi(a,\Delta\lambda)\|=\left\|\int_a^r \mathrm{D}_t U(\tau,t)\right\|\le\int_a^r(\omega(\varepsilon)(\|\xi(\tau,\Delta\lambda)\|+M)+\|\xi(\tau,\Delta\lambda)\|)\,\mathrm{d}h(\tau)$$

$$= \omega(\varepsilon)M(h(r)-h(a)) + (1+\omega(\varepsilon))\int_a^r \|\xi(\tau,\Delta\lambda)\|\,\mathrm{d}h(\tau)$$

$$\le \omega(\varepsilon)M(h(b)-h(a)) + (1+\omega(\varepsilon))\int_a^r \|\xi(\tau,\Delta\lambda)\|\,\mathrm{d}h(\tau).$$

It follows that

$$\|\xi(r,\Delta\lambda)\| \le \|\xi(r,\Delta\lambda)-\xi(a,\Delta\lambda)\| + \|\xi(a,\Delta\lambda)\|$$

$$\le \varepsilon + \omega(\varepsilon)M(h(b)-h(a)) + (1+\omega(\varepsilon))\int_a^r \|\xi(\tau,\Delta\lambda)\|\,\mathrm{d}h(\tau).$$

Finally, Gronwall's inequality leads to the estimate

$$\|\xi(r,\Delta\lambda)\| \le (\varepsilon + \omega(\varepsilon)M(h(b)-h(a)))e^{(1+\omega(\varepsilon))(h(r)-h(a))}$$

$$\le (\varepsilon + \omega(\varepsilon)M(h(b)-h(a)))e^{(1+\omega(\varepsilon))(h(b)-h(a))}.$$

Since $\lim_{\varepsilon\to 0+}\omega(\varepsilon)=0$, we see that if $\Delta\lambda\to 0$, then $\xi(r,\Delta\lambda)\to 0$ uniformly for $r\in[a,b]$. □

In the simplest case when $l=n$, $\Lambda\subset B$ and $x^0(\lambda)=\lambda$ for every $\lambda\in\Lambda$, the previous theorem says that solutions of

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \mathrm{D}_t F(x,\tau,t), \quad x(a)=\lambda$$

are differentiable with respect to λ, and the derivative $Z(t)=x_\lambda(t,\lambda_0)$, $t\in[a,b]$, is the unique solution of the generalized differential equation

$$Z(s) = I + \int_a^s \mathrm{D}_t[F_x(x(\tau,\lambda_0),\tau,t)Z(\tau)], \quad s\in[a,b].$$

Remark 4.2. In Theorem 4.1, we are assuming the existence of a $\rho>0$ such that for every $\lambda\in\mathbb{R}^l$ satisfying $\|\lambda-\lambda_0\|<\rho$, the initial value problem (9) has a solution $t\mapsto x(t,\lambda)$ defined on $[a,b]$ and taking values in B. For equations whose right-hand side F does not depend on τ, this assumption can be replaced by the following simple condition:

$$\|F(x,t)-F(x,s)\| \le |k(t)-k(s)|, \quad s,t\in[a,b], \quad x\in B. \tag{11}$$

Let us explain why this condition is sufficient. (We are still assuming that conditions 1–6 from Theorem 4.1 are satisfied. In particular, we are assuming that the initial value problem (9) has a solution corresponding to $\lambda=\lambda_0$.) Observe that if $c\in[a,b)$ and $\|y-x(c+,\lambda_0)\|<\eta/2$, then $y\in B$. (Choose $\delta>0$ such that $\|x(c+,\lambda_0)-x(c+\delta,\lambda_0)\|<\eta/2$; then $\|y-x(c+\delta,\lambda_0)\|\le\|y-x(c+,\lambda_0)\|+\|x(c+,\lambda_0)-x(c+\delta,\lambda_0)\|<\eta$, and thus $y\in B$ by assumption 6.)

Following the first part of proof of Theorem 4.1, we observe that there is a $\delta>0$ such that if $\|\lambda-\lambda_0\|<\delta$ and if the solution $t\mapsto x(t,\lambda)$ exists on $[a,c]\subseteq[a,b]$, then

$$\|x(t,\lambda)-x(t,\lambda_0)\| < \frac{\eta}{4}\cdot\min\left(1,\frac{1}{k(b)-k(a)+1}\right), \quad t\in[a,c].$$

If $\lambda\in\mathbb{R}^l$ satisfies $\|\lambda-\lambda_0\|<\delta$ and the solution $t\mapsto x(t,\lambda)$ exists on $[a,c]\subseteq[a,b]$ with $c\in[a,b)$, then

$$\|\,(x(c,\lambda)+F(x(c,\lambda),c+)-F(x(c,\lambda),c)) - (x(c,\lambda_0)+F(x(c,\lambda_0),c+)-F(x(c,\lambda_0),c))\,\|$$

$$\le \|x(c,\lambda)-x(c,\lambda_0)\| + \|F(x(c,\lambda),c+)-F(x(c,\lambda),c)-F(x(c,\lambda_0),c+)+F(x(c,\lambda_0),c)\|$$

$$< \eta/4 + \|x(c,\lambda)-x(c,\lambda_0)\|(k(c+)-k(c)) < \eta/4 + \|x(c,\lambda)-x(c,\lambda_0)\|(k(b)-k(a)) < \eta/2$$

(we have used assumption 5). By Theorem 2.3, $x(c+,\lambda_0)=x(c,\lambda_0)+F(x(c,\lambda_0),c+)-F(x(c,\lambda_0),c)$. Thus the previous inequality and assumption 6 imply $x(c,\lambda)+F(x(c,\lambda),c+)-F(x(c,\lambda),c)\in B$.

All assumptions of the local existence theorem for generalized differential equations (see [8, Theorem 4.2]) are satisfied (we need (11) at this moment), and thus the solution $t\mapsto x(t,\lambda)$ can be extended to a larger interval $[a,d]$, $d\in(c,b]$. Consequently, for $\|\lambda-\lambda_0\|<\delta$, the solution must exist on the whole interval $[a,b]$.

Remark 4.3. Assume that the set B from Theorem 4.1 is convex. Then it is easy to see that assumption 5 in this theorem is redundant. Indeed, the mean-value theorem for vector-valued functions and assumption 3 lead to the estimate

$$\|F(x,\tau,t) - F(x,\tau,s) - F(y,\tau,t) + F(y,\tau,s)\|$$

$$\leq \left(\int_0^1 \|F_x(ux+(1-u)y,\tau,t) - F_x(ux+(1-u)y,\tau,s)\|\,\mathrm{d}u\right) \cdot \|x-y\| \leq |h(t)-h(s)| \cdot \|x-y\|$$

for all $s,t,\tau \in [a,b]$, $x,y \in B$, i.e., assumption 5 is satisfied with $k = h$.

With the help of Theorem 4.1, it is easy to obtain an even more general theorem for equations where not only the initial condition, but also the right-hand side of the equation depends on the parameter λ. The proof is inspired by a similar proof of Theorem 8.49 in [3].

Theorem 4.4. *Let $B \subset \mathbb{R}^n$ be an open set, $\lambda_0 \in \mathbb{R}^l$, $\rho > 0$, $\Lambda = \{\lambda \in \mathbb{R}^l; \|\lambda - \lambda_0\| < \rho\}$, $x^0 : \Lambda \to B$, $F : B \times [a,b]^2 \times \Lambda \to \mathbb{R}^n$. Assume that F is regulated and left-continuous in the third variable, and that for every $\lambda \in \Lambda$, the initial value problem*

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \mathrm{D}_t F(x,\tau,t,\lambda), \quad x(a) = x^0(\lambda)$$

has a solution in B; let $x(t,\lambda)$ be the value of that solution at $t \in [a,b]$. Moreover, let the following conditions be satisfied:

1. *For every fixed pair $(t,\tau) \in [a,b]^2$, the function $(x,\lambda) \mapsto F(x,\tau,t,\lambda)$ is continuously differentiable on $B \times \Lambda$.*
2. *The function x^0 is differentiable at λ_0.*
3. *There exists a left-continuous nondecreasing function $h : [a,b] \to \mathbb{R}$ such that*

   $$\|F_x(x,\tau,t,\lambda) - F_x(x,\tau,s,\lambda)\| \leq |h(t)-h(s)|,$$

   $$\|F_\lambda(x,\tau,t,\lambda) - F_\lambda(x,\tau,s,\lambda)\| \leq |h(t)-h(s)|$$

   for all $s,t,\tau \in [a,b]$, $x \in B$, $\lambda \in \Lambda$.
4. *There exists a continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ such that $\omega(0) = 0$ and*

   $$\|F_x(x,\tau,t,\lambda_1) - F_x(x,\tau,s,\lambda_1) - F_x(y,\tau,t,\lambda_2) + F_x(y,\tau,s,\lambda_2)\| \leq \omega(\|x-y\| + \|\lambda_1-\lambda_2\|) \cdot |h(t)-h(s)|,$$

   $$\|F_\lambda(x,\tau,t,\lambda_1) - F_\lambda(x,\tau,s,\lambda_1) - F_\lambda(y,\tau,t,\lambda_2) + F_\lambda(y,\tau,s,\lambda_2)\| \leq \omega(\|x-y\| + \|\lambda_1-\lambda_2\|) \cdot |h(t)-h(s)|$$

   for all $s,t,\tau \in [a,b]$, $x,y \in B$, $\lambda_1,\lambda_2 \in \Lambda$.
5. *There exists a left-continuous nondecreasing function $k : [a,b] \to \mathbb{R}$ such that*

   $$\|F(x,\tau,t,\lambda_1) - F(x,\tau,s,\lambda_1) - F(y,\tau,t,\lambda_2) + F(y,\tau,s,\lambda_1)\| \leq (\|x-y\| + \|\lambda_1-\lambda_2\|) \cdot |k(t)-k(s)|$$

   for all $s,t,\tau \in [a,b]$, $x,y \in B$, $\lambda_1,\lambda_2 \in \Lambda$.
6. *There exists a number $\eta > 0$ such that if $x \in \mathbb{R}^n$ satisfies $\|x - x(t,\lambda_0)\| < \eta$ for some $t \in [a,b]$, then $x \in B$ (i.e., the η-neighborhood of the solution $t \mapsto x(t,\lambda_0)$ is contained in B).*

Then the function $\lambda \mapsto x(t,\lambda)$ is differentiable at λ_0, uniformly for all $t \in [a,b]$. Moreover, its derivative $Z(t) = x_\lambda(t,\lambda_0)$, $t \in [a,b]$, is the unique solution of the generalized differential equation

$$Z(s) = x^0_\lambda(\lambda_0) + \int_a^s \mathrm{D}_t[F_x(x(\tau,\lambda_0),\tau,t,\lambda_0)Z(\tau) + F_\lambda(x(\tau,\lambda_0),\tau,t,\lambda_0)], \quad s \in [a,b].$$

Proof. Let $\tilde{B} = B \times \Lambda$. Without loss of generality, assume that all finite-dimensional spaces we are working with are equipped with the L_1 norm. In particular, when $(x, \lambda) \in \tilde{B}$, then

$$\|(x,\lambda)\| = \sum_{i=1}^{n} |x_i| + \sum_{j=1}^{l} |\lambda_j| = \|x\| + \|\lambda\|.$$

Define $\tilde{F} : \tilde{B} \times [a,b]^2 \to \mathbb{R}^{n+l}$ by

$$\tilde{F}((x,\lambda),\tau,t) = (F(x,\tau,t,\lambda),0,\ldots,0) \in \mathbb{R}^{n+l}, \quad x \in B, \ \ \lambda \in \Lambda, \ \ t,\tau \in [a,b],$$

and $y^0 : \Lambda \to \tilde{B}$ by

$$y^0(\lambda) = (x^0(\lambda),\lambda), \quad \lambda \in \Lambda.$$

From these definitions, it is clear that for every $\lambda \in \Lambda$, the function

$$y(t,\lambda) = (x(t,\lambda),\lambda), \quad t \in [a,b],$$

is a solution of the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t} y = \mathrm{D}_t \tilde{F}(y,\tau,t), \quad y(a) = y^0(\lambda)$$

(note that by the definition of $\tilde{F}$, the last l components of every solution are constant on $[a,b]$).

Without loss of generality, assume that $\eta < \rho$. If $(x,\lambda) \in \tilde{B}$ is such that $\|(x,\lambda) - (x(t,\lambda_0),\lambda_0)\| < \eta$ for some $t \in [a,b]$, then $\|x - x(t,\lambda_0)\| < \eta$ and $\|\lambda - \lambda_0\| < \eta$, i.e., $x \in B$ and $\lambda \in \Lambda$. In other words, the η-neighborhood of the solution $t \mapsto y(t,\lambda_0)$ is contained in $\overline{B}$.

The derivative of $\tilde{F}$ with respect to y is the $(n+l) \times (n+l)$ matrix

$$\tilde{F}_y(y,\tau,t) = \begin{pmatrix} \frac{\partial \tilde{F}_1}{\partial y_1}(y,\tau,t) & \cdots & \frac{\partial \tilde{F}_1}{\partial y_{n+l}}(y,\tau,t) \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{F}_{n+l}}{\partial y_1}(y,\tau,t) & \cdots & \frac{\partial \tilde{F}_{n+l}}{\partial y_{n+l}}(y,\tau,t) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(x,\tau,t,\lambda) & \cdots & \frac{\partial F_1}{\partial x_n}(x,\tau,t,\lambda) & \frac{\partial F_1}{\partial \lambda_1}(x,\tau,t,\lambda) & \cdots & \frac{\partial F_1}{\partial \lambda_l}(x,\tau,t,\lambda) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(x,\tau,t,\lambda) & \cdots & \frac{\partial F_n}{\partial x_n}(x,\tau,t,\lambda) & \frac{\partial F_n}{\partial \lambda_1}(x,\tau,t,\lambda) & \cdots & \frac{\partial F_n}{\partial \lambda_l}(x,\tau,t,\lambda) \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $y = (x,\lambda) \in B \times \Lambda$ and $t,\tau \in [a,b]$.

Similarly, the derivative of y^0 with respect to λ at λ_0 is the $(n+l) \times l$ matrix

$$y^0_\lambda(\lambda_0) = \begin{pmatrix} \frac{\partial x^0_1}{\partial \lambda_1}(\lambda_0) & \cdots & \frac{\partial x^0_1}{\partial \lambda_l}(\lambda_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial x^0_n}{\partial \lambda_1}(\lambda_0) & \cdots & \frac{\partial x^0_n}{\partial \lambda_l}(\lambda_0) \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}.$$

Using assumptions 3, 4 and 5, it is not difficult to see that $\tilde{F}$ and $\tilde{F}_y$ satisfy assumptions 3, 4 and 5 of Theorem 4.1. For example, let $s,t,\tau \in [a,b]$, $y_1,y_2 \in \tilde{B}$, where $y_1 = (x_1,\lambda_1)$ and $y_2 = (x_2,\lambda_2)$. Then

$$\|\tilde{F}_y(y_1,\tau,t) - \tilde{F}_y(y_1,\tau,s) - \tilde{F}_y(y_2,\tau,t) + \tilde{F}_y(y_2,\tau,s)\|$$

$$= \|F_x(x_1,\tau,t,\lambda_1) - F_x(x_1,\tau,s,\lambda_1) - F_x(x_2,\tau,t,\lambda_2) + F_x(x_2,\tau,s,\lambda_2)\|$$
$$+\|F_\lambda(x_1,\tau,t,\lambda_1) - F_\lambda(x_1,\tau,s,\lambda_1) - F_\lambda(x_2,\tau,t,\lambda_2) + F_\lambda(x_2,\tau,s,\lambda_2)\|$$
$$\le 2\omega(\|x_1 - x_2\| + \|\lambda_1 - \lambda_2\|) \cdot |h(t) - h(s)| = \omega(\|y_1 - y_2\|) \cdot |2h(t) - 2h(s)|,$$

which verifies assumption 4 of Theorem 4.1.

Now, according to Theorem 4.1, the function $\lambda \mapsto y(t,\lambda)$ is differentiable at λ_0, uniformly for all $t \in [a,b]$, and its derivative $\tilde{Z}(t) = y_\lambda(t,\lambda_0)$, $t \in [a,b]$, is the unique solution of the generalized differential equation

$$\tilde{Z}(s) = y^0_\lambda(\lambda_0) + \int_a^s \mathrm{D}_t[\tilde{F}_y(y(\tau,\lambda_0),\tau,t)\tilde{Z}(\tau)], \quad s \in [a,b].$$

Let $Z(t) = x_\lambda(t,\lambda_0)$, $t \in [a,b]$; observe that Z is the submatrix of $\tilde{Z}$ corresponding to the first n rows. Also, note that the last l rows of $\tilde{Z}$ form the identity matrix. Thus it follows that

$$Z(s) = x^0_\lambda(\lambda_0) + \int_a^s \mathrm{D}_t[F_x(x(\tau,\lambda_0),\tau,t,\lambda_0)Z(\tau) + F_\lambda(x(\tau,\lambda_0),\tau,t,\lambda_0)], \quad s \in [a,b]. \qquad \square$$

5 Relation to other types of equations

In this section, we show that for impulsive differential equations and for dynamic equations on time scales, differentiability of solutions with respect to initial conditions follows from our Theorem 4.1. (Similarly, it can be shown that differentiability with respect to parameters follows from Theorem 4.4). The reason is that both types of equations can be rewritten as generalized equations, whose right-hand sides do not depend on τ.

Assume that $r > 0$ is a fixed number. We restrict ourselves to the case $B = \{x \in \mathbb{R}^n;\ \|x\| < r\}$; we use $\overline{B}$ to denote the closure of B.

Lemma 5.1. *Let μ be the Lebesgue-Stieltjes measure generated by a left-continuous nondecreasing function $g : [a,b] \to \mathbb{R}$ (i.e., $\mu([c,d)) = g(d) - g(c)$ for every interval $[c,d) \subset [a,b]$). Assume that $f : \overline{B} \times [a,b] \to \mathbb{R}^{m\times n}$ satisfies the following conditions:*

- *For every $x \in \overline{B}$, the function $s \mapsto f(x,s)$ is measurable on $[a,b]$ with respect to the measure μ.*
- *There exists a μ-measurable function $M : [a,b] \to \mathbb{R}$ such that $\int_a^b M(s)\,\mathrm{d}\mu < +\infty$ and*

$$\|f(x,s)\| \le M(s), \quad x \in \overline{B}, \quad s \in [a,b].$$

- *For every $s \in [a,b]$, the function $x \mapsto f(x,s)$ is continuous in $\overline{B}$.*

Consider the function F given by

$$F(x,t) = \int_{[a,t)} f(x,s)\,\mathrm{d}\mu = \int_a^t f(x,s)\,\mathrm{d}g(s), \quad x \in \overline{B}, \quad t \in [a,b].$$

Then the following statements are true:

1. *There exists a nondecreasing left-continuous function $h : [a,b] \to \mathbb{R}$ and a continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ such that $\omega(0) = 0$ and*

$$\|F(x,t) - F(x,s)\| \le |h(t) - h(s)|, \quad s,t \in [a,b], \quad x \in \overline{B},$$

$$\|F(x,t) - F(x,s) - F(y,t) + F(y,s)\| \le \omega(\|x - y\|)|h(t) - h(s)|, \quad s,t \in [a,b], \quad x,y \in \overline{B}.$$

2. *If $x : [a,b] \to \overline{B}$, $Z : [a,b] \to \mathbb{R}^{n\times l}$ are regulated functions, then*

$$\int_a^b \mathrm{D}_t[F(x(\tau),t)Z(\tau)] = \int_a^b f(x(\tau),\tau)Z(\tau)\,\mathrm{d}g(\tau). \tag{12}$$

Proof. For the first statement, see Proposition 5.9 in [8] and the references given there.

Let us prove the second statement. According to Proposition 5.12 in [8], we have

$$\int_a^b \mathrm{D}_t F(x(\tau),t) = \int_a^b f(x(\tau),\tau)\,\mathrm{d}g(\tau) \tag{13}$$

for every regulated function $x : [a,b] \to B$. Let $[\alpha,\beta] \subseteq [a,b]$ and assume that $Z : [\alpha,\beta] \to \mathbb{R}^{n\times l}$ is constant on (α,β). Then, by (13) and Theorem 2.3,

$$\int_\alpha^\beta \mathrm{D}_t[F(x(\tau),t)Z(\tau)]$$

$$= \lim_{\varepsilon\to 0+}\left(\int_{\alpha+\varepsilon}^{\beta-\varepsilon} \mathrm{D}_t[F(x(\tau),t)Z(\tau)] + \int_\alpha^{\alpha+\varepsilon} \mathrm{D}_t[F(x(\tau),t)Z(\tau)] + \int_{\beta-\varepsilon}^{\beta} \mathrm{D}_t[F(x(\tau),t)Z(\tau)]\right)$$

$$= \lim_{\varepsilon\to 0+}\left(\int_{\alpha+\varepsilon}^{\beta-\varepsilon} f(x(\tau),\tau)Z(\tau)\,\mathrm{d}g(\tau)\right) + (F(x(\alpha),\alpha+)-F(x(\alpha),\alpha))Z(\alpha) + (F(x(\beta),\beta)-F(x(\beta),\beta-))Z(\beta)$$

$$= \lim_{\varepsilon\to 0+}\left(\int_{\alpha+\varepsilon}^{\beta-\varepsilon} f(x(\tau),\tau)Z(\tau)\,\mathrm{d}g(\tau)\right) + f(x(\alpha),\alpha)Z(\alpha)(g(\alpha+)-g(\alpha)) + f(x(\beta),\beta)Z(\beta)(g(\beta)-g(\beta-))$$

$$= \int_\alpha^\beta f(x(\tau),\tau)Z(\tau)\,\mathrm{d}g(\tau).$$

This shows that (12) is satisfied for all step functions $Z : [a,b] \to \mathbb{R}^{n\times l}$. For a general regulated function Z, let $\{Z_k\}_{k=1}^\infty$ be a sequence of step functions that is uniformly convergent to Z. Then

$$\int_a^b \mathrm{D}_t[F(x(\tau),t)Z(\tau)] = \lim_{k\to\infty}\int_a^b \mathrm{D}_t[F(x(\tau),t)Z_k(\tau)]$$

$$= \lim_{k\to\infty}\int_a^b f(x(\tau),\tau)Z_k(\tau)\,\mathrm{d}g(\tau) = \int_a^b f(x(\tau),\tau)Z(\tau)\,\mathrm{d}g(\tau),$$

where the first equality follows from Lemma 3.3 and the last equality from [8, Corollary 1.32]. □

Let us start by considering differential equations with impulses. Assume that C is an open neighborhood of $\overline{B}$, $f : C\times[a,b] \to \mathbb{R}^n$ is a continuous function whose derivative f_x exists and is continuous on $C\times[a,b]$, and $I^1,\dots,I^k : C \to \mathbb{R}^n$ are continuously differentiable functions. Then it is known (see [8, Chapter 5]) that the impulsive differential equation

$$\left.\begin{array}{rcl} x'(t) &=& f(x(t),t),\quad t\in[a,b]\backslash\{t_1,\dots,t_k\},\\ x(t_i+)-x(t_i) &=& I^i(x(t_i)),\quad i\in\{1,\dots,k\},\\ x(a) &=& x^0(\lambda), \end{array}\right\} \tag{14}$$

whose solutions are assumed to be left-continuous, is equivalent to the generalized differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \mathrm{D}_t F(x,t),\quad t\in[a,b],\qquad x(a)=x^0(\lambda),$$

where $F(x,t) = F^1(x,t) + F^2(x,t)$ and

$$F^1(x,t) = \int_a^t f(x,s)\,\mathrm{d}s,\quad F^2(x,t) = \sum_{i=1}^k I^i(x)\chi_{(t_i,\infty)}(t)$$

(the symbol χ_A denotes the characteristic function of a set $A\subset\mathbb{R}$). More precisely, $x : [a,b] \to \overline{B}$ is a solution of the impulsive equation (14) if and only if it is a solution of the generalized equation (see [8, Theorem 5.20]). Now, F^1 and F^2 are differentiable with respect to x and

$$F^1_x(x,t) = \int_a^t f_x(x,s)\,\mathrm{d}s,\quad F^2_x(x,t) = \sum_{i=1}^k I^i_x(x)\chi_{(t_i,\infty)}(t).$$

By the first part of Lemma 5.1 (where we take $g(s) = s$ and μ is the Lebesgue measure), we obtain the existence of functions $h_1 : [a, b] \to \mathbb{R}$ and $\omega_1 : [0, \infty) \to [0, \infty)$ such that

$$\|F_x^1(x, t) - F_x^1(x, s)\| \leq |h_1(t) - h_1(s)|, \quad s, t \in [a, b], \ \ x \in \overline{B},$$

$$\|F_x^1(x, t) - F_x^1(x, s) - F_x^1(y, t) + F_x^1(y, s)\| \leq \omega_1(\|x - y\|)|h_1(t) - h_1(s)|, \quad s, t \in [a, b], \ \ x, y \in \overline{B}.$$

By continuity, there exists a $K \geq 1$ such that $\|I_x^i(x)\| \leq K$ for all $x \in \overline{B}$, $i \in \{1, \dots, k\}$. Let $h_2(t) = K \sum_{i=1}^{k} \chi_{(t_i, \infty)}(t)$ for $t \in [a, b]$, and let ω_2 be the common modulus of continuity of the mappings $I^1, \dots, I^k$ on $\overline{B}$. Then a simple calculation reveals that

$$\|F_x^2(x, t) - F_x^2(x, s)\| \leq |h_2(t) - h_2(s)|, \quad s, t \in [a, b], \ \ x \in \overline{B},$$

$$\|F_x^2(x, t) - F_x^2(x, s) - F_x^2(y, t) + F_x^2(y, s)\| \leq \omega_2(\|x - y\|)|h_2(t) - h_2(s)|, \quad s, t \in [a, b], \ \ x, y \in \overline{B}.$$

Consequently, the function $F_x = F_x^1 + F_x^2$ satisfies assumptions 3, 4 of Theorem 4.1 with $h = h_1 + h_2$ and $\omega = \omega_1 + \omega_2$. By Remark 4.3, assumption 5 is satisfied with $k = h$. It follows that solutions of the impulsive equation (14) are differentiable with respect to λ, uniformly on $[a, b]$. (Actually, the whole procedure still works under weaker hypotheses on f; the crucial thing is to ensure that f and f_x satisfy the assumptions of Lemma 5.1.)

To obtain an equation for the derivative $Z(s) = x_\lambda(s, \lambda_0)$, we make use of the fact that

$$F_x(x, t) = \int_a^t \tilde{f}_x(x, s) \, \mathrm{d}g(s), \quad t \in [a, b],$$

where $g(s) = s + \sum_{i=1}^{k} \chi_{(t_i, \infty)}(s)$, and

$$\tilde{f}_x(x, t) = \begin{cases} f_x(x, t) & \text{if } t \in [a, b] \backslash \{t_1, \dots, t_k\}, \\ I_x^i(x) & \text{if } t = t_i \text{ for some } i \in \{1, \dots, k\} \end{cases}$$

(see e.g. [1, Remark 3.12]). By Theorem 4.1 and the second part of Lemma 5.1,

$$Z(s) = x_\lambda^0(\lambda_0) + \int_a^s \mathrm{D}_t[F_x(x(\tau, \lambda_0), t)Z(\tau)] = x_\lambda^0(\lambda_0) + \int_a^s \tilde{f}_x(x(\tau, \lambda_0), \tau)Z(\tau) \, \mathrm{d}g(\tau), \quad s \in [a, b].$$

This integral equation can be rewritten back (see again [1, Remark 3.12]) as the impulsive equation

$$\begin{aligned} Z'(t) &= f_x(x(t, \lambda_0), t)Z(t), \quad t \in [a, b] \backslash \{t_1, \dots, t_k\}, \\ Z(t_i+) - Z(t_i) &= I_x^i(x(t_i, \lambda_0))Z(t_i), \quad i \in \{1, \dots, k\}, \\ x(a) &= x_\lambda^0(\lambda_0), \end{aligned}$$

which agrees with the result from [6].

Next, let us turn our attention to dynamic equations on time scales. Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$, $a < b$. We use the notation $[a, b]_\mathbb{T} = [a, b] \cap \mathbb{T}$. For every $t \in [a, b]$, let

$$t^* = \inf\{s \in [a, b]_\mathbb{T}; s \geq t\}.$$

Given an arbitrary function $f : [a, b]_\mathbb{T} \to \mathbb{R}^n$, we define a function $f^* : [a, b] \to \mathbb{R}^n$ by

$$f^*(t) = f(t^*), \quad t \in [a, b].$$

Similarly, for every function $f : C \times [a, b]_\mathbb{T} \to \mathbb{R}^n$, where $C \subset \mathbb{R}^n$, let $f^* : C \times [a, b] \to \mathbb{R}^n$ be defined by

$$f^*(x, t) = f(x, t^*), \quad t \in [a, b], \ \ x \in C.$$

Assume that C is an open neighborhood of $\overline{B}$ and $f : C \times [a, b]_\mathbb{T} \to \mathbb{R}^n$ satisfies the following conditions:

- For every $t \in [a, b]_\mathbb{T}$, the function $x \mapsto f(x, t)$ is continuously differentiable on C.

- For every continuous function $x : [a,b]_{\mathbb{T}} \to \overline{B}$, the functions $t \mapsto f(x(t),t)$ and $t \mapsto f_x(x(t),t)$ are rd-continuous.
- f_x is bounded in $\overline{B} \times [a,b]_{\mathbb{T}}$.

A consequence of these conditions is that f is bounded in $\overline{B} \times [a,b]_{\mathbb{T}}$ (use the estimate $\|f(x,t)\| \le \|f(x,t) - f(0,t)\| + \|f(0,t)\|$ and apply the mean-value theorem in the first norm).

Under these assumptions, it is known (see [11, Theorem 12]) that the dynamic equation

$$x^{\Delta}(t) = f(x(t),t), \quad t \in [a,b]_{\mathbb{T}}, \qquad x(a) = x^0(\lambda) \tag{15}$$

is equivalent to the generalized ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x = \mathrm{D}_t F(x,t), \quad t \in [a,b], \qquad x(a) = x^0(\lambda), \tag{16}$$

where

$$F(x,t) = \int_a^t f^*(x,s)\,\mathrm{d}g(s),$$

and $g(s) = s^*$ for every $s \in [a,b]$. More precisely, if $x : [a,b]_{\mathbb{T}} \to \overline{B}$ is a solution of (15), then the function $x^* : [a,b] \to \overline{B}$ is a solution of (16). Conversely, every solution $y : [a,b] \to \overline{B}$ of (16) has the form $y = x^*$, where $x : [a,b]_{\mathbb{T}} \to \overline{B}$ is a solution of (15).

We have

$$F_x(x,t) = \int_a^t f_x^*(x,s)\,\mathrm{d}g(s).$$

By the first part of Lemma 5.1, there exist functions $h : [a,b] \to \mathbb{R}$ and $\omega : [0,\infty) \to [0,\infty)$ such that

$$\|F_x(x,t) - F_x(x,s)\| \le |h(t) - h(s)|, \quad s,t \in [a,b], \;\; x \in \overline{B},$$

$$\|F_x(x,t) - F_x(x,s) - F_x(y,t) + F_x(y,s)\| \le \omega(\|x-y\|)|h(t) - h(s)|, \quad s,t \in [a,b], \;\; x,y \in \overline{B}.$$

Thus, F_x satisfies assumptions 3, 4 of Theorem 4.1. By Remark 4.3, assumption 5 is satisfied with $k = h$. This means that solutions of the dynamic equation (15) are differentiable with respect to λ, uniformly on $[a,b]_{\mathbb{T}}$.

Let $Z(s) = x_\lambda(s^*, \lambda_0)$ be the corresponding derivative at λ_0. By Theorem 4.1 and the second part of Lemma 5.1,

$$Z(s) = x_\lambda^0(\lambda_0) + \int_a^s \mathrm{D}_t[F_x(x(\tau^*,\lambda_0),t)Z(\tau)] = x_\lambda^0(\lambda_0) + \int_a^s f_x^*(x(\tau^*,\lambda_0),\tau)Z(\tau)\,\mathrm{d}g(\tau), \quad s \in [a,b].$$

Consequently (see [11, Theorem 5]),

$$Z(s) = x_\lambda^0(\lambda_0) + \int_a^s f_x(x(\tau,\lambda_0),\tau)Z(\tau)\,\Delta\tau, \quad s \in [a,b]_{\mathbb{T}},$$

and therefore

$$Z^{\Delta}(t) = f_x(x(t,\lambda_0),t)Z(t), \quad t \in [a,b]_{\mathbb{T}}, \qquad Z(a) = x_\lambda^0(\lambda_0),$$

which agrees with the result obtained in [2].

6 Conclusion

Besides impulsive differential equations and dynamic equations on time scales, our differentiability results are also applicable to the so-called measure differential equations of the form

$$x(t) = x(a) + \int_a^t f(x(s),s)\,\mathrm{d}s + \int_a^t g(x(s),s)\,\mathrm{d}u(s), \quad t \in [a,b],$$

where u is a left-continuous function with bounded variation. It was shown in [8, Chapter 5] that under certain assumptions, this equation is equivalent to the generalized ordinary differential equation whose right-hand side is

$$F(x,t) = \int_a^t f(x,s)\,\mathrm{d}s + \int_a^t g(x,s)\,\mathrm{d}u(s).$$

As in the previous section, a simple application of Lemma 5.1 shows that under suitable assumptions on f and g, the hypotheses of Theorem 4.1 are satisfied, i.e., solutions of measure differential equations are differentiable with respect to initial conditions.

An interesting open question is whether the results in this paper can be extended to generalized equations whose solutions take values in infinite-dimensional Banach spaces. Numerous authors have already investigated equations of this type (see e.g. [5, 7]). For example, it is known that under certain assumptions, measure functional differential equations are equivalent to generalized ordinary differential equations with vector-valued solutions (see e.g. [1, 10] and the references there). Therefore, differentiability results for the latter type of equations would be directly applicable in the study of functional differential equations.

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