Generalized elementary functions

Giselle Antunes Monteiro* and Antonín Slavík†

Abstract

We use the theory of generalized linear differential equations to introduce new definitions of the exponential, hyperbolic and trigonometric functions. We derive some basic properties of these generalized functions, and show that the time scale elementary functions with Lebesgue integrable arguments represent a special case of our definitions.

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1 Introduction

There are many equivalent ways of introducing the classical exponential function; one possibility is to define the exponential as the unique solution of the initial-value problem $z'(t) = z(t)$, $z(0) = 1$. More generally, for every continuous function p defined on the real line, the initial-value problem $z'(t) = p(t)z(t)$, $z(t_0) = 1$, which can be written in the equivalent integral form

$$z(t) = 1 + \int_{t_0}^{t} p(s)z(s)\,\mathrm{d}s, \tag{1.1}$$

has the unique solution $z(t) = e^{\int_{t_0}^{t} p(s)\,\mathrm{d}s}$. In this paper, we replace Eq. (1.1) by the more general equation

$$z(t) = 1 + \int_{t_0}^{t} z(s)\,\mathrm{d}P(s), \tag{1.2}$$

where the integral on the right-hand side is the Kurzweil-Stieltjes integral (see the next section). We define the generalized exponential function $e_{\mathrm{d}P}$ as the unique solution of Eq. (1.2) and study its properties. Obviously, if P is continuously differentiable with $P' = p$, then Eq. (1.2) reduces back to Eq. (1.1). On the other hand, Eq. (1.2) is much more general and makes sense even if P is discontinuous. We point out that Eq. (1.2) is a generalized linear differential equation in the sense of J. Kurzweil's definition [10]. Therefore, we can use the existing theory of generalized differential equations (see [13, 14]) in our study of the generalized exponential function.

In Section 2, we start by recalling some basic facts about the Kurzweil-Stieltjes integral and generalized linear ordinary differential equations. Next, we prove an existence-uniqueness theorem for equations with complex-valued coefficients and solutions. In Section 3, we define the generalized exponential function and investigate its properties. For example, we show that the product of two exponentials $e_{\mathrm{d}P}e_{\mathrm{d}Q}$ gives the exponential of a function denoted by $P \oplus Q$; when P, Q are continuous, we have $P \oplus Q = P + Q$, but otherwise $P \oplus Q$ is more complicated and takes into account the discontinuities of P and Q. In

*Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic. E-mail: gam@math.cas.cz

†Charles University in Prague, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8, Czech Republic. E-mail: slavik@karlin.mff.cuni.cz.

Section 4, we use the exponential to introduce the generalized hyperbolic and trigonometric functions. Finally, in Section 5, we demonstrate that our generalized elementary functions include the time scale elementary functions as a special case. At the same time, we show how the usual definitions of the time scale exponential, hyperbolic and trigonometric functions can be extended from rd-continuous to Lebesgue Δ-integrable arguments.

2 Preliminaries

We need the concept of the Kurzweil-Stieltjes integral, which represents a special case of the Kurzweil integral introduced in [10] under the name "generalized Perron integral".

Consider a pair of functions $g : [a, b] \to \mathbb{R}^{n\times n}$ and $f : [a, b] \to \mathbb{R}^n$. We say that f is Kurzweil-Stieltjes integrable with respect to g, if there exists a vector $I \in \mathbb{R}^n$ such that for every $\varepsilon > 0$, there is a function $\delta : [a, b] \to \mathbb{R}^+$ such that

$$\left\| \sum_{j=1}^{m} (g(s_j) - g(s_{j-1}))f(\tau_j) - I \right\| < \varepsilon \tag{2.1}$$

for every partition of $[a, b]$ with division points $a = s_0 < s_1 < \cdots < s_m = b$ and tags $\tau_j \in [s_{j-1}, s_j]$, $j \in \{1, \dots, m\}$, satisfying

$$[s_{j-1}, s_j] \subset (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)), \quad j \in \{1, 2, \dots, m\}.$$

In this case, the vector I is called the Kurzweil-Stieltjes integral (or the Perron-Stieltjes integral), and we use the notation $\int_a^b \mathrm{d}[g]\, f = I$.

In this paper, the case when $n > 1$ is needed only in the formulation of Theorem 2.6, which is then used to prove Theorem 2.7. For $n = 1$, the order of multiplication in (2.1) does not matter, i.e.,

$$\sum_{j=1}^{m} (g(s_j) - g(s_{j-1}))f(\tau_j) = \sum_{j=1}^{m} f(\tau_j)(g(s_j) - g(s_{j-1})),$$

and we occasionally use the simpler notation $\int_a^b f\, \mathrm{d}g$ in place of $\int_a^b \mathrm{d}[g]\, f$.

Basic properties of the Kurzweil-Stieltjes integral can be found in [13, 17].

Throughout this paper, we work with regulated functions defined on a compact interval $[a, b]$ and use the following notation:

$$\Delta^+ g(t) = \begin{cases} g(t+) - g(t) & \text{if } t \in [a, b), \\ 0 & \text{if } t = b, \end{cases} \qquad \Delta^- g(t) = \begin{cases} g(t) - g(t-) & \text{if } t \in (a, b], \\ 0 & \text{if } t = a. \end{cases}$$

Also, we let $\Delta g(t) = \Delta^+ g(t) + \Delta^- g(t)$.

The following theorem describes the properties of the indefinite Kurzweil-Stieltjes integral and can be found in [17, Proposition 2.16].

Theorem 2.1. *Consider a pair of functions $f, g : [a, b] \to \mathbb{R}$ such that g is regulated and $\int_a^b f\, \mathrm{d}g$ exists. Then, for every $t_0 \in [a, b]$, the function*

$$h(t) = \int_{t_0}^{t} f\, \mathrm{d}g, \quad t \in [a, b]$$

is regulated and satisfies

$$\begin{aligned} h(t+) &= h(t) + f(t)\Delta^+ g(t), \quad t \in [a, b), \\ h(t-) &= h(t) - f(t)\Delta^- g(t), \quad t \in (a, b]. \end{aligned}$$

If $I \subset \mathbb{R}$ is an interval, $h : I \to \mathbb{R}$ is a function which is zero except a countable set $\{t_1, t_2, \dots\} \subset I$, and the sum $S = \sum_i h(t_i)$ is absolutely convergent, we use the notation $S = \sum_{x\in I} h(x)$. The next lemma is taken over from [17, Proposition 2.12].

Lemma 2.2. *Let $f : [a, b] \to \mathbb{R}$ be a function which is zero except a countable set $\{t_1, t_2, \dots\} \subset [a, b]$ and $\sum_i f(t_i)$ is absolutely convergent. Then, for every regulated function $g : [a, b] \to \mathbb{R}$, we have*

$$\int_a^b f \,\mathrm{d}g = \sum_{x\in[a,b]} f(x)\Delta g(x).$$

It is well known that every function $h : [a, b] \to \mathbb{R}$ with bounded variation has at most countably many discontinuities, and $\sum_{x\in[a,b]}(|\Delta^+ h(x)| + |\Delta^- h(x)|)$ is finite. Hence, both $f = \Delta^+ h$ and $f = \Delta^- h$ satisfy the assumptions of the previous lemma; we will use this observation later.

The following integration by parts formula for the Kurzweil-Stieltjes integral can be found in [17, Theorem 2.15].

Theorem 2.3. *If $f, g : [a, b] \to \mathbb{R}$ are regulated and at least one of them has bounded variation, then*

$$\int_a^b f \,\mathrm{d}g + \int_a^b g \,\mathrm{d}f = f(b)g(b) - f(a)g(a) + \sum_{x\in[a,b]} (\Delta^- f(x)\Delta^- g(x) - \Delta^+ f(x)\Delta^+ g(x)).$$

Remark 2.4. For our purposes, it will be sometimes more convenient to rewrite the last sum as follows:

$$\sum_{x\in[a,b]} (\Delta^- f(x)\Delta^- g(x) - \Delta^+ f(x)\Delta^+ g(x))$$

$$= \sum_{x\in[a,b]} (\Delta^- f(x)(\Delta^- g(x)+\Delta^+ g(x)) - (\Delta^+ f(x)+\Delta^- f(x))\Delta^+ g(x)) = \sum_{x\in[a,b]} (\Delta^- f(x)\Delta g(x) - \Delta f(x)\Delta^+ g(x))$$

The next substitution theorem for the Kurzweil-Stieltjes integral was proved in [17, Theorem 2.19].

Theorem 2.5. *Assume that $h : [a, b] \to \mathbb{R}$ is bounded and $f, g : [a, b] \to \mathbb{R}$ are such that $\int_a^b f \,\mathrm{d}g$ exists. Then*

$$\int_a^b h(x) \,\mathrm{d}\left[\int_a^x f(z) \,\mathrm{d}g(z)\right] = \int_a^b h(x)f(x) \,\mathrm{d}g(x),$$

whenever either side of the equation exists.

We need to extend the definition of the Kurzweil-Stieltjes integral to complex-valued functions. Given a pair of functions $f, g : [a, b] \to \mathbb{C}$ with real parts f_1, g_1 and imaginary parts f_2, g_2, we define

$$\int_a^b f \,\mathrm{d}g = \int_a^b (f_1 + if_2) \,\mathrm{d}(g_1 + ig_2) = \int_a^b f_1 \,\mathrm{d}g_1 - \int_a^b f_2 \,\mathrm{d}g_2 + i\left(\int_a^b f_1 \,\mathrm{d}g_2 + \int_a^b f_2 \,\mathrm{d}g_1\right) \tag{2.2}$$

whenever the integrals on the right-hand side exist.

All results mentioned in this section (integration by parts, substitution, etc.) are still valid for complex-valued functions. We leave the verification of this fact up to the reader: In all cases, it is enough to rewrite the integrals of complex-valued functions using the definition (2.2), then apply the corresponding "real-valued" theorem, and finally return back to integration of complex-valued functions.

The following statement is the existence and uniqueness theorem for generalized linear differential equations (see [13, Theorem 6.5] or [14, Theorem III.1.4]).

Theorem 2.6. *Consider a function $A : [a,b] \to \mathbb{R}^{n\times n}$, which has bounded variation on $[a,b]$. Let $t_0 \in [a,b]$ and assume that $I + \Delta^+ A(t)$ is invertible for every $t \in [a,t_0)$, and $I - \Delta^- A(t)$ is invertible for every $t \in (t_0,b]$. Then, for every $\tilde{x} \in \mathbb{R}^n$, there exists a unique function $x : [a,b] \to \mathbb{R}^n$ such that*

$$x(t) = \tilde{x} + \int_{t_0}^{t} \mathrm{d}[A]\, x, \quad t \in [a,b].$$

Moreover, x has bounded variation on $[a,b]$.

Before we introduce the exponential function, we need the following existence and uniqueness theorem for equations with complex-valued coefficients and solutions. The statement can be found in [7, Theorem 3.1], but without proof; for completeness, we include the proof here.

Theorem 2.7. *Consider a function $P : [a,b] \to \mathbb{C}$, which has bounded variation on $[a,b]$. Let $t_0 \in [a,b]$ and assume that $1 + \Delta^+ P(t) \neq 0$ for every $t \in [a,t_0)$, and $1 - \Delta^- P(t) \neq 0$ for every $t \in (t_0,b]$. Then, for every $\tilde{z} \in \mathbb{C}$, there exists a unique function $z : [a,b] \to \mathbb{C}$ such that*

$$z(t) = \tilde{z} + \int_{t_0}^{t} \mathrm{d}[P]\, z, \quad t \in [a,b]. \tag{2.3}$$

The function z has bounded variation on $[a,b]$. If P and $\tilde{z}$ are real, then z is real as well.

Proof. We decompose all complex quantities into real and imaginary parts as follows: $P = P_1 + iP_2$, $z = z_1 + iz_2$, and $\tilde{z} = \tilde{z}_1 + i\tilde{z}_2$. Now, we see that equation (2.3) is equivalent to the following system of two equations with real coefficients:

$$\begin{aligned} z_1(t) &= \tilde{z}_1 + \int_{t_0}^{t} \mathrm{d}[P_1]\, z_1 - \int_{t_0}^{t} \mathrm{d}[P_2]\, z_2 \\ z_2(t) &= \tilde{z}_2 + \int_{t_0}^{t} \mathrm{d}[P_1]\, z_2 + \int_{t_0}^{t} \mathrm{d}[P_2]\, z_1 \end{aligned}$$

The system can be also written in the vector form

$$u(t) = \tilde{u} + \int_{t_0}^{t} \mathrm{d}[A] u, \quad t \in [a,b] \tag{2.4}$$

with $\tilde{u} = (\tilde{z}_1, \tilde{z}_2)$,

$$u(t) = (z_1(t), z_2(t)), \quad A(t) = \begin{pmatrix} P_1(t) & -P_2(t) \\ P_2(t) & P_1(t) \end{pmatrix}, \quad t \in [a,b].$$

Since P has bounded variation on $[a,b]$, it is clear that A has the same property. The condition $1 + \Delta^+ P(t) \neq 0$ implies

$$1 + \Delta^+ P_1(t) \neq 0 \ \text{ or } \ \Delta^+ P_2(t) \neq 0, \quad t \in [a,t_0),$$

and similarly $1 - \Delta^- P(t) \neq 0$ implies

$$1 - \Delta^- P_1(t) \neq 0 \ \text{ or } \ \Delta^- P_2(t) \neq 0, \quad t \in (t_0,b].$$

In view of this, we have

$$\det(I + \Delta^+ A(t)) = (1 + \Delta^+ P_1(t))^2 + (\Delta^+ P_2(t))^2 \neq 0, \quad t \in [a,t_0),$$

$$\det(I - \Delta^- A(t)) = (1 - \Delta^- P_1(t))^2 + (\Delta^- P_2(t))^2 \neq 0, \quad t \in (t_0,b].$$

Hence, existence and uniqueness of solution to the equation (2.3) follows from Theorem 2.6.

If P and $\tilde{z}$ are real, the equation for z_2 simplifies to $z_2(t) = \int_{t_0}^{t} \mathrm{d}[P_1]\, z_2$, whose solution is identically zero and therefore z is real. □

3 Exponential function

We are now ready to introduce the generalized exponential function. Throughout the rest of this paper, we work with a fixed compact interval $[a,b] \subset \mathbb{R}$. The whole theory still works with other types of bounded intervals (open, half-open), as well as unbounded ones; in the latter case, bounded variation functions should be replaced by functions with locally bounded variation.

Definition 3.1. Consider a function $P : [a,b] \to \mathbb{C}$, which has bounded variation on $[a,b]$. Let $t_0 \in [a,b]$ and assume that $1 + \Delta^+ P(t) \neq 0$ for every $t \in [a,t_0)$, and $1 - \Delta^- P(t) \neq 0$ for every $t \in (t_0,b]$. Then we define the generalized exponential function $t \mapsto e_{\mathrm{d}P}(t,t_0)$, $t \in [a,b]$, as the unique solution $z : [a,b] \to \mathbb{C}$ of the generalized linear differential equation

$$z(t) = 1 + \int_{t_0}^{t} z(s)\,\mathrm{d}P(s).$$

Note that the exponential is a real-valued function whenever P is real. Also, it is clear that for $P(s) = s$, our definition reduces to the classical exponential function: $e_{\mathrm{d}P}(t,t_0) = e^{t-t_0}$.

Our notion of the exponential function represents a special case of the fundamental matrix corresponding to a system of generalized linear differential equations; this more general concept has been studied in [13, Chapter 6]. In particular, properties 2 to 6 from the next theorem can be found in [13, Theorem 6.15]. However, the results in the rest of this section are completely new, and make a significant use of the fact that we are dealing with scalar equations only.

In each of the following statements, we assume that the function P is such that all exponentials appearing in the given identity are well defined. For example, in the fifth statement, it is necessary to assume that $1 + \Delta^+ P(t) \neq 0$ for every $t \in [a, \max(s,r))$, and $1 - \Delta^- P(t) \neq 0$ for every $t \in (\min(s,r), b]$.

Theorem 3.2. *Let $P : [a,b] \to \mathbb{C}$ be a function with bounded variation. The generalized exponential function has the following properties:*

1. *If P is constant, then $e_{\mathrm{d}P}(t,t_0) = 1$ for every $t \in [a,b]$.*
2. *$e_{\mathrm{d}P}(t,t) = 1$ for every $t \in [a,b]$.*
3. *The function $t \mapsto e_{\mathrm{d}P}(t,t_0)$ is regulated on $[a,b]$ and satisfies*

$$\begin{aligned}
\Delta^+ e_{\mathrm{d}P}(t,t_0) &= \Delta^+ P(t) e_{\mathrm{d}P}(t,t_0), \quad t \in [a,b),\\
\Delta^- e_{\mathrm{d}P}(t,t_0) &= \Delta^- P(t) e_{\mathrm{d}P}(t,t_0), \quad t \in (a,b],\\
e_{\mathrm{d}P}(t+,t_0) &= (1 + \Delta^+ P(t)) e_{\mathrm{d}P}(t,t_0), \quad t \in [a,b),\\
e_{\mathrm{d}P}(t-,t_0) &= (1 - \Delta^- P(t)) e_{\mathrm{d}P}(t,t_0), \quad t \in (a,b].
\end{aligned}$$

4. *The function $t \mapsto e_{\mathrm{d}P}(t,t_0)$ has bounded variation on $[a,b]$.*
5. *$e_{\mathrm{d}P}(t,s) e_{\mathrm{d}P}(s,r) = e_{\mathrm{d}P}(t,r)$ for every t, s, $r \in [a,b]$.*
6. *$e_{\mathrm{d}P}(t,s) = (e_{\mathrm{d}P}(s,t))^{-1}$ for every t, $s \in [a,b]$.*
7. *$\overline{e_{\mathrm{d}P}(t,t_0)} = e_{\mathrm{d}\overline{P}}(t,t_0)$ for every $t \in [a,b]$, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.*
8. *If P is continuous, then $e_{\mathrm{d}P}(t,t_0) = e^{P(t)-P(t_0)}$ for every $t \in [a,b]$.*

Proof. The first two statements are obvious. The third statement is a consequence of Theorem 2.1, and the fourth statement follows from Theorem 2.7.

To prove the fifth statement, note that, given arbitrary r, $s\in[a,b]$, we have

$$\begin{aligned} e_{\mathrm{d}P}(t,r)&=1+\int_r^t e_{\mathrm{d}P}(\tau,r)\,\mathrm{d}P(\tau)=1+\int_r^s e_{\mathrm{d}P}(\tau,r)\,\mathrm{d}P(\tau)+\int_s^t e_{\mathrm{d}P}(\tau,r)\,\mathrm{d}P(\tau)\\ &=e_{\mathrm{d}P}(s,r)+\int_s^t e_{\mathrm{d}P}(\tau,r)\,\mathrm{d}P(\tau),\end{aligned}$$

for every $t\in[a,b]$. Hence, the function $y(t)=e_{\mathrm{d}P}(t,r)$, $t\in[a,b]$, is a solution of the generalized linear differential equation

$$x(t)=\tilde{x}+\int_s^t x(s)\,\mathrm{d}P(s),\quad t\in[a,b],\quad \text{where }\ \tilde{x}=e_{\mathrm{d}P}(s,r).$$

On the other hand, it is not hard to see that $z(t)=e_{\mathrm{d}P}(t,s)\tilde{x}$, $t\in[a,b]$, is also a solution of the same equation. By the uniqueness of solutions (see Theorem 2.7), we have $y(t)=z(t)$, $t\in[a,b]$, which proves the fifth statement.

The sixth statement is a direct consequence of previous one. Indeed, for t, $s\in[a,b]$, we obtain

$$e_{\mathrm{d}P}(t,s)e_{\mathrm{d}P}(s,t)=e_{\mathrm{d}P}(t,t)=1.$$

By the definition of the exponential function, we have

$$e_{\mathrm{d}P}(t,t_0)=1+\int_{t_0}^t e_{\mathrm{d}P}(s,t_0)\,\mathrm{d}P(s).$$

Taking the complex conjugate of both sides, we get

$$\overline{e_{\mathrm{d}P}(t,t_0)}=1+\int_{t_0}^t \overline{e_{\mathrm{d}P}(s,t_0)}\,\mathrm{d}\overline{P}(s),$$

which proves the seventh statement.

In the eighth statement, assume first that P is continuously differentiable with $P'=p$. The function $z(t)=e_{\mathrm{d}P}(t,t_0)$ is the unique solution of the equation

$$z(t)=1+\int_{t_0}^t z(s)\,\mathrm{d}P(s)=1+\int_{t_0}^t z(s)\,\mathrm{d}\left(P(t_0)+\int_{t_0}^s p(s)\,\mathrm{d}s\right)=1+\int_{t_0}^t z(s)p(s)\,\mathrm{d}s$$

(we have used the substitution theorem). Differentiation of the equality above gives $z'(t)=z(t)p(t)$ and the unique solution of this equation satisfying $z(t_0)=1$ is given by $z(t)=e^{\int_{t_0}^t p(s)\,\mathrm{d}s}=e^{P(t)-P(t_0)}$.

Now, consider the general case when P is merely continuous. Because P has bounded variation, we can write $P=P^1-P^2$, where P^1,P^2 are nondecreasing and continuous. For $j\in\{1,2\}$, there exists a sequence $\{P_n^j\}_{n=1}^\infty$ of nondecreasing polynomials which is uniformly convergent to P^j. (For example, in the well-known constructive proof of the Weierstrass approximation theorem involving the Bernstein polynomials, the approximating polynomials corresponding to a nondecreasing function are nondecreasing; see [4, Theorem 6.3.3].) For every $n\in\mathbb{N}$, the function $P_n=P_n^1-P_n^2$ satisfies $e_{\mathrm{d}P_n}(t,t_0)=e^{P_n(t)-P_n(t_0)}$. Hence, $\lim_{n\to\infty}e_{\mathrm{d}P_n}(t,t_0)=e^{P(t)-P(t_0)}$. On the other hand, thanks to the monotonicity and uniform convergence, the functions P_n^1 and P_n^2, $n\in\mathbb{N}$, have uniformly bounded variations, and the same holds for the functions P_n, $n\in\mathbb{N}$. Thus, applying the continuous dependence theorem for generalized linear differential equations (see [7, Theorem 4.1] or [11, Theorem 3.4]), we have $\lim_{n\to\infty}e_{\mathrm{d}P_n}(t,t_0)=e_{\mathrm{d}P}(t,t_0)$, which completes the proof. □

In the next theorem, we show that the product of two exponential functions again leads to an exponential function.

Theorem 3.3. *Assume that $P, Q : [a,b] \to \mathbb{C}$ have bounded variation, $(1+\Delta^+P(t))(1+\Delta^+Q(t)) \neq 0$ for every $t \in [a,t_0)$, and $(1-\Delta^-P(t))(1-\Delta^-Q(t)) \neq 0$ for every $t \in (t_0,b]$. Then*

$$e_{\mathrm{d}P}(t,t_0)e_{\mathrm{d}Q}(t,t_0) = e_{\mathrm{d}(P\oplus Q)}(t,t_0), \quad t \in [a,b],$$

where

$$(P \oplus Q)(t) = P(t) + Q(t) + \int_{t_0}^{t} \Delta^+Q(s)\,\mathrm{d}P(s) - \int_{t_0}^{t} \Delta^-P(s)\,\mathrm{d}Q(s),$$

or equivalently

$$(P \oplus Q)(t) = P(t) + Q(t) + \sum_{s\in[t_0,t)} \Delta^+Q(s)\Delta^+P(s) - \sum_{s\in(t_0,t]} \Delta^-Q(s)\Delta^-P(s).$$

Proof. By Lemma 2.2, we have

$$\int_{t_0}^{t} \Delta^+Q(s)\,\mathrm{d}P(s) - \int_{t_0}^{t} \Delta^-P(s)\,\mathrm{d}Q(s) = \sum_{s\in[t_0,t]} (\Delta^+Q(s)\Delta P(s) - \Delta^-P(s)\Delta Q(s))$$

$$= \sum_{s\in[t_0,t]} (\Delta^+Q(s)(\Delta^+P(s) + \Delta^-P(s)) - \Delta^-P(s)(\Delta^+Q(s) + \Delta^-Q(s)))$$

$$= \sum_{s\in[t_0,t]} (\Delta^+Q(s)\Delta^+P(s) - \Delta^-P(s)\Delta^-Q(s)).$$

(For $t_0 > t$, all sums of the form $\sum_{s\in[t_0,t]} h(s)$ should be interpreted as $-\sum_{s\in[t,t_0]} h(s)$.) According to our convention, $\Delta^+Q(t) = \Delta^+P(t) = 0$ and $\Delta^-Q(t_0) = \Delta^-P(t_0) = 0$. Hence, the two definitions of $P \oplus Q$ are equivalent.

For $t \in [a,b]$, let

$$R(t) = \sum_{s\in(t_0,t]} \Delta^-Q(s)\Delta^-P(s) \quad \text{and} \quad T(t) = \sum_{s\in[t_0,t)} \Delta^+Q(s)\Delta^+P(s).$$

These functions have bounded variation and

$$\Delta^-R(t) = \Delta^-Q(t)\Delta^-P(t), \quad \Delta^+R(t) = 0,$$
$$\Delta^-T(t) = 0, \quad \Delta^+T(t) = \Delta^+Q(t)\Delta^+P(t).$$

In view of this, it is clear that $P \oplus Q$ has bounded variation on $[a,b]$. Moreover,

$$\begin{aligned} 1 - \Delta^-(P\oplus Q)(t) &= 1 - \Delta^-P(t) - \Delta^-Q(t) - \Delta^-T(t) + \Delta^-R(t) \\ &= 1 - \Delta^-P(t) - \Delta^-Q(t) + \Delta^-Q(t)\Delta^-P(t) \\ &= \big(1 - \Delta^-P(t)\big)\big(1 - \Delta^-Q(t)\big) \neq 0, \end{aligned} \tag{3.1}$$

for $t \in (t_0,b]$. Proceeding in a similar way, we can show that

$$1 + \Delta^+(P\oplus Q)(t) = \big(1+\Delta^+P(t)\big)\big(1+\Delta^+Q(t)\big) \neq 0, \quad t \in [a,t_0). \tag{3.2}$$

Therefore, the exponential function $t \mapsto e_{\mathrm{d}(P\oplus Q)}(t,t_0)$ is well defined.

For $t \in [a,b]$, integration by parts gives

$$e_{\mathrm{d}P}(t,t_0)e_{\mathrm{d}Q}(t,t_0) = e_{\mathrm{d}P}(t_0,t_0)e_{\mathrm{d}Q}(t_0,t_0) + \int_{t_0}^{t} e_{\mathrm{d}P}(s,t_0)\,\mathrm{d}[e_{\mathrm{d}Q}(s,t_0)] + \int_{t_0}^{t} e_{\mathrm{d}Q}(s,t_0)\,\mathrm{d}[e_{\mathrm{d}P}(s,t_0)]$$

$$+\sum_{s\in[t_0,t]}(\Delta e_{\mathrm{d}P}(s,t_0)\Delta^+ e_{\mathrm{d}Q}(s,t_0)-\Delta^- e_{\mathrm{d}P}(s,t_0)\Delta e_{\mathrm{d}Q}(s,t_0)).$$

Using the substitution theorem, we have

$$\int_{t_0}^t e_{\mathrm{d}P}(s,t_0)\,\mathrm{d}[e_{\mathrm{d}Q}(s,t_0)]=\int_{t_0}^t e_{\mathrm{d}P}(s,t_0)\,\mathrm{d}\left[1+\int_{t_0}^s e_{\mathrm{d}Q}(u,t_0)\,\mathrm{d}Q(u)\right]=\int_{t_0}^t e_{\mathrm{d}P}(s,t_0)e_{\mathrm{d}Q}(s,t_0)\,\mathrm{d}Q(s),$$

$$\int_{t_0}^t e_{\mathrm{d}Q}(s,t_0)\,\mathrm{d}[e_{\mathrm{d}P}(s,t_0)]=\int_{t_0}^t e_{\mathrm{d}Q}(s,t_0)\,\mathrm{d}\left[1+\int_{t_0}^s e_{\mathrm{d}P}(u,t_0)\,\mathrm{d}P(u)\right]=\int_{t_0}^t e_{\mathrm{d}Q}(s,t_0)e_{\mathrm{d}P}(s,t_0)\,\mathrm{d}P(s).$$

Also, Lemma 2.2, Theorem 3.2 and the substitution theorem imply

$$\sum_{s\in[t_0,t]}(\Delta e_{\mathrm{d}P}(s,t_0)\Delta^+ e_{\mathrm{d}Q}(s,t_0)-\Delta^- e_{\mathrm{d}P}(s,t_0)\Delta e_{\mathrm{d}Q}(s,t_0))$$

$$=\int_{t_0}^t \Delta^+ e_{\mathrm{d}Q}(s,t_0)\,\mathrm{d}[e_{\mathrm{d}P}(s,t_0)]-\int_{t_0}^t \Delta^- e_{\mathrm{d}P}(s,t_0)\,\mathrm{d}[e_{\mathrm{d}Q}(s,t_0)]$$

$$=\int_{t_0}^t \Delta^+ Q(s)e_{\mathrm{d}Q}(s,t_0)e_{\mathrm{d}P}(s,t_0)\,\mathrm{d}P(s)-\int_{t_0}^t \Delta^- P(s)e_{\mathrm{d}P}(s,t_0)e_{\mathrm{d}Q}(s,t_0)\,\mathrm{d}Q(s)$$

$$=\int_{t_0}^t e_{\mathrm{d}Q}(s,t_0)e_{\mathrm{d}P}(s,t_0)\,\mathrm{d}\left[\int_{t_0}^s \Delta^+ Q(u)\,\mathrm{d}P(u)-\int_{t_0}^s \Delta^- P(u)\,\mathrm{d}Q(u)\right].$$

By combining the previous results, we obtain

$$e_{\mathrm{d}P}(t,t_0)e_{\mathrm{d}Q}(t,t_0)=1+\int_{t_0}^t e_{\mathrm{d}P}(s,t_0)e_{\mathrm{d}Q}(s,t_0)\,\mathrm{d}\left[P(s)+Q(s)+\int_{t_0}^s \Delta^+ Q(u)\,\mathrm{d}P(u)-\int_{t_0}^s \Delta^- P(u)\,\mathrm{d}Q(u)\right],$$

which proves the relation $e_{\mathrm{d}P}(t,t_0)e_{\mathrm{d}Q}(t,t_0)=e_{\mathrm{d}(P\oplus Q)}(t,t_0)$. □

Similarly, the reciprocal value of an exponential function is again an exponential function.

Theorem 3.4. *Assume that $P:[a,b]\to\mathbb{C}$ has bounded variation, $1+\Delta^+P(t)\neq 0$ for every $t\in[a,b)$, and $1-\Delta^-P(t)\neq 0$ for every $t\in(a,b]$. Then*

$$(e_{\mathrm{d}P}(t,t_0))^{-1}=e_{\mathrm{d}(\ominus P)}(t,t_0),\quad t\in[a,b],$$

where

$$(\ominus P)(t)=-P(t)+\sum_{s\in[t_0,t)}\frac{(\Delta^+P(s))^2}{1+\Delta^+P(s)}-\sum_{s\in(t_0,t]}\frac{(\Delta^-P(s))^2}{1-\Delta^-P(s)}.$$

Proof. For $t\in[a,b]$, let

$$R_1(t)=\sum_{s\in[t_0,t)}\frac{(\Delta^+P(s))^2}{1+\Delta^+P(s)},\quad R_2(t)=\sum_{s\in(t_0,t]}\frac{(\Delta^-P(s))^2}{1-\Delta^-P(s)}.$$

These functions have bounded variation on $[a,b]$ and satisfy

$$\Delta^-R_1(t)=0,\quad \Delta^+R_1(t)=\frac{(\Delta^+P(t))^2}{1+\Delta^+P(t)},\quad \Delta^+R_2(t)=0,\quad \Delta^-R_2(t)=\frac{(\Delta^-P(t))^2}{1-\Delta^-P(t)}. \tag{3.3}$$

Thus, $\ominus P$ has bounded variation on $[a,b]$ and

$$1+\Delta^+(\ominus P)(t)=1-\Delta^+P(t)+\frac{(\Delta^+P(t))^2}{1+\Delta^+P(t)}=\frac{1}{1+\Delta^+P(t)}\neq 0,\quad t\in[a,t_0), \tag{3.4}$$

$$1-\Delta^-(\ominus P)(t)=1+\Delta^-P(t)+\frac{(\Delta^-P(t))^2}{1-\Delta^-P(t)}=\frac{1}{1-\Delta^-P(t)}\neq 0,\quad t\in(t_0,b], \tag{3.5}$$

which implies that the exponential function $e_{\mathrm{d}(\ominus P)}$ is well defined.

Using the relations (3.3) together with the definition of $\oplus$ given in Theorem 3.3, we obtain

$$\begin{aligned}(P\oplus(\ominus P))(t)&=P(t)-P(t)+R_1(t)-R_2(t)+\sum_{s\in[t_0,t)}\Delta^+(-P+R_1-R_2)(s)\Delta^+P(s)\\&\quad-\sum_{s\in(t_0,t]}\Delta^-(-P+R_1-R_2)(s)\Delta^-P(s)=R_1(t)-R_2(t)\\&\quad+\sum_{s\in[t_0,t)}\left(-(\Delta^+P(s))^2+\frac{(\Delta^+P(s))^3}{1+\Delta^+P(s)}\right)-\sum_{s\in(t_0,t]}\left(-(\Delta^-P(s))^2-\frac{(\Delta^-P(s))^3}{1-\Delta^-P(s)}\right)\\&=R_1(t)-R_2(t)-\sum_{s\in[t_0,t)}\frac{(\Delta^+P(s))^2}{1+\Delta^+P(s)}+\sum_{s\in(t_0,t]}\frac{(\Delta^-P(s))^2}{1-\Delta^-P(s)}=0.\end{aligned}$$

Now, it follows from Theorems 3.3 and 3.2 that $e_{\mathrm{d}P}(t,t_0)e_{\mathrm{d}(\ominus P)}(t,t_0)=e_{\mathrm{d}(P\oplus(\ominus P))}(t,t_0)=1$. □

The preceding two theorems have the following algebraic interpretation: Let $BV_*([a,b],\mathbb{R})$ be the class consisting of all functions $P:[a,b]\to\mathbb{R}$ that have bounded variation on $[a,b]$ and satisfy $1+\Delta^+P(t)\neq 0$ for every $t\in[a,b)$, and $1-\Delta^-P(t)\neq 0$ for every $t\in(a,b]$. Given a pair of functions $P,Q\in BV_*([a,b],\mathbb{R})$, write $P\sim Q$ if and only if $P-Q$ is a constant function. Clearly, the relation $\sim$ is an equivalence. Now, the quotient set $BV_*([a,b],\mathbb{R})/\sim$ is a commutative group equipped with the binary operation $\oplus$. The zero element of this group is the class of all constant functions, and inverse elements are obtained using the $\ominus$ operation. Moreover, it is not difficult to verify that

$$\begin{aligned}((P\oplus Q)\oplus R)(t)&=(P\oplus(Q\oplus R))(t)=P(t)+Q(t)+R(t)\\&+\sum_{s\in[t_0,t)}\big(\Delta^+Q(s)\Delta^+P(s)+\Delta^+R(s)\Delta^+P(s)+\Delta^+Q(s)\Delta^+R(s)\big)+\sum_{s\in[t_0,t)}\Delta^+P(s)\Delta^+Q(s)\Delta^+R(s)\\&-\sum_{s\in(t_0,t]}\big(\Delta^-Q(s)\Delta^-P(s)+\Delta^-R(s)\Delta^-P(s)+\Delta^-Q(s)\Delta^-R(s)\big)+\sum_{s\in(t_0,t]}\Delta^-P(s)\Delta^-Q(s)\Delta^-R(s),\end{aligned}$$

i.e., the operation $\oplus$ is associative.

The next theorem provides some information about the sign of the exponential function.

Theorem 3.5. *Consider a function $P:[a,b]\to\mathbb{R}$, which has bounded variation on $[a,b]$ and satisfies $1+\Delta^+P(t)\neq 0$ for every $t\in[a,b)$, and $1-\Delta^-P(t)\neq 0$ for every $t\in(a,b]$. Then, for every $t_0\in[a,b]$, the following statements hold:*

1. *$e_{\mathrm{d}P}(t,t_0)\neq 0$ for all $t\in[a,b]$.*
2. *If $1+\Delta^+P(t)<0$, then $e_{\mathrm{d}P}(t,t_0)e_{\mathrm{d}P}(t+,t_0)<0$.*
3. *If $1-\Delta^-P(t)<0$, then $e_{\mathrm{d}P}(t,t_0)e_{\mathrm{d}P}(t-,t_0)<0$.*
4. *If $1+\Delta^+P(t)>0$ and $1-\Delta^-P(t)>0$, then $t\mapsto e_{\mathrm{d}P}(t,t_0)$ does not change sign in the neighborhood of t.*

Proof. If $e_{\mathrm{d}P}(t,t_0)=0$ for a certain $t\in[a,b]$, we can use Theorem 3.2 to obtain $1=e_{\mathrm{d}P}(t_0,t_0)=e_{\mathrm{d}P}(t_0,t)e_{\mathrm{d}P}(t,t_0)=0$, which is a contradiction. The second and third statement follow immediately from the third part of Theorem 3.2. Finally, if $1+\Delta^+P(t)>0$ and $1-\Delta^-P(t)>0$, then $e_{\mathrm{d}P}(t+,t_0)$ and $e_{\mathrm{d}P}(t-,t_0)$ have the same sign as $e_{\mathrm{d}P}(t,t_0)$, which proves the fourth statement. □

According to the previous theorem, the exponential function changes sign at all points t such that $1+\Delta^+P(t)<0$ or $1-\Delta^-P(t)<0$. Since P has bounded variation, we conclude that every finite interval can contain only a finite number of points where the exponential function changes its sign.

In view of Theorem 3.5, it makes sense to introduce the class $BV_+([a,b],\mathbb{R})$ consisting of all functions $P:[a,b]\to\mathbb{R}$ that have bounded variation on $[a,b]$ and satisfy $1+\Delta^+P(t)>0$ for every $t\in[a,b)$, and $1-\Delta^-P(t)>0$ for every $t\in(a,b]$.

Theorem 3.6. *The elements of $BV_+([a,b],\mathbb{R})$ have the following properties:*

1. *If $P\in BV_+([a,b],\mathbb{R})$, then $e_{\mathrm{d}P}(t,t_0)>0$ for all $t,t_0\in[a,b]$.*
2. *If $P,Q\in BV_+([a,b],\mathbb{R})$, then $P\oplus Q\in BV_+([a,b],\mathbb{R})$.*
3. *If $P\in BV_+([a,b],\mathbb{R})$, then $\ominus P\in BV_+([a,b],\mathbb{R})$.*

Proof. The first statement follows from the fourth part of Theorem 3.5 and the fact that $e_{\mathrm{d}P}(t_0,t_0)$ is positive. The second statement is a consequence of the formulas (3.1) and (3.2), which were obtained in the proof of Theorem 3.3:

$$1-\Delta^-(P\oplus Q)(t)=\big(1-\Delta^-P(t)\big)\big(1-\Delta^-Q(t)\big),$$

$$1+\Delta^+(P\oplus Q)(t)=\big(1+\Delta^+P(t)\big)\big(1+\Delta^+Q(t)\big),$$

Similarly, the third statement is a consequence of the formulas (3.4) and (3.5) from the proof of Theorem 3.4:

$$1+\Delta^+(\ominus P)(t)=\frac{1}{1+\Delta^+P(t)},\quad 1-\Delta^-(\ominus P)(t)=\frac{1}{1-\Delta^-P(t)}.\qquad\square$$

According to the previous theorem, the set $BV_+([a,b],\mathbb{R})/\sim$ is a subgroup of $BV_*([a,b],\mathbb{R})/\sim$.

4 Hyperbolic and trigonometric functions

Definition 4.1. Consider a function $P:[a,b]\to\mathbb{C}$, which has bounded variation on $[a,b]$. Let $t_0\in[a,b]$ and assume that $1-(\Delta^+P(t))^2\neq 0$ for every $t\in[a,t_0)$, and $1-(\Delta^-P(t))^2\neq 0$ for every $t\in(t_0,b]$. Then we define the hyperbolic functions $t\mapsto\cosh_{\mathrm{d}P}(t,t_0)$ and $t\mapsto\sinh_{\mathrm{d}P}(t,t_0)$, $t\in[a,b]$, by the formulas

$$\cosh_{\mathrm{d}P}(t,t_0)=\frac{e_{\mathrm{d}P}(t,t_0)+e_{\mathrm{d}(-P)}(t,t_0)}{2},\quad \sinh_{\mathrm{d}P}(t,t_0)=\frac{e_{\mathrm{d}P}(t,t_0)-e_{\mathrm{d}(-P)}(t,t_0)}{2}.$$

Note that the condition $1-(\Delta^+P(t))^2\neq 0$ is equivalent to $(1+\Delta^+P(t))(1+\Delta^+(-P)(t))\neq 0$, and $1-(\Delta^-P(t))^2\neq 0$ is equivalent to $(1-\Delta^-P(t))(1-\Delta^-(-P)(t))\neq 0$. Therefore, $e_{\mathrm{d}P}$ and $e_{\mathrm{d}(-P)}$ are well defined.

Obviously, the two hyperbolic functions are real if P is real, and for $P(s)=s$, we obtain the classical hyperbolic functions: $\cosh_{\mathrm{d}P}(t,t_0)=\cosh(t-t_0)$, $\sinh_{\mathrm{d}P}(t,t_0)=\sinh(t-t_0)$. More generally, if P is continuous, then $\cosh_{\mathrm{d}P}(t,t_0)=\cosh(P(t)-P(t_0))$ and $\sinh_{\mathrm{d}P}(t,t_0)=\sinh(P(t)-P(t_0))$.

Theorem 4.2. *Let $P:[a,b]\to\mathbb{C}$ be a bounded variation function satisfying $1-(\Delta^+P(t))^2\neq 0$ for every $t\in[a,t_0)$, and $1-(\Delta^-P(t))^2\neq 0$ for every $t\in(t_0,b]$. The generalized hyperbolic functions have the following properties:*

1. $\cosh_{\mathrm{d}P}(t_0,t_0)=1$*,* $\sinh_{\mathrm{d}P}(t_0,t_0)=0$*.*
2. $\cosh_{\mathrm{d}P}(t,t_0)=1+\int_{t_0}^t\sinh_{\mathrm{d}P}(s,t_0)\,\mathrm{d}P(s)$*,* $t\in[a,b]$*.*
3. $\sinh_{\mathrm{d}P}(t,t_0)=\int_{t_0}^t\cosh_{\mathrm{d}P}(s,t_0)\,\mathrm{d}P(s)$*,* $t\in[a,b]$*.*

4. $\cosh^2_{\mathrm{d}P}(t,t_0) - \sinh^2_{\mathrm{d}P}(t,t_0) = e_{\mathrm{d}Q}(t,t_0)$*, $t \in [a,b]$, where*

$$Q(t) = (P \oplus (-P))(t) = \int_{t_0}^{t} (\Delta^- P(s) - \Delta^+ P(s))\,\mathrm{d}P(s) = \sum_{s\in(t_0,t]} (\Delta^- P(s))^2 - \sum_{s\in[t_0,t)} (\Delta^+ P(s))^2.$$

Proof. The first statement is obvious. Using the definition of the exponential function, we obtain

$$\cosh_{\mathrm{d}P}(t,t_0) = \frac{1}{2}\left(1 + \int_{t_0}^{t} e_{\mathrm{d}P}(s,t_0)\,\mathrm{d}P(s) + 1 + \int_{t_0}^{t} e_{\mathrm{d}(-P)}(s,t_0)\,\mathrm{d}(-P)(s)\right)$$

$$= 1 + \frac{1}{2}\int_{t_0}^{t} (e_{\mathrm{d}P}(s,t_0) - e_{\mathrm{d}(-P)}(s,t_0))\,\mathrm{d}P(s) = 1 + \int_{t_0}^{t} \sinh_{\mathrm{d}P}(s,t_0)\,\mathrm{d}P(s).$$

The third identity can be proved similarly. To verify the fourth one, observe that

$$\cosh^2_{\mathrm{d}P}(t,t_0) - \sinh^2_{\mathrm{d}P}(t,t_0) = \left(\frac{e_{\mathrm{d}P}(t,t_0) + e_{\mathrm{d}(-P)}(t,t_0)}{2}\right)^2 - \left(\frac{e_{\mathrm{d}P}(t,t_0) - e_{\mathrm{d}(-P)}(t,t_0)}{2}\right)^2$$

$$= e_{\mathrm{d}P}(t,t_0)e_{\mathrm{d}(-P)}(t,t_0) = e_{\mathrm{d}(P\oplus(-P))}(t,t_0).$$

From Theorem 3.3, we have

$$(P \oplus (-P))(t) = -\int_{t_0}^{t} \Delta^+ P(s)\,\mathrm{d}P(s) + \int_{t_0}^{t} \Delta^- P(s)\,\mathrm{d}P(s) = -\sum_{s\in[t_0,t)} (\Delta^+ P(s))^2 + \sum_{s\in(t_0,t]} (\Delta^- P(s))^2.$$

□

Definition 4.3. Consider a function $P : [a,b] \to \mathbb{C}$, which has bounded variation on $[a,b]$. Let $t_0 \in [a,b]$ and assume that $1 + (\Delta^+ P(t))^2 \neq 0$ for every $t \in [a,t_0)$, and $1 + (\Delta^- P(t))^2 \neq 0$ for every $t \in (t_0,b]$. Then we define the generalized trigonometric functions $t \mapsto \cos_{\mathrm{d}P}(t,t_0)$ and $t \mapsto \sin_{\mathrm{d}P}(t,t_0)$, $t \in [a,b]$, by the formulas

$$\begin{aligned} \cos_{\mathrm{d}P}(t,t_0) &= \frac{e_{\mathrm{d}(iP)}(t,t_0) + e_{\mathrm{d}(-iP)}(t,t_0)}{2} = \cosh_{\mathrm{d}(iP)}(t,t_0), \\ \sin_{\mathrm{d}P}(t,t_0) &= \frac{e_{\mathrm{d}(iP)}(t,t_0) - e_{\mathrm{d}(-iP)}(t,t_0)}{2i} = -i\sinh_{\mathrm{d}(iP)}(t,t_0). \end{aligned}$$

Note that the condition $1 + (\Delta^+ P(t))^2 \neq 0$ is equivalent to $(1 + \Delta^+(iP)(t))(1 + \Delta^+(-iP)(t)) \neq 0$, and $1 + (\Delta^- P(t))^2 \neq 0$ is equivalent to $(1 - \Delta^-(iP)(t))(1 - \Delta^-(-iP)(t)) \neq 0$. Therefore, $e_{\mathrm{d}(iP)}$ and $e_{\mathrm{d}(-iP)}$ are well defined. When P is a real function, both conditions are always satisfied.

Again, it is easy to see that for $P(s) = s$, our definitions coincide with the classical trigonometric functions: $\cos_{\mathrm{d}P}(t,t_0) = \cos(t-t_0)$, $\sin_{\mathrm{d}P}(t,t_0) = \sin(t-t_0)$. Also, if P is continuous, then $\cos_{\mathrm{d}P}(t,t_0) = \cos(P(t) - P(t_0))$ and $\sin_{\mathrm{d}P}(t,t_0) = \sin(P(t) - P(t_0))$.

If P is real, the trigonometric functions are real as well: By the seventh statement of Theorem 3.2, $e_{\mathrm{d}(iP)} + e_{\mathrm{d}(-iP)} = e_{\mathrm{d}(iP)} + \overline{e_{\mathrm{d}(iP)}}$, which is purely real. Similarly, $e_{\mathrm{d}(iP)} - e_{\mathrm{d}(-iP)} = e_{\mathrm{d}(iP)} - \overline{e_{\mathrm{d}(iP)}}$, which is purely imaginary.

Theorem 4.4. *Let $P : [a,b] \to \mathbb{C}$ be a bounded variation function satisfying $1 + (\Delta^+ P(t))^2 \neq 0$ for every $t \in [a,t_0)$, and $1 + (\Delta^- P(t))^2 \neq 0$ for every $t \in (t_0,b]$. The generalized trigonometric functions have the following properties:*

1. $\cos_{\mathrm{d}P}(t_0,t_0) = 1$, $\sin_{\mathrm{d}P}(t_0,t_0) = 0$.

2. $\cos_{\mathrm{d}P}(t,t_0) = 1 - \int_{t_0}^{t} \sin_{\mathrm{d}P}(s,t_0)\,\mathrm{d}P(s)$, $t \in [a,b]$.

3. $\sin_{\mathrm{d}P}(t,t_0)=\int_{t_0}^t \cos_{\mathrm{d}P}(s,t_0)\,\mathrm{d}P(s)$, $t\in[a,b]$.

4. $\cos^2_{\mathrm{d}P}(t,t_0)+\sin^2_{\mathrm{d}P}(t,t_0)=e_{\mathrm{d}Q}(t,t_0)$, $t\in[a,b]$, *where*

$$Q(t)=(iP\oplus(-iP))(t)=\int_{t_0}^t(\Delta^+P(s)-\Delta^-P(s))\,\mathrm{d}P(s)=\sum_{s\in[t_0,t)}(\Delta^+P(s))^2-\sum_{s\in(t_0,t]}(\Delta^-P(s))^2.$$

Proof. The first statement is obvious. Using the definition of the exponential function, we get

$$\begin{aligned}\cos_{\mathrm{d}P}(t,t_0)&=\frac{1}{2}\left(1+\int_{t_0}^t e_{\mathrm{d}(iP)}(s,t_0)\,\mathrm{d}[iP(s)]+1+\int_{t_0}^t e_{\mathrm{d}(-iP)}(s,t_0)\,\mathrm{d}[-iP(s)]\right)\\&=1+\frac{i}{2}\int_{t_0}^t(e_{\mathrm{d}(iP)}(s,t_0)-e_{\mathrm{d}(-iP)}(s,t_0))\,\mathrm{d}P(s)\\&=1+\frac{i}{2}\int_{t_0}^t(2i\,\sin_{\mathrm{d}P}(s,t_0))\,\mathrm{d}P(s)=1-\int_{t_0}^t\sin_{\mathrm{d}P}(s,t_0)\,\mathrm{d}P(s).\end{aligned}$$

Similarly, we can obtain the third identity. To prove the last identity, we observe that

$$\begin{aligned}\cos^2_{\mathrm{d}P}(t,t_0)+\sin^2_{\mathrm{d}P}(t,t_0)&=\left(\frac{e_{\mathrm{d}(iP)}(t,t_0)+e_{\mathrm{d}(-iP)}(t,t_0)}{2}\right)^2+\left(\frac{e_{\mathrm{d}(iP)}(t,t_0)-e_{\mathrm{d}(-iP)}(t,t_0)}{2i}\right)^2\\&=e_{\mathrm{d}(iP)}(t,t_0)\,e_{\mathrm{d}(-iP)}(t,t_0)=e_{\mathrm{d}(iP\oplus(-iP))}(t,t_0).\end{aligned}$$

From Theorem 3.3, we have

$$(iP\oplus(-iP))(t)=\int_{t_0}^t\Delta^+P(s)\,\mathrm{d}P(s)-\int_{t_0}^t\Delta^-P(s)\,\mathrm{d}P(s)=\sum_{s\in[t_0,t)}(\Delta^+P(s))^2-\sum_{s\in(t_0,t]}(\Delta^-P(s))^2.$$

□

5 Time scale elementary functions

In this section, we demonstrate that the definitions and properties of the elementary functions on time scales correspond to special cases of our earlier results. We assume some basic familiarity with the time scale calculus; in particular, we need the concepts of the Δ-derivative and Δ-integral (in the sense of Lebesgue and Henstock-Kurzweil), as well as the definitions of the elementary functions; see [1, 2, 8, 12].

Let $\mathbb{T}$ be a time scale. We use the symbol $[a,b]_{\mathbb{T}}$ to denote the interval $[a,b]\cap\mathbb{T}$. Consider a point $t_0\in[a,b]_{\mathbb{T}}$ and an rd-continuous function $p:[a,b]_{\mathbb{T}}\to\mathbb{R}$ such that $1+\mu(t)p(t)\neq 0$ for all $t\in[a,t_0)_{\mathbb{T}}$. The time scale exponential function $t\mapsto e_p(t,t_0)$ is usually defined as the unique solution of the initial-value problem

$$\begin{aligned}x^\Delta(t)&=p(t)x(t),\quad t\in[a,b]_{\mathbb{T}},\\x(t_0)&=1.\end{aligned}$$

(There exist alternative definitions of the exponential function; see [3] for a nice overview.) For our purposes, it is more convenient to work with the equivalent integral form

$$x(t)=1+\int_{t_0}^t p(s)x(s)\Delta s,\quad t\in[a,b]_{\mathbb{T}}.$$

At this point, we need the relationship between the Δ-integral and Kurzweil-Stieltjes integral, which was described in [15] and later refined in [5]. Given a real number $t\leq\sup\mathbb{T}$, let $t^*=\inf\{s\in\mathbb{T};s\geq t\}$. Since $\mathbb{T}$ is a closed set, we have $t^*\in\mathbb{T}$. Further, given a function $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$, we consider its extension $f^*:[a,b]\to\mathbb{R}$ given by $f^*(t)=f(t^*)$. The next statement combines Theorems 4.2 and 4.5 from [5].

Theorem 5.1. *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be an arbitrary function. Define $g(s) = s^*$ for every $s \in [a,b]$. Then the Kurzweil-Henstock Δ-integral $\int_a^b f(t)\Delta t$ exists if and only if the Kurzweil-Stieltjes integral $\int_a^b f^*(t)\,\mathrm{d}g(t)$ exists; in this case, both integrals have the same value. Moreover, if*

$$\begin{aligned} F_1(t) &= \int_a^t f(s)\Delta s, \quad t \in [a,b]_{\mathbb{T}}, \\ F_2(t) &= \int_a^t f^*(s)\,\mathrm{d}g(s), \quad t \in [a,b], \end{aligned}$$

then $F_2(t) = F_1^*(t)$ for every $t \in [a,b]$.

It is useful to note that the value of $\int_a^b f^*\,\mathrm{d}g$ does not change if we replace f^* by a different function which coincides with f on $[a,b]_{\mathbb{T}}$. This is the content of the next proposition from [6, Theorem 4.2].

Theorem 5.2. *Let $g(s) = s^*$ for every $s \in [a,b]$ and consider a pair of functions $f_1, f_2 : [a,b] \to \mathbb{R}$ such that $f_1(t) = f_2(t)$ for every $t \in [a,b]_{\mathbb{T}}$. If $\int_a^b f_1\,\mathrm{d}g$ exists, then $\int_a^b f_2\,\mathrm{d}g$ exists as well and both integrals have the same value.*

Lemma 5.3. *Let $p : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a Lebesgue Δ-integrable function satisfying $1 + \mu(t)p(t) \neq 0$ for every $t \in [a,t_0)_{\mathbb{T}}$. Then, the function*

$$P(t) = \int_{t_0}^t p^*(s)\,\mathrm{d}g(s), \quad t \in [a,b],$$

where $g(s) = s^*$ for every $s \in [a,b]$, has the following properties:

1. *P has bounded variation on $[a,b]$.*
2. *P is left-continuous at all points $t \in (a,b]$.*
3. *P is right-continuous at all points $t \in [a,b) \setminus \mathbb{T}$, $\Delta^+ P(t) = p(t)\mu(t)$ for every $t \in [a,b)_{\mathbb{T}}$, and $1 + \Delta^+ P(t) \neq 0$ for every $t \in [a,t_0)$.*

Proof. The Lebesgue Δ-integrability of p implies that both p and $|p|$ are Henstock-Kurzweil Δ-integrable [12, Theorem 2.19]. By Theorem 5.1, the Kurzweil-Stieltjes integral $\int_a^b |p^*(s)|\,\mathrm{d}g(s)$ exists. If $a = \tau_0 < \tau_1 < \cdots < \tau_m = b$ is an arbitrary partition of $[a,b]$, then

$$\sum_{i=1}^m |P(\tau_i) - P(\tau_{i-1})| \leq \sum_{i=1}^m \int_{\tau_{i-1}}^{\tau_i} |p^*(s)|\,\mathrm{d}g(s) = \int_a^b |p^*(s)|\,\mathrm{d}g(s),$$

which proves the first statement. The second and third statement follow from Theorem 2.1, because g is left-continuous on $(a,b]$, right-continuous on $[a,b) \setminus \mathbb{T}$, and $\Delta^+ P(t) = p(t)\Delta^+ g(t) = p(t)\mu(t)$ for $t \in [a,b)_{\mathbb{T}}$. □

We now return back to the exponential functions.

Theorem 5.4. *Let $p : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a Lebesgue Δ-integrable function satisfying $1 + \mu(t)p(t) \neq 0$ for every $t \in [a,t_0)_{\mathbb{T}}$. Then, there exists a unique solution of the equation*

$$x(t) = 1 + \int_{t_0}^t p(s)x(s)\Delta s, \quad t \in [a,b]_{\mathbb{T}}. \tag{5.1}$$

The solution is given by $x(t) = e_{\mathrm{d}P}(t,t_0)$, $t \in [a,b]_{\mathbb{T}}$, where $P(t) = \int_{t_0}^t p^*(s)\,\mathrm{d}g(s)$ and $g(s) = s^*$.

Proof. As a consequence of Theorem 5.1, we see that Eq. (5.1) is equivalent to

$$x^*(t) = 1 + \int_{t_0}^{t} p^*(s)x^*(s)\,\mathrm{d}g(s), \quad t \in [a,b].$$

By the substitution theorem, the last equation is equivalent to the generalized linear differential equation

$$x^*(t) = 1 + \int_{t_0}^{t} x^*(s)\,\mathrm{d}P(s), \quad t \in [a,b].$$

According to Lemma 5.3, the function P satisfies the assumptions of the existence and uniqueness theorem for generalized linear differential equations, and we conclude that $x^*(t) = e_{\mathrm{d}P}(t,t_0)$, $t \in [a,b]$. □

By the previous theorem, we have the following relationship between the two exponential functions:

$$e_p(t,t_0) = e_{\mathrm{d}P}(t,t_0), \quad t \in [a,b]_{\mathbb{T}}.$$

Moreover, the theorem shows that the time scale exponential function, defined as the unique solution of Eq. (5.1), makes a perfect sense for functions p which are not rd-continuous but merely Lebesgue Δ-integrable.

The next theorem lists some well-known properties of the time scale exponential function; the proof for the case when p is rd-continuous can be found in [1, Theorem 2.36]. Our goal is to show that these statements follow from the results in Section 3, and hence the properties of the exponential function are preserved for Lebesgue Δ-integrable functions p.

In the statements below, we use the convention that $\sigma(t) = t$ and $\mu(t) = 0$ at the right endpoint $t = b$. Also, we assume that all exponentials appearing in the identities are well defined.

Theorem 5.5. *Let $p : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a Lebesgue Δ-integrable function. The time scale exponential function has the following properties:*

1. $e_0(t,t_0) = 1$ *for every* $t \in [a,b]_{\mathbb{T}}$.
2. $e_p(t,t) = 1$ *for every* $t \in [a,b]_{\mathbb{T}}$.
3. $e_p(\sigma(t),t_0) = (1+\mu(t)p(t))e_p(t,t_0)$ *for every* $t \in [a,b]_{\mathbb{T}}$.
4. $e_p(t,t_0)e_q(t,t_0) = e_r(t,t_0)$ *for every* $t \in [a,b]_{\mathbb{T}}$*, where* $r(t) = p(t)+q(t)+\mu(t)p(t)q(t)$.
5. $e_p(t,t_0)^{-1} = e_u(t,t_0)$ *for every* $t \in [a,b]_{\mathbb{T}}$*, where* $u(t) = -p(t)(1+p(t)\mu(t))^{-1}$.

Proof. The first two statements are obvious. To prove the third statement, observe that, for $t \in [a,b]_{\mathbb{T}}$,

$$e_p(\sigma(t),t_0) = e_{\mathrm{d}P}(\sigma(t),t_0) = e_{\mathrm{d}P}(t+,t_0) = (1+\Delta^+P(t))e_{\mathrm{d}P}(t,t_0) = (1+\mu(t)p(t))e_p(t,t_0).$$

In the fourth statement, we have

$$e_p(t,t_0)e_q(t,t_0) = e_{\mathrm{d}P}(t,t_0)e_{\mathrm{d}Q}(t,t_0) = e_{\mathrm{d}(P\oplus Q)}(t,t_0),$$

where $P(t) = \int_{t_0}^{t} p^*(s)\,\mathrm{d}g(s)$ and $Q(t) = \int_{t_0}^{t} q^*(s)\,\mathrm{d}g(s)$. Hence, the formula for $P \oplus Q$ reduces to

$$(P\oplus Q)(t) = P(t)+Q(t)+\int_{t_0}^{t}\Delta^+Q(s)\,\mathrm{d}P(s) = P(t)+Q(t)+\int_{t_0}^{t}\Delta^+Q(s)p^*(s)\,\mathrm{d}g(s)$$

$$= \int_{t_0}^{t}(p^*(s)+q^*(s)+q^*(s)\Delta^+g(s)p^*(s))\,\mathrm{d}g(s) = \int_{t_0}^{t}(p^*(s)+q^*(s)+q^*(s)p^*(s)\mu^*(s))\,\mathrm{d}g(s) = \int_{t_0}^{t} r^*(s)\,\mathrm{d}g(s).$$

In the next-to-last integral, we have replaced by $\Delta^+ g(s)$ by $\Delta^+ g(s^*) = \mu^*(s)$, which is correct thanks to Theorem 5.2. Now, it follows that $e_{\mathrm{d}(P\oplus Q)}(t,t_0) = e_r(t,t_0)$.

Finally, we have

$$e_p(t,t_0)^{-1} = e_{\mathrm{d}P}(t,t_0)^{-1} = e_{\mathrm{d}(\ominus P)}(t,t_0),$$

Using the formula for $\ominus P$ together with Lemma 2.2, we get

$$(\ominus P)(t) = -P(t) + \sum_{s\in[t_0,t)} \frac{(\Delta^+ P(s))^2}{1+\Delta^+ P(s)} = -P(t) + \int_{t_0}^t \frac{\Delta^+ P(s)}{1+\Delta^+ P(s)}\,\mathrm{d}P(s)$$

$$= -\int_{t_0}^t p^*(s)\,\mathrm{d}g(s) + \int_{t_0}^t \frac{p^*(s)\mu^*(s)}{1+p^*(s)\mu^*(s)}\,\mathrm{d}P(s) = -\int_{t_0}^t p^*(s)\,\mathrm{d}g(s) + \int_{t_0}^t \frac{p^*(s)^2\mu^*(s)}{1+p^*(s)\mu^*(s)}\,\mathrm{d}g(s)$$

$$= -\int_{t_0}^t p^*(s)\left(1 - \frac{p^*(s)\mu^*(s)}{1+p^*(s)\mu^*(s)}\right)\mathrm{d}g(s) = -\int_{t_0}^t \frac{p^*(s)}{1+p^*(s)\mu^*(s)}\,\mathrm{d}g(s) = \int_{t_0}^t u^*\,\mathrm{d}g(s),$$

and hence $e_{\mathrm{d}(\ominus P)}(t,t_0) = e_u(t,t_0)$, $t\in[a,b]_{\mathbb{T}}$. $\square$

Remark 5.6. In the time scale literature, the functions r and u appearing in the fourth and fifth statement of the previous theorem are usually denoted by $p\oplus q$ and $\ominus p$. To avoid confusion, we emphasize that in our paper, the symbols $\oplus$ and $\ominus$ have a different meaning.

Before we finish our discussion of the time scale exponential function, we remark that Theorems 3.5 and 3.6 might be used to derive some information about the sign of $e_p(t,t_0)$; we leave it up to the reader to check that these results are in agreement with Theorems 2.44, 2.48 and Lemma 2.47 from [1].

We now proceed to the time scale hyperbolic functions, which are defined in a natural way using the exponential function (cf. [1, Definition 3.17]). Hence, we immediately see their relation to our generalized hyperbolic functions:

$$\cosh_p = \frac{e_p + e_{-p}}{2} = \frac{e_{\mathrm{d}P} + e_{\mathrm{d}(-P)}}{2} = \cosh_{\mathrm{d}P},$$

$$\sinh_p = \frac{e_p - e_{-p}}{2} = \frac{e_{\mathrm{d}P} - e_{\mathrm{d}(-P)}}{2} = \sinh_{\mathrm{d}P},$$

where $P(t) = \int_{t_0}^t p^*(s)\,\mathrm{d}g(s)$. We already know that the right-hand sides are defined only if P satisfies the conditions $1-(\Delta^- P(t))^2 \neq 0$, $t\in(t_0,b]$, and $1-(\Delta^+ P(t))^2 \neq 0$, $t\in[a,t_0)$. For our function P, the first condition is always satisfied, while the second reduces to $1-p(t)^2\mu(t)^2 \neq 0$, $t\in[a,t_0)_{\mathbb{T}}$; this coincides with the condition in [1, Definition 3.17].

When defining the time scale hyperbolic functions, it is usually assumed that p is a rd-continuous function. However, it follows from our discussion concerning the exponential function that it is enough if p is Lebesgue Δ-integrable.

The next theorem summarizes some basic facts about the hyperbolic functions; in a slightly different form, the statement can be found in [1, Lemma 3.18]. We show that these identities are consequences of the results from Section 3, and hence the properties of the hyperbolic functions are still valid for Lebesgue Δ-integrable functions p.

Theorem 5.7. *Let $p : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a Lebesgue Δ-integrable function satisfying $1-p(t)^2\mu(t)^2 \neq 0$, $t\in[a,t_0)_{\mathbb{T}}$. The time scale hyperbolic functions have the following properties:*

1. $\cosh_p(t,t_0) = 1 + \int_{t_0}^t p(s)\sinh_p(s,t_0)\,\Delta s$, $t\in[a,b]_{\mathbb{T}}$.

2. $\sinh_p(t,t_0) = \int_{t_0}^t p(s)\cosh_p(s,t_0)\,\Delta s$, $t\in[a,b]_{\mathbb{T}}$.

3. $\cosh_p^2(t,t_0) - \sinh_p^2(t,t_0) = e_{-p^2\mu}(t,t_0)$, $t\in[a,b]_{\mathbb{T}}$.

Proof. The first and second statement follow from Theorems 4.2 and 5.1. The third identity can be obtained as follows:

$$\cosh_p^2(t,t_0) - \sinh_p^2(t,t_0) = \cosh_{\mathrm{d}P}^2(t,t_0) - \sinh_{\mathrm{d}P}^2(t,t_0) = e_{\mathrm{d}Q}(t,t_0),$$

where

$$\begin{aligned} Q(t) &= -\int_{t_0}^t \Delta^+ P(s)\,\mathrm{d}P(s) = -\int_{t_0}^t p^*(s)\Delta^+ g(s) p^*(s)\,\mathrm{d}g(s) \\ &= -\int_{t_0}^t p^*(s)^2 \Delta^+ g(s^*)\,\mathrm{d}g(s) = -\int_{t_0}^t p^*(s)^2 \mu^*(s)\,\mathrm{d}g(s). \end{aligned}$$

It follows that $e_{\mathrm{d}Q}(t,t_0) = e_{-p^2\mu}(t,t_0)$, $t \in [a,b]_{\mathbb{T}}$. □

Finally, we have the following relation between the time scale trigonometric functions (see [1, Definition 3.25]) and our generalized trigonometric functions:

$$\cos_p = \frac{e_{ip} + e_{-ip}}{2} = \frac{e_{\mathrm{d}(iP)} + e_{\mathrm{d}(-iP)}}{2} = \cos_{\mathrm{d}P},$$

$$\sin_p = \frac{e_{ip} - e_{-ip}}{2i} = \frac{e_{\mathrm{d}(iP)} - e_{\mathrm{d}(-iP)}}{2i} = \sin_{\mathrm{d}P},$$

where $P(t) = \int_{t_0}^t p^*(s)\,\mathrm{d}g(s)$. The conditions $1 + (\Delta^+ P(t))^2 \neq 0$, $t \in [a,t_0)$, and $1 + (\Delta^- P(t))^2 \neq 0$, $t \in (t_0,b]$ from the definition of the generalized trigonometric functions reduce to the single condition $1 + p(t)^2\mu(t)^2 \neq 0$, $t \in [a,t_0)_{\mathbb{T}}$, which agrees with [1, Definition 3.25].

The following theorem summarizes the basic properties of these functions (cf. [1, Lemma 3.26]). We leave it up to the reader to verify that the identities are immediate consequences of Theorems 4.4 and 5.1. Again, these formulas hold in the general case when p is Lebesgue Δ-integrable.

Theorem 5.8. *Let $p : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a Lebesgue Δ-integrable function satisfying $1 + p(t)^2\mu(t)^2 \neq 0$, $t \in [a,t_0)_{\mathbb{T}}$. The time scale trigonometric functions have the following properties:*

1. $\cos_p(t,t_0) = 1 + \int_{t_0}^t p(s)\sin_p(s,t_0)\,\Delta s$, $t \in [a,b]_{\mathbb{T}}$.
2. $\sin_p(t,t_0) = \int_{t_0}^t p(s)\cos_p(s,t_0)\,\Delta s$, $t \in [a,b]_{\mathbb{T}}$.
3. $\cos_p^2(t,t_0) - \sin_p^2(t,t_0) = e_{p^2\mu}(t,t_0)$, $t \in [a,b]_{\mathbb{T}}$.

6 Conclusion

Our definition of the exponential function is a non-constructive one as it relies on the existence-uniqueness theorem for generalized linear differential equations. A different approach could be based on product integration theory (see [13, Chapter 7]), which expresses the solution of a generalized linear equation as a limit of certain products.

Various authors have extended the definitions of the time scale elementary functions to matrix-valued arguments (see e.g. [1, Chapter 5], [9], [16]). Similarly, given a matrix-valued function $P : [a,b] \to \mathbb{R}^{n\times n}$, it is natural to define the generalized exponential function $t \mapsto e_{\mathrm{d}P}(t,t_0)$ as the unique solution of the generalized linear differential equation

$$Z(t) = I + \int_{t_0}^t \mathrm{d}[P]\,Z, \quad t \in [a,b],$$

where I is the identity matrix. According to Theorem 2.6, it is enough to assume that A has bounded variation on $[a,b]$, $I + \Delta^+ A(t)$ is invertible for every $t \in [a,t_0)$, and $I - \Delta^- A(t)$ is invertible for every

$t \in (t_0, b]$. In fact, this definition of the exponential function coincides with the notion of a fundamental matrix; some of its properties can be found in [13, Theorem 6.15]. Unfortunately, the behavior of this matrix-valued exponential function is not as nice as in the scalar case. For example, to obtain an analogue of Theorem 3.3, it is necessary to impose certain commutativity conditions on P and Q. With this in mind, we have restricted our attention to scalar functions only.

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