Explicit solutions to dynamic diffusion-type equations and their time integrals

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Abstract

This paper deals with solutions of diffusion-type partial dynamic equations on discrete-space domains. We provide two methods for finding explicit solutions, examine their asymptotic behavior and time integrability. These properties depend significantly not only on the underlying time structure but also on the dimension and symmetry of the problem. Throughout the paper, the results are interpreted in the context of random walks and related stochastic processes.

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1 Introduction

Semidiscrete partial differential equations have attracted attention of researchers in several applied areas where the discrete space occurs naturally, e.g. in biology [4], signal and image processing [13], and stochastic processes [7]. Nonetheless, to our knowledge there is no systematic theory of semidiscrete partial differential equations. In this work, we study solutions of partial dynamic equations of diffusion type on domains with discrete space and general time structure (continuous, discrete and other). We consider the equation

$$u^{\Delta_t}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t), \quad x \in \mathbb{Z},\ t \in \mathbb{T}, \tag{1.1}$$

where $a, b, c \in \mathbb{R}$ are constants and $\mathbb{T}$ is a time scale (arbitrary closed subset of $\mathbb{R}$). The symbol u^{Δ_t} denotes the partial Δ-derivative with respect to t, which coincides with the standard partial derivative u_t when $\mathbb{T} = \mathbb{R}$, or the forward partial difference $\Delta_t u$ when $\mathbb{T} = \mathbb{Z}$. Since the differences with respect to x are not used, we omit the lower index t in u^{Δ_t} and write u^{Δ} only. The time scale calculus is used as a tool to obtain general results from which the corresponding statements for discrete and semidiscrete diffusion follow easily. Readers who are not familiar with the basic principles and notations of this theory are kindly asked to consult Stefan Hilger's original paper [10] or the survey [3]. This paper contributes to recent efforts of several researchers who have studied partial dynamic equations (e.g. [1, 2, 11, 12, 19]).

The present work is a free continuation of our recent paper [18], where we started to develop a systematic theory for equations of the form (1.1). Note that if $b = -2a = -2c$, the equation represents the space-discretized version of the classical diffusion equation (therefore, we talk about diffusion-type equations). Also, if $b = -a$ and $c = 0$ (or $b = -c$ and $a = 0$), we get the transport equation with discrete space. Another motivation for questions studied in this paper comes from the connection of (1.1) with Markov processes. Indeed, consider a one-dimensional discrete-time random walk on $\mathbb{Z}$. Let p, q, $r \in [0, 1]$ be the probabilites of going left, standing still, and going right, respectively (so that $p + q + r = 1$). If $u(x, t)$ is the probability of visiting point x at time t, we get $u(x, t+1) = pu(x+1, t)+qu(x, t)+ru(x-1, t)$, or the equivalent diffusion-type equation

$$u^{\Delta}(x,t) = pu(x+1,t) + (q-1)u(x,t) + ru(x-1,t), \quad x \in \mathbb{Z},\ t \in \mathbb{N}_0,$$

which, coupled with the initial condition $u(0, 0) = 1$ and $u(x, 0) = 0$ for $x \neq 0$, describes the random walk starting from the origin.

Next, consider a continuous-time random walk on $\mathbb{Z}$. Assume that in a time interval $[t, t + h]$, the probabilities of going left and right are $ph + o(h)$ and $rh + o(h)$, respectively. It follows that $u(x, t + h) = (ph + o(h))u(x, t + 1) + (1 - ph - rh + o(h))u(x, t) + (rh + o(h))u(x, t - 1)$. By subtracting $u(x, t)$, dividing by h and passing to the limit $h \to 0$, we get the diffusion-type equation

$$u^{\Delta}(x,t) = pu(x+1,t) + (-p-r)u(x,t) + ru(x-1,t), \quad x \in \mathbb{Z},\ t \in \mathbb{R}_0^+.$$

Finally, for a general time scale $\mathbb{T}$, solutions of (1.1) can be regarded as heterogeneous stochastic processes. This interesting relationship is discussed throughout the paper and illustrates our results.

In Section 2, we briefly summarize the main results from [18]. In Section 3, we present two methods for finding explicit solutions of (1.1) once a particular time scale is given. These methods are then used in Section 4 to examine the asymptotic behavior of solutions as well as finiteness of their time integrals. We calculate the exact values of these integrals for $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$, and discover the surprising fact that they coincide. In Section 5, multidimensional diffusion equations are briefly considered and we prove a slight generalization of G. Pólya's famous result [17] on the recurrence of symmetric random walks in $\mathbb{Z}^N$.

2 Preliminaries

Let us start with a short overview of the main results from [18], which will be used later. We consider dynamic diffusion-type equations of the form

$$u^{\Delta}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t), \quad x \in \mathbb{Z},\ t \in \mathbb{T}, \tag{2.1}$$

where a, b, c are real numbers. The graininess of $\mathbb{T}$ influences the behavior of solutions in a substantial way, and some of the results presented in this section assume that the graininess is sufficiently small. The paper [18] contains a wealth of examples showing that these graininess conditions are indeed necessary.

The first result is an existence-uniqueness theorem for Eq. (2.1). Assume that X is a Banach space, $t_0 \in \mathbb{T}$, and A is a bounded linear operator on X such that $I + A\mu(t)$ is invertible for every $t \in (-\infty, t_0)_{\mathbb{T}}$, where μ stands for the graininess function. Recall that the time scale exponential function $t \mapsto e_A(t, t_0)$ is defined as the unique solution of the initial-value problem

$$\begin{array}{rcl} x^{\Delta}(t) & = & Ax(t), \quad t \in \mathbb{T}, \\ x(t_0) & = & I. \end{array} \tag{2.2}$$

We use the symbol $\ell^\infty(\mathbb{Z})$ to denote the space of all bounded real sequences $\{u_n\}_{n\in\mathbb{Z}}$.

Theorem 2.1. *Consider an interval* $[T_1, T_2]_{\mathbb{T}} \subset \mathbb{T}$ *and a point* $t_0 \in [T_1, T_2]_{\mathbb{T}}$. *Let* $u^0 \in \ell^\infty(\mathbb{Z})$. *Assume that* $\mu(t) < \frac{1}{|a|+|b|+|c|}$ *for every* $t \in [T_1, t_0)_{\mathbb{T}}$. *Let the operator* $A : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ *be given by*

$$A(\{u_n\}_{n\in\mathbb{Z}}) = \{au_{n+1} + bu_n + cu_{n-1}\}_{n\in\mathbb{Z}}.$$

Also, define the function $U : [T_1, T_2]_{\mathbb{T}} \to \ell^\infty(\mathbb{Z})$ by $U(t) = e_A(t, t_0)u^0$, $t \in [T_1, T_2]_{\mathbb{T}}$. Then

$$u(x, t) = U(t)_x, \quad x \in \mathbb{Z},\ t \in [T_1, T_2]_{\mathbb{T}},$$

is the unique bounded solution of Eq. (2.1) *on $\mathbb{Z} \times [T_1, T_2]_{\mathbb{T}}$ such that $u(x, t_0) = u_x^0$ for every $x \in \mathbb{Z}$.*

The superposition principle allows us to easily find explicit solutions for general initial conditions.

Theorem 2.2. *Let $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ be the unique bounded solution of Eq.* (2.1) *corresponding to the initial condition*

$$u(x, t_0) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

If $\{c_k\}_{k\in\mathbb{Z}}$ is an arbitrary bounded real sequence, then

$$v(x, t) = \sum_{k\in\mathbb{Z}} c_k u(x - k, t)$$

is the unique bounded solution of Eq. (2.1) *corresponding to the initial condition $v(x, t_0) = c_x$, $x \in \mathbb{Z}$.*

The next theorem shows that for solutions of Eq. (2.1) with $a + b + c = 0$, the sum $\sum_{x\in\mathbb{Z}} u(x, t)$ is the same for all t.

Theorem 2.3. *Let $u : \mathbb{Z} \times [T_1, T_2]_{\mathbb{T}} \to \mathbb{R}$ be a bounded solution of Eq.* (2.1) *with $a + b + c = 0$. Assume that the following conditions are satisfied:*

- *For a certain $t_0 \in [T_1, T_2]_{\mathbb{T}}$, the sum $\sum_{x\in\mathbb{Z}} |u(x, t_0)|$ is finite.*
- *$\mu(t) < \frac{1}{|a|+|b|+|c|}$ for every $t \in [T_1, t_0)_{\mathbb{T}}$.*

Then $\sum_{x\in\mathbb{Z}} u(x, t) = \sum_{x\in\mathbb{Z}} u(x, t_0)$ for every $t \in [T_1, T_2]_{\mathbb{T}}$.

The next statement represents the minimum and maximum principles.

Theorem 2.4. *Let a, b, c be such that $a, c \geq 0$, $b \leq 0$. Consider a bounded solution $u : \mathbb{Z} \times [T_1, T_2]_{\mathbb{T}} \to \mathbb{R}$ of Eq.* (2.1)*. Moreover, assume that $\mu(t) \leq -1/b$ for every $t \in [T_1, T_2)_{\mathbb{T}}$. Then the following statements are true for all $K \geq 0$:*

- *If $a + b + c \geq 0$ and $u(x, T_1) \geq K$ for every $x \in \mathbb{Z}$, then $u(x, t) \geq K$ for all $t \in [T_1, T_2]_{\mathbb{T}}$, $x \in \mathbb{Z}$.*
- *If $a + b + c \leq 0$ and $u(x, T_1) \leq K$ for every $x \in \mathbb{Z}$, then $u(x, t) \leq K$ for all $t \in [T_1, T_2]_{\mathbb{T}}$, $x \in \mathbb{Z}$.*

The final two results apply to problems with symmetric right-hand sides. In this case, symmetric initial conditions lead to symmetric solutions.

Theorem 2.5. *Let $u : \mathbb{Z} \times [T_1, T_2]_{\mathbb{T}} \to \mathbb{R}$ be a bounded solution of Eq.* (2.1) *with $a = c$. Assume that the following conditions are satisfied:*

- *For a certain $t_0 \in [T_1, T_2]_{\mathbb{T}}$, we have $u(x, t_0) = u(-x, t_0)$ for every $x \in \mathbb{N}$.*
- *$\mu(t) < \frac{1}{2|a|+|b|}$ for every $t \in [T_1, t_0)_{\mathbb{T}}$.*

Then $u(x, t) = u(-x, t)$ for every $t \in [T_1, T_2]_{\mathbb{T}}$ and $x \in \mathbb{N}$.

We conclude by characterizing the maxima of solutions for certain fixed values of time. Recall that two adjacent time scale intervals $[t_0, t_0 + t]_{\mathbb{T}}$ and $[t_0 + t, t_0 + 2t]_{\mathbb{T}}$ are isometric, if the following conditions are satisfied:

- If $\tau \in [t_0, t_0 + t]_{\mathbb{T}}$, then $\tau + t \in [t_0 + t, t_0 + 2t]_{\mathbb{T}}$.
- If $\tau \in [t_0, t_0 + t)_{\mathbb{T}}$, then $\mu(\tau) = \mu(\tau + t)$.

Theorem 2.6. *Assume that the intervals $[t_0, t_0 + t]_{\mathbb{T}}$, $[t_0 + t, t_0 + 2t]_{\mathbb{T}}$ are isometric, and $u : \mathbb{Z} \times [t_0, t_0 + 2t]_{\mathbb{T}} \to \mathbb{R}$ is the unique bounded solution of Eq. (2.1) with $a = c$ corresponding to the initial condition*

$$u(x, t_0) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Then $|u(x, t_0 + 2t)| \leq u(0, t_0 + 2t)$ for every $x \in \mathbb{Z}$.

3 Explicit solutions

In this section, we derive explicit formulas for the unique bounded solution of the initial-value problem

$$\begin{aligned} u^{\Delta}(x, t) &= au(x + 1, t) + bu(x, t) + cu(x - 1, t), \\ u(x, t_0) &= \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases} \end{aligned} \tag{3.1}$$

where a, b, c are arbitrary real numbers. By Theorem 2.2, the knowledge of this particular solution enables us to obtain solutions corresponding to arbitrary initial conditions. In the following text, we discuss two methods of solving the initial-value problem (3.1); interested readers might also consult the paper [12], which demonstrates a different method for solving partial dynamic equations based on the Laplace Δ-transform.

3.1 Generating functions

Our first method is based on the use of generating functions. Given a sequence $\{u_n\}_{n\in\mathbb{Z}}$, its generating function is the series $U(z) = \sum_{n=-\infty}^{\infty} u_n z^n$. Depending on the context, the series can be interpreted either as a classical Laurent series, or as a formal Laurent series.

Assume that u is the solution of the initial-value problem (3.1), and let $F(z, t) = \sum_{x=-\infty}^{\infty} u(x, t) z^x$; in other words, for every fixed $t \in \mathbb{T}$, the function $z \mapsto F(z, t)$ is the generating function of $\{u(x, t)\}_{x\in\mathbb{Z}}$. Using (3.1), we see that

$$F^{\Delta}(z, t) = \sum_{x=-\infty}^{\infty} u^{\Delta}(x, t) z^x = \sum_{x=-\infty}^{\infty} (au(x + 1, t) + bu(x, t) + cu(x - 1, t)) z^x$$

$$= a \sum_{x=-\infty}^{\infty} u(x + 1, t) z^x + b \sum_{x=-\infty}^{\infty} u(x, t) z^x + c \sum_{x=-\infty}^{\infty} u(x - 1, t) z^x = (a/z + b + cz) F(z, t),$$

and $F(z, t_0) = \sum_{x=-\infty}^{\infty} u(x, t_0) z^x = 1$. We obtained a first-order linear dynamic equation for the function F. Its solution is given by the time scale exponential function

$$F(z, t) = e_{a/z+b+cz}(t, t_0).$$

By the definition of F, we know that $u(x, t)$ is the coefficient of z^x in the series expansion of $F(z, t)$.

To sum up, given a particular time scale $\mathbb{T}$, it is enough to calculate the value of $e_{a/z+b+cz}(t, t_0)$, find its Laurent series expansion with respect to z, and look at the coefficient of z^x to find an explicit formula for $u(x, t)$. Finally, since our calculation was purely formal, it might be necessary to check the validity of the solution obtained in this way. However, if $\mathbb{T}$ consists of isolated points $t_0 < t_1 < t_2 < \cdots$, it is not difficult to see that for $t = t_k$, the solution of (3.1) vanishes outside the region $x \in \{-k, \dots, k\}$; this phenomenon can be interpreted as a causality principle for Eq. (3.1). Hence, the sum $F(z,t) = \sum_{x=-\infty}^{\infty} u(x,t)z^x$ has only a finite number of nonzero terms, and all steps in our generating function method (e.g. term by term differentiation) are well justified.

We now illustrate the whole procedure by several examples.

Example 3.1. For $\mathbb{T} = \mathbb{R}$, we obtain the semidiscrete equation

$$u_t(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t). \tag{3.2}$$

Assume that $t_0 = 0$. The time scale exponential function $e_{a/z+b+cz}(t,t_0)$ reduces to the classical exponential function $e^{(a/z+b+cz)t}$. Therefore, our generating function method gives

$$F(z,t) = e^{(a/z+b+cz)t} = e^{bt}e^{(a/z+cz)t}.$$

To obtain the series expansion of F, we need the identity

$$e^{w/2(z+1/z)} = \sum_{x=-\infty}^{\infty} I_x(w)z^x \tag{3.3}$$

(see [16, formula 10.35.1]), where $I_x(w) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+x+1)k!}\left(\frac{w}{2}\right)^{2k+x}$ is the modified Bessel function of the first kind. It follows that

$$F(z,t) = e^{bt}e^{(a/z+cz)t} = e^{bt}e^{\sqrt{ac}\left(\frac{\sqrt{a}}{\sqrt{cz}}+\frac{\sqrt{cz}}{\sqrt{a}}\right)t} = e^{bt}\sum_{x=-\infty}^{\infty} I_x(2t\sqrt{ac})\left(\sqrt{\frac{c}{a}}\,z\right)^x, \tag{3.4}$$

which leads to the result

$$u(x,t) = e^{bt}I_x(2t\sqrt{ac})\left(\sqrt{\frac{c}{a}}\right)^x, \quad x \in \mathbb{Z},\ t \in \mathbb{R}. \tag{3.5}$$

One can easily verify its correctness by substituting into Eq. (3.2) and using the identity $I_x'(t) = \frac{1}{2}(I_{x-1}(t) + I_{x+1}(t))$ (see [16, formula 10.29.1]).

Note that for symmetric right-hand sides with $a = c$, the solution simplifies to

$$u(x,t) = e^{bt}I_x(2at), \quad x \in \mathbb{Z},\ t \in \mathbb{R}.$$

This function is even with respect to x (see [16, formula 10.27.1]), which agrees with Theorem 2.5. For every fixed $t \geq 0$, the solution attains its maximum value at $x = 0$ (see [16, paragraph 10.37]), which agrees with Theorem 2.6. Also, note the infinite speed propagation phenomenon, which is well known from the classical diffusion equation with continuous space and time. Figure 1 shows the solution corresponding to $a = c = 1$ and $b = -2$.

In Eq. (3.4), we were tacitly assuming that $ac \neq 0$. For $a = 0$ (which happens e.g. for the transport equation), the generating function simplifies to

$$F(z,t) = e^{bt}e^{czt} = e^{bt}\sum_{x=0}^{\infty}\frac{(ct)^x}{x!}z^x,$$

and it follows that $u(x,t) = e^{bt}\frac{(ct)^x}{x!}$ for $x \geq 0$, and $u(x,t) = 0$ for $x < 0$. Similarly, for $c = 0$, we obtain $u(x,t) = e^{bt}\frac{(at)^{-x}}{(-x)!}$ for $x \leq 0$, and $u(x,t) = 0$ for $x > 0$. These formulas generalize the result from [19, Lemma 4.1].

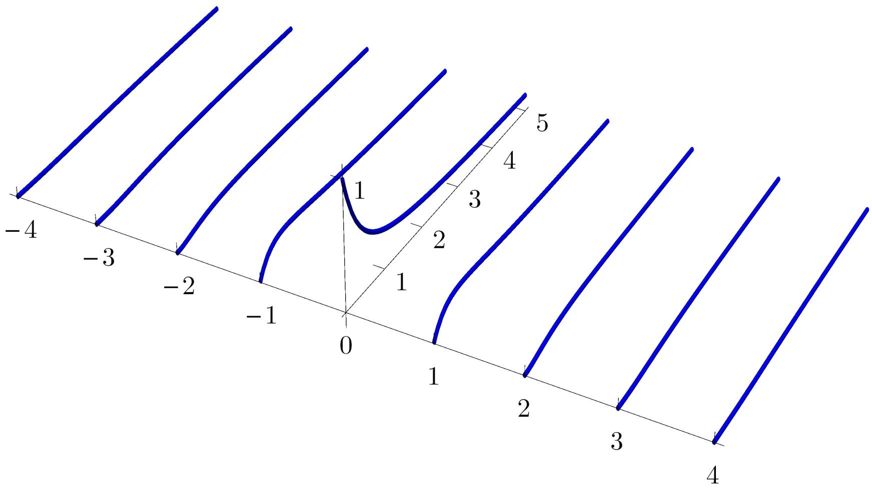

Figure 1: Bounded solution of the semidiscrete diffusion equation

Next, let us consider the discrete problem with $\mathbb{T} = \mathbb{Z}$. We start the following technical lemma. The symbol $\binom{t}{t_1,\dots,t_n}$ stands for the multinomial coefficient, which is equal to $\frac{t!}{t_1!\cdots t_n!}$ when $t, t_1, \dots, t_n$ are nonnegative integers, and otherwise is zero.

Lemma 3.2. *For every $t \in \mathbb{N}_0$ and $x \in \mathbb{Z}$, the coefficient of z^x in $(n + a/z + b + cz)^t$ is*

$$\sum_{j=0}^{t} \binom{t}{j, t-2j-x, j+x} a^j (b+n)^{t-2j-x} c^{j+x}.$$

Proof. By the multinomial theorem, we have

$$(n + a/z + b + cz)^t = (a/z + (b+n) + cz)^t = \sum_{t_1+t_2+t_3=t} \binom{t}{t_1, t_2, t_3} a^{t_1} (b+n)^{t_2} c^{t_3} z^{t_3-t_1}. \tag{3.6}$$

We need the coefficient of z^x in (3.6), which corresponds to $t_3 - t_1 = x$. Taking into account this equality and the condition $t_1 + t_2 + t_3 = t$, we get $t_3 = t_1 + x$ and $t_2 = t - 2t_1 - x$. Consequently, the coefficient of z^x is

$$\sum_{j=0}^{t} \binom{t}{j, t-2j-x, j+x} a^j (b+n)^{t-2j-x} c^{j+x}, \quad x \in \mathbb{Z},\ t \in \mathbb{N}_0 \tag{3.7}$$

(we have renamed t_1 to j). Finally, note that the powers of z in (3.6) have exponents $x \in \{-t, \dots, t\}$, but the resulting formula (3.7) is valid for all $x \in \mathbb{Z}$ (when $|x| > t$, all multinomial coefficients are zero). $\square$

Note that trinomial coefficients satisfy $\binom{t_1+t_2+t_3}{t_1,t_2,t_3} = \binom{t_1+t_2+t_3}{t_1+t_2}\binom{t_1+t_2}{t_1}$. This means that the trinomial coefficient $\binom{t}{j,t-2j-x,j+x}$ from Lemma 3.2 can be also written in other forms such as $\binom{t}{2j+x}\binom{2j+x}{j}$, $\binom{t}{j}\binom{t-j}{j+x}$, etc.

Example 3.3. Consider the time scale $\mathbb{T} = \mathbb{Z}$ and let $t_0 = 0$. In this case, we obtain the purely discrete equation

$$u(x, t+1) - u(x, t) = au(x+1, t) + bu(x, t) + cu(x-1, t).$$

It is known (see [3, Example 2.50]) that for $\mathbb{T} = \mathbb{Z}$, we have $e_\alpha(t,0) = (1+\alpha)^t$. Hence,

$$F(z,t) = e_{a/z+b+cz}(t,0) = (1 + a/z + b + cz)^t, \quad t \in \mathbb{Z}.$$

Recalling that $u(x,t)$ is the coefficient of z^x in $F(z,t)$ and using Lemma 3.2, we get

$$u(x,t) = \sum_{j=0}^{t} \binom{t}{j, t-2j-x, j+x} a^j (b+1)^{t-2j-x} c^{j+x}, \quad x \in \mathbb{Z},\, t \in \mathbb{N}_0. \tag{3.8}$$

In the special case when $a = c = 1$ and $b = -2$, the formula simplifies to

$$u(x,t) = \sum_{j=0}^{t} \binom{t}{j, t-2j-x, j+x} (-1)^{t-2j-x} = (-1)^{t+x} \sum_{j=0}^{t} \binom{t}{j, t-2j-x, j+x}.$$

The next table shows the values of $u(x,t)$ for $t \in \{0,\dots,7\}$:

0	1	−7	28	−77	161	−266	357	−393	357	−266	161	−77	28	−7	1	0
0	0	1	−6	21	−50	90	−126	141	−126	90	−50	21	−6	1	0	0
0	0	0	1	−5	15	−30	45	−51	45	−30	15	−5	1	0	0	0
0	0	0	0	1	−4	10	−16	19	−16	10	−4	1	0	0	0	0
0	0	0	0	0	1	−3	6	−7	6	−3	1	0	0	0	0	0
0	0	0	0	0	0	1	−2	3	−2	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	−1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0

Note that the values are symmetric with respect to $x = 0$, which agrees with Theorem 2.5. For $t \in \{0, 2, 4, 6, \dots\}$, the solution attains its maximum value at $x = 0$, which agrees with Theorem 2.6. Also, the table confirms that the propagation speed is finite (causality principle).

Another way to simplify equation (3.8) is to consider the case $b = -1$. The term $(b+1)^{t-2j-x}$ in (3.8) is nonzero only if $t-2j-x = 0$, which can happen only if $t-x$ is even. In this case, we have $j = (t-x)/2$, and formula (3.8) reduces to

$$u(x,t) = \binom{t}{\frac{t-x}{2}, 0, \frac{t+x}{2}} a^{(t-x)/2} c^{(t+x)/2} = \binom{t}{\frac{t+x}{2}} a^{(t-x)/2} c^{(t+x)/2}.$$

This result is well known from probability theory (see [20, p. 2]) in the case when $a, c \ge 0$ and $a + c = 1$, which corresponds to the one-dimensional random walk with transition probabilities a and c.

Finally, if $a = 0$ (transport equation), formula (3.8) simplifies to

$$u(x,t) = \binom{t}{0, t-x, x} (b+1)^{t-x} c^x = \binom{t}{x} (b+1)^{t-x} c^x.$$

Similary, for $c = 0$, we obtain $u(x,t) = \binom{t}{-x}(b+1)^{t+x} a^{-x}$. These formulas generalize the result from [19, Lemma 5.1].

To conclude, we consider the harmonic time scale.

Example 3.4. Consider the time scale $\mathbb{T} = \{H_n,\, n \in \mathbb{N}_0\}$, where $H_0 = 0$ and $H_n = \sum_{k=1}^{n} \frac{1}{k}$ are the harmonic numbers. Assume that $t_0 = 0$. It is known (see [3, Example 2.53]) that the values of the time scale exponential function are the binomial coefficients: $e_\alpha(H_n, 0) = \binom{n+\alpha}{n}$. In particular, we have

$$F(z, H_n) = e_{a/z+b+cz}(H_n, 0) = \binom{n + a/z + b + cz}{n}, \quad n \in \mathbb{N}_0.$$

Using the identity (see [16, formula 26.8.7])

$$x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n} s(n,l)x^l, \quad x\in\mathbb{R},\, n\in\mathbb{N}_0,$$

where $s(n,l)$ are the Stirling numbers of the first kind, we obtain

$$F(z,H_n)=\frac{1}{n!}\sum_{l=0}^{n} s(n,l)(n+a/z+b+cz)^l. \tag{3.9}$$

By Lemma 3.2, the coefficient of z^x in $(n+a/z+b+cz)^l$ is

$$\sum_{j=0}^{l}\binom{l}{j,\,l-2j-x,\,j+x}a^j(b+n)^{l-2j-x}c^{j+x}$$

for every $x\in\{-l,\dots,l\}$. Using this information together with Eq. (3.9), we find that the solution of our initial-value problem has the form

$$u(x,H_n)=\frac{1}{n!}\sum_{l=|x|}^{n}\sum_{j=0}^{l} s(n,l)\binom{l}{j,\,l-2j-x,\,j+x}a^j(b+n)^{l-2j-x}c^{j+x}, \quad x\in\mathbb{Z},\, n\in\mathbb{N}_0.$$

As an illustration, consider the simple case when $a=c=1$, $b=-2$. Then

$$u(x,H_n)=\frac{1}{n!}\sum_{l=|x|}^{n}\sum_{j=0}^{l} s(n,l)\binom{l}{j,\,l-2j-x,\,j+x}(n-2)^{l-2j-x}, \quad x\in\mathbb{Z},\, n\in\mathbb{N}_0.$$

The next table shows the values of $u(x,t)$ for $t\in\{H_0,\dots,H_7\}$:

0	$\frac{1}{5040}$	$\frac{1}{360}$	$\frac{11}{720}$	$\frac{2}{45}$	$\frac{1}{12}$	$\frac{11}{90}$	$\frac{127}{840}$	$\frac{29}{180}$	$\frac{127}{840}$	$\frac{11}{90}$	$\frac{1}{12}$	$\frac{2}{45}$	$\frac{11}{720}$	$\frac{1}{360}$	$\frac{1}{5040}$	0
0	0	$\frac{1}{720}$	$\frac{1}{80}$	$\frac{31}{720}$	$\frac{1}{12}$	$\frac{89}{720}$	$\frac{37}{240}$	$\frac{59}{360}$	$\frac{37}{240}$	$\frac{89}{720}$	$\frac{1}{12}$	$\frac{31}{720}$	$\frac{1}{80}$	$\frac{1}{720}$	0	0
0	0	0	$\frac{1}{120}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{19}{120}$	$\frac{1}{6}$	$\frac{19}{120}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{120}$	0	0	0
0	0	0	0	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{24}$	0	0	0	0
0	0	0	0	0	$\frac{1}{6}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{6}$	0	0	0	0	0
0	0	0	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0
0	0	0	0	0	0	0	1	-1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0

3.2 Infinite series representation

We now proceed to the second method of solving the initial-value problem (3.1). In the next theorem, we obtain a "Taylor series" representation for the unique bounded solution. (For a discussion of Taylor series on time scales, see [2]. From a formal point of view, series solutions to countable linear systems of dynamic equations have been studied in [14].) We need the fact that for $t\geq t_0$, the time scale exponential $e_A(t,t_0)$ can be expressed as

$$e_A(t,t_0)=\sum_{k=0}^{\infty}A^k h_k(t,t_0), \quad t\in[t_0,\infty)_{\mathbb{T}}, \tag{3.10}$$

where $h_k:\mathbb{T}^2\to\mathbb{R}$, $k\in\mathbb{N}_0$, are the time scale polynomials (see [3]).

Theorem 3.5. *The unique bounded solution of the initial-value problem* (3.1) *is*

$$u(x,t)=\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\binom{k}{l,k-2l-x,l+x}a^l b^{k-2l-x}c^{l+x}\right)h_k(t,t_0),\quad x\in\mathbb{Z},\ t\in[t_0,\infty)_{\mathbb{T}}. \tag{3.11}$$

In the special case when $a=c$ and $a+b+c=0$, we have

$$u(x,t)=\sum_{k=0}^{\infty}\binom{2k}{x+k}(-1)^{x+k}a^k h_k(t,t_0),\quad x\in\mathbb{Z},\ t\in[t_0,\infty)_{\mathbb{T}}. \tag{3.12}$$

Proof. If we identify the solution $u:\mathbb{Z}\times\mathbb{T}\to\mathbb{R}$ with the vector-valued function $U:\mathbb{T}\to\ell^\infty(\mathbb{Z})$ given by $U(t)=\{u(x,t)\}_{x\in\mathbb{Z}}$, Theorem 2.1 says that $U(t)=e_A(t,t_0)U(t_0)$, where the operator $A:\ell^\infty(\mathbb{Z})\to\ell^\infty(\mathbb{Z})$ is given by

$$A(\{u_n\}_{n\in\mathbb{Z}})=\{au_{n+1}+bu_n+cu_{n-1}\}_{n\in\mathbb{Z}}.$$

Recalling the formula (3.10) for the time scale exponential function, we obtain

$$e_A(t,t_0)=\sum_{k=0}^{\infty}(A^kU(t_0))h_k(t,t_0),\quad t\in[t_0,\infty)_{\mathbb{T}}. \tag{3.13}$$

In our case, $U(t_0)$ is the sequence $(\dots,0,0,1,0,0,\dots)$. Let $A^kU(t_0)=\{V(x,k)\}_{x\in\mathbb{Z}}$. We now apply the generating function method to find an expression for $V(x,k)$. To this end, let $V_k(z)=\sum_{x=-\infty}^{\infty}V(x,k)z^x$. Then

$$V(x,k+1)=aV(x+1,k)+bV(x,k)+cV(x-1,k),$$

and consequently

$$V_{k+1}(z)=a/zV_k(z)+bV_k(z)+czV_k(z)=(a/z+b+cz)V_k(z).$$

Since $V_0(z)=1$, it follows that

$$V_k(z)=(a/z+b+cz)^k.$$

Recall that $V(x,k)$ coincides with the coefficient of z^x in $V_k(z)$. According to Lemma 3.2, we get

$$V(x,k)=\sum_{l=0}^{k}\binom{k}{l,k-2l-x,l+x}a^l b^{k-2l-x}c^{l+x}. \tag{3.14}$$

Returning back to Eq. (3.13), we conclude that the solution to the initial-value problem (3.1) is given by the formula (3.11).

Now, consider the case when $a=c$ and $a+b+c=0$, i.e., $b=-2a$. Instead of substituting these values into Eq. (3.11) and trying to simplify, it is easier to return to the formula for $V_k(z)$ and observe that

$$V_k(z)=(a/z-2a+az)^k=\frac{a^k(1-2z+z^2)^k}{z^k}=\frac{a^k(z-1)^{2k}}{z^k}.$$

The coefficient of z^x in $V_k(z)$ is the same as the coefficient of z^{x+k} in $a^k(z-1)^{2k}$, which is equal to $a^k\binom{2k}{x+k}(-1)^{2k-(x+k)}=\binom{2k}{x+k}(-1)^{x+k}a^k$. Using Eq. (3.13), we obtain the identity (3.12). □

In contrast to our first method, the identities (3.11) and (3.12) provide closed-form formulas for the solution. However, they require the knowledge of the time scale polynomials h_k, which are usually more difficult to determine than the time scale exponential function needed in the generating function method. For example, there seems to be no simple closed form for h_k when $\mathbb{T}=\{H_n,\ n\in\mathbb{N}_0\}$. Also, the use of (3.11) often leads to formulas which are more complicated than the ones obtained using generating functions (although the results have to be equivalent, it can be difficult to verify this fact; on the other hand this equivalence might lead to some interesting identities).

For example, when $\mathbb{T} = \mathbb{R}$ and $t_0 = 0$, we have $h_k(t, 0) = \frac{t^k}{k!}$, and the formula

$$u(x,t) = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} \binom{k}{l, k-2l-x, l+x} a^l b^{k-2l-x} c^{l+x} \right) \frac{t^k}{k!}$$

provided by Eq. (3.11) is not nearly as nice as the closed form (3.5) obtained in Example 3.1.

Similarly, for $\mathbb{T} = \mathbb{Z}$ and $t_0 = 0$, it is known that $h_k(t, 0) = \binom{t}{k}$ (see [3, Example 1.102]). Now, Eq. (3.11) gives the formula

$$u(x,t) = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} \binom{k}{l, k-2l-x, l+x} a^l b^{k-2l-x} c^{l+x} \right) \binom{t}{k}.$$

Since $\binom{t}{k} = 0$ for $k > t$, the infinite series reduces to the finite sum

$$\sum_{k=0}^{t} \left(\sum_{l=0}^{k} \binom{k}{l, k-2l-x, l+x} a^l b^{k-2l-x} c^{l+x} \right) \binom{t}{k}.$$

Still, this double sum is more complicated than the formula obtained in Example 3.3.

On the other hand, there are situations when formulas (3.11) and (3.12) might be helpful. For the time scale $\mathbb{T} = 2^{\mathbb{N}_0} = \{2^n;\, n \in \mathbb{N}_0\}$ and $t_0 = 1$, the exponential function needed in the generating function method has the form

$$e_{a/z+b+cz}(2^m, 1) = \prod_{k=0}^{m-1} \left(1 + 2^k \left(a/z + b + cz\right)\right), \quad m \in \mathbb{N}_0$$

(see [3, Example 2.55]), and there seems to be no simple closed form for the coefficient of z^x. In the next example, we use Theorem 3.5 to obtain solution formulas for the more general time scale $q^{\mathbb{N}_0}$, with $q > 1$.

Example 3.6. Consider the time scale $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n;\, n \in \mathbb{N}_0\}$, where $q > 1$ is a fixed real number. Assume that $t_0 = 1$. It is known (see [3, Example 1.104]) that

$$h_k(t, 1) = \prod_{n=0}^{k-1} \frac{t - q^n}{\sum_{j=0}^{n} q^j}, \quad t \in \mathbb{T}.$$

Substitution into Eq. (3.11) gives the formula

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \binom{k}{l, k-2l-x, l+x} a^l b^{k-2l-x} c^{l+x} \prod_{n=0}^{k-1} \frac{t - q^n}{\sum_{j=0}^{n} q^j}.$$

In fact, the first sum has only finitely many nonzero terms. The reason is that every $t \in \mathbb{T}$ has the form q^m, and hence the product inside the double sum is zero whenever k exceeds m. Thus, we see that

$$u(x, q^m) = \sum_{k=0}^{m} \sum_{l=0}^{k} \binom{k}{l, k-2l-x, l+x} a^l b^{k-2l-x} c^{l+x} \prod_{n=0}^{k-1} \frac{q^m - q^n}{\sum_{j=0}^{n} q^j}, \quad m \in \mathbb{N}_0.$$

In the simple case when $a = c = 1$ and $b = -2$, we obtain

$$u(x, q^m) = \sum_{k=0}^{m} \binom{2k}{x+k} (-1)^{x+k} \prod_{n=0}^{k-1} \frac{q^m - q^n}{\sum_{j=0}^{n} q^j}, \quad m \in \mathbb{N}_0.$$

For example, when $q = 2$, the formula has the particularly nice form

$$u(x, 2^m) = \sum_{k=0}^{m} \binom{2k}{x+k} (-1)^{x+k} \prod_{n=0}^{k-1} \frac{2^m - 2^n}{2^{n+1} - 1}, \quad m \in \mathbb{N}_0.$$

The next table shows the values of $u(x, t)$ for $t \in \{2^0, \dots, 2^5\}$:

0	1024	−8256	31448	−74458	121327	−142169	121327	−74458	31448	−8256	1024	0
0	0	64	−392	1142	−2049	2471	−2049	1142	−392	64	0	0
0	0	0	8	−34	71	−89	71	−34	8	0	0	0
0	0	0	0	2	−5	7	−5	2	0	0	0	0
0	0	0	0	0	1	−1	1	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0

Example 3.7. Consider Eq. (3.1) with $a = 0$. Then the formula (3.11) simplifies to

$$u(x, t) = \sum_{j=0}^{\infty} \binom{j}{0, j - x, x} b^{j-x} c^x h_j(t, t_0) = \sum_{j=0}^{\infty} \binom{j}{x} b^{j-x} c^x h_j(t, t_0).$$

In particular, the solution of the transport equation $u^{\Delta}(x, t) = -ku(x, t) + ku(x - 1, t)$ is

$$u(x, t) = \sum_{j=0}^{\infty} \binom{j}{x} (-1)^{j+x} k^j h_j(t, t_0), \quad x \in \mathbb{Z}, \ t \in [t_0, \infty)_{\mathbb{T}}.$$

4 Asymptotic behavior and summability

Our next goal is to study the asymptotic behavior of nonnegative solutions to the initial-value problem (3.1). We use the explicit solutions and techniques obtained in the previous sections. Since the general solution (3.11) is difficult to analyze, we focus on the discrete and continuous case only. We also discuss the time integrability of the solutions in the symmetric/nonsymmetric case, and its relationship with random walks.

4.1 Discrete problem

We start with the discrete case $\mathbb{T} = \mathbb{Z}$:

$$\begin{aligned} u(x, t + 1) - u(x, t) &= au(x + 1, t) + bu(x, t) + cu(x - 1, t), \\ u(x, 0) &= \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases} \end{aligned} \tag{4.1}$$

We know from Example 3.3 that the unique solution of the problem (4.1) is

$$u(x, t) = \sum_{j=0}^{t} \binom{t}{j, t - 2j - x, j + x} a^j (b + 1)^{t-2j-x} c^{j+x}, \quad x \in \mathbb{Z}, \, t \in \mathbb{N}_0. \tag{4.2}$$

In order to guarantee that the solution remains nonnegative, we restrict our attention to the case when $a, c \geq 0$ and $b \geq -1$.

The simplest situation occurs for $b = -1$, when the solution reduces to

$$u(x,t) = \begin{cases} \binom{t}{\frac{t+x}{2}} a^{(t-x)/2} c^{(t+x)/2} & \text{if } t+x \text{ is even,} \\ 0 & \text{if } t+x \text{ is odd.} \end{cases} \tag{4.3}$$

Using this formula, it is not difficult to obtain information about the asymptotic behavior and summability of the function $t \mapsto u(x,t)$.

Theorem 4.1. *Let u be the unique solution of (4.1) with $b = -1$ and $a, c > 0$. For every fixed $x \in \mathbb{Z}$, we have*

$$u(x,t) \sim \frac{\sqrt{2}}{\sqrt{\pi t}} \left(\sqrt{\frac{c}{a}}\right)^x (2\sqrt{ac})^t$$

for $t \to \infty$ and having the same parity as x.

Proof. The formula (4.3) implies that we may focus on t's which have the same parity as x. Then,

$$u(x,t) = \left(\sqrt{\frac{c}{a}}\right)^x (\sqrt{ac})^t \frac{t!}{\left(\frac{t+x}{2}\right)!\left(\frac{t-x}{2}\right)!}, \quad t \geq |x|.$$

Using the Stirling formula, we obtain the asymptotic estimate

$$\frac{t!}{\left(\frac{t+x}{2}\right)!\left(\frac{t-x}{2}\right)!} \sim \frac{\sqrt{2\pi t}}{\sqrt{\pi(t+x)}\sqrt{\pi(t-x)}} \frac{\left(\frac{t}{e}\right)^t}{\left(\frac{t+x}{2e}\right)^{\frac{t+x}{2}}\left(\frac{t-x}{2e}\right)^{\frac{t-x}{2}}} \sim \frac{\sqrt{2}}{\sqrt{\pi t}} \frac{(2t)^t}{(t^2-x^2)^{t/2}\left(\frac{t+x}{t-x}\right)^{x/2}} \sim \frac{\sqrt{2}}{\sqrt{\pi t}} 2^t,$$

which proves the statement. □

Theorem 4.1 has the following simple consequence: When $b = -1$ and $x \in \mathbb{Z}$ is arbitrary, the sum $\sum_{t=0}^{\infty} u(x,t)$ is finite if and only if $2\sqrt{ac} < 1$. The next theorem generalizes this statement to the case when $b \neq -1$.

Theorem 4.2. *Let u be the unique solution of (4.1) with $b \geq -1$ and $a, c \geq 0$. Then, for every $x \in \mathbb{Z}$, the sum $\sum_{t=0}^{\infty} u(x,t)$ is finite if and only if $2\sqrt{ac} + b < 0$. In particular, when $a + b + c = 0$, the sum is finite if and only if $a \neq c$.*

Proof. Using the explicit formulas from the end of Example 3.3, it is easy to check that $\sum_{t=0}^{\infty} u(x,t)$ is finite if $a = 0$ or $c = 0$; we leave this up to the reader and proceed to the case $ac \neq 0$. We calculate $\sum_{t=0}^{\infty} u(x,t)$ using the explicit formula (4.2):

$$\sum_{t=0}^{\infty} u(x,t) = \sum_{t=0}^{\infty} \sum_{j=0}^{t} \binom{t}{j, t-2j-x, j+x} a^j (b+1)^{t-2j-x} c^{j+x} \tag{4.4}$$

Since all terms in the double sum (4.4) are nonnegative, we can change the summation order and obtain

$$\sum_{t=0}^{\infty} u(x,t) = \sum_{j=0}^{\infty} a^j c^{j+x} S_j,$$

where

$$S_j = \binom{2j+x}{j, 0, j+x}(b+1)^0 + \binom{2j+x+1}{j, 1, j+x}(b+1)^1 + \cdots = \sum_{k=0}^{\infty} \binom{2j+x+k}{j, k, j+x}(b+1)^k.$$

Since $\binom{2j+x+k}{j,k,j+x} = \binom{2j+x}{j,j+x}\binom{2j+x+k}{2j+x}$, we have

$$S_j = \binom{2j+x}{j, j+x} \sum_{k=0}^{\infty} \binom{2j+x+k}{2j+x} (b+1)^k.$$

By the ratio test, the sum is finite when $b \in (-2, 0)$ and infinite for $b \in (-\infty, -2) \cup (0, \infty)$. For $b = 0$, the sum is also divergent because its terms do not approach zero. Note that for $b \geq 0$, the condition $2\sqrt{ac} + b < 0$ is not satisfied. Taking into account our assumption $b \geq -1$, we see that it is enough to focus on the case $b \in [-1, 0)$. Using the well-known identity $\sum_{k=0}^{\infty} \binom{r+k}{r} q^k = \frac{1}{(1-q)^{1+r}}$ (see [16, formula 26.3.4]), we get

$$S_j = \binom{2j+x}{j, j+x} \frac{1}{(-b)^{1+2j+x}}, \quad j \in \mathbb{N}_0,$$

and therefore

$$\sum_{t=0}^{\infty} u(x,t) = \sum_{j=0}^{\infty} a^j c^{j+x} \binom{2j+x}{j, j+x} \frac{1}{(-b)^{1+2j+x}} = \frac{c^x}{(-b)^{x+1}} \sum_{j=0}^{\infty} \left(\frac{ac}{b^2}\right)^j \binom{2j+x}{j, j+x}. \tag{4.5}$$

Note that

$$\frac{\binom{2(j+1)+x}{j+1, j+x+1}}{\binom{2j+x}{j,j+x}} = \frac{(2j+x+2)(2j+x+1)}{(j+x+1)(j+1)} \to 4$$

for $j \to \infty$. Thus, by the ratio test, the infinite series in (4.5) converges if $\frac{4ac}{b^2} < 1$ and diverges if $\frac{4ac}{b^2} > 1$. If $\frac{4ac}{b^2} = 1$, we use the Stirling formula to get the estimate

$$\left(\frac{ac}{b^2}\right)^j \binom{2j+x}{j, j+x} \sim \frac{1}{4^j} \frac{(2j+x)^{2j+x}}{(j+x)^{j+x} j^j} \sqrt{\frac{2j+x}{j(j+x)2\pi}}, \quad j \to \infty. \tag{4.6}$$

It is not difficult to verify that

$$\begin{aligned} (2j+x)^j &\sim (2j)^j e^{x/2}, \\ (2j+x)^{j+x} &\sim (2j+2x)^{j+x} e^{-x/2} = 2^{j+x} (j+x)^{j+x} e^{-x/2}, \\ \sqrt{\frac{2j+x}{j(j+x)2\pi}} &\sim \frac{1}{\sqrt{\pi j}} \end{aligned}$$

for $j \to \infty$. Using these estimates together with (4.6), we obtain

$$\left(\frac{ac}{b^2}\right)^j \binom{2j+x}{j, j+x} \sim \frac{2^x}{\sqrt{\pi j}},$$

and it follows from the limit comparison test that $\sum_{t=0}^{\infty} u(x,t)$ does not converge.

To sum up, we have proved that $\sum_{t=0}^{\infty} u(x,t)$ is finite if and only if $\frac{4ac}{b^2} < 1$, i.e., if and only if $2\sqrt{ac} < -b$. When $a + b + c = 0$, it follows from the AM-GM inequality that $2\sqrt{ac} \leq a + c = -b$ with the equality occuring if and only if $a = c$. $\square$

Next, we compute the exact value of the time sums in the cases when they are finite.

Theorem 4.3. *Let u be the unique solution of (4.1) with $b \geq -1$, $a, c \geq 0$, and $2\sqrt{ac} + b < 0$. Then for every $x \in \mathbb{Z}$, we have the following results:*

- *If $ac \neq 0$, then*

$$\sum_{t=0}^{\infty} u(x,t) = \begin{cases} \dfrac{(-b-\sqrt{b^2-4ac})^x}{2^x a^x \sqrt{b^2-4ac}} & \textit{for } x \geq 0, \\ \dfrac{(-b-\sqrt{b^2-4ac})^{-x}}{2^{-x} c^{-x} \sqrt{b^2-4ac}} & \textit{for } x \leq 0. \end{cases}$$

- *If $a = 0$, then*

$$\sum_{t=0}^{\infty} u(x,t) = \begin{cases} c^x(-b)^{-1-x} & \textit{for } x \geq 0, \\ 0 & \textit{for } x < 0. \end{cases}$$

- *If $c = 0$, then*

$$\sum_{t=0}^{\infty} u(x,t) = \begin{cases} 0 & \textit{for } x > 0, \\ a^{-x}(-b)^{-1+x} & \textit{for } x \leq 0. \end{cases}$$

In particular, if $a+b+c=0$, we obtain the following results:

- *If $c > a > 0$, then*

$$\sum_{t=0}^{\infty} u(x,t) = \begin{cases} \frac{\left(\frac{c}{a}\right)^x}{c-a} & \textit{for } x \leq 0, \\ \frac{1}{c-a} & \textit{for } x \geq 0. \end{cases}$$

- *If $a > c > 0$, then*

$$\sum_{t=0}^{\infty} u(x,t) = \begin{cases} \frac{1}{a-c} & \textit{if } x \leq 0, \\ \frac{\left(\frac{c}{a}\right)^x}{a-c} & \textit{if } x \geq 0. \end{cases}$$

- *If $a = 0$, then*

$$\sum_{t=0}^{\infty} u(x,t) = \begin{cases} \frac{1}{c} & \textit{for } x \geq 0, \\ 0 & \textit{for } x < 0. \end{cases}$$

- *If $c = 0$, then*

$$\sum_{t=0}^{\infty} u(x,t) = \begin{cases} 0 & \textit{for } x > 0, \\ \frac{1}{a} & \textit{for } x \leq 0. \end{cases}$$

Proof. Using the explicit formulas from the end of Example 3.3, it is easy to verify the statements for $a = 0$ and $c = 0$. We proceed to the case $ac \neq 0$. In the proof of the last theorem, we arrived to the formula

$$\sum_{t=0}^{\infty} u(x,t) = \frac{c^x}{(-b)^{x+1}} \sum_{j=0}^{\infty} \left(\frac{ac}{b^2}\right)^j \binom{2j+x}{j, j+x}.$$

Recall our convention that $\binom{2j+x}{j,j+x}$ equals $\frac{(2j+x)!}{j!(j+x)!} = \binom{2j+x}{j}$ if $j + x \geq 0$, and is zero otherwise. Consider the case $x \geq 0$. We use the identity

$$\sum_{j=0}^{\infty} \binom{tj+r}{j} q^j = \frac{\mathcal{B}_t(q)^r}{1 - t + t\mathcal{B}_t(q)^{-1}}, \quad t, r \in \mathbb{Z},$$

where $\mathcal{B}_t(q) = \sum_{j=0}^{\infty} \binom{tj+1}{j} \frac{1}{tj+1} q^j$ is the generalized binomial series (see formulas (5.58) and (5.61) in [8]). By letting $t = 2$, $r = x$ and $q = \frac{ac}{b^2}$, we get

$$\sum_{t=0}^{\infty} u(x,t) = \frac{c^x}{(-b)^{x+1}} \frac{\mathcal{B}_2(q)^x}{-1 + 2\mathcal{B}_2(q)^{-1}} = \frac{c^x}{(-b)^{x+1}} \frac{\mathcal{B}_2(q)^{x+1}}{2 - \mathcal{B}_2(q)}.$$

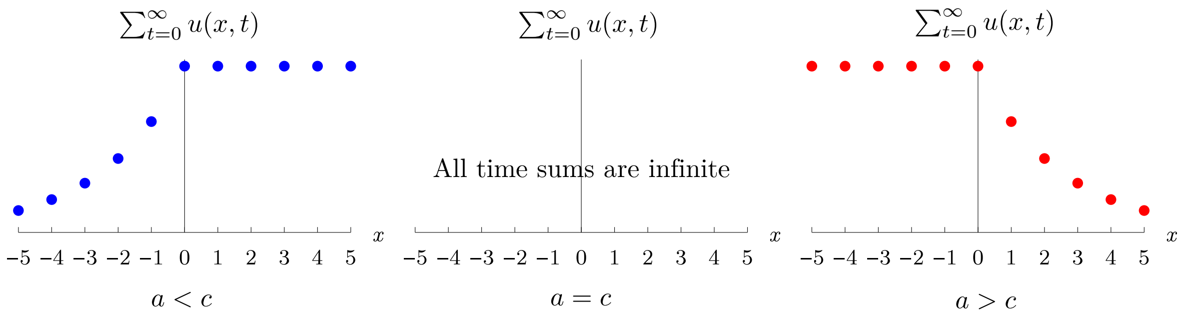

Figure 2: Time sums for the discrete diffusion equation with $a+b+c=0$ (cf. Theorem 4.3)

However, it is known that $\mathcal{B}_2$ can be expressed in the closed form $\mathcal{B}_2(q) = \frac{1-\sqrt{1-4q}}{2q}$ for $q \neq 0$ (see identity (5.68) in [8]). It follows that

$$\mathcal{B}_2(q) = \frac{1-\sqrt{1-4\frac{ac}{b^2}}}{2\frac{ac}{b^2}} = \frac{b+\sqrt{b^2-4ac}}{2\frac{ac}{b}},$$

and therefore

$$\begin{aligned}\sum_{t=0}^{\infty} u(x,t) &= \frac{c^x}{(-b)^{x+1}} \frac{\mathcal{B}_2(q)^{x+1}}{2-\mathcal{B}_2(q)} = \frac{c^x}{(-b)^{x+1}} \frac{(b+\sqrt{b^2-4ac})^{x+1}}{(2\frac{ac}{b})^{x+1}} \frac{2\frac{ac}{b}}{4\frac{ac}{b}-b-\sqrt{b^2-4ac}} \\ &= \frac{(-b-\sqrt{b^2-4ac})^{x+1}}{2^x a^x (4ac-b^2-b\sqrt{b^2-4ac})} = \frac{(-b-\sqrt{b^2-4ac})^x}{2^x a^x \sqrt{b^2-4ac}},\end{aligned}$$

which completes the proof for $x \geq 0$.

For $x \leq 0$, it is enough to note that $u(x,t) = v(-x,t)$, where v is the unique solution of the problem

$$\begin{aligned} v(x,t+1) - v(x,t) &= cv(x+1,t) + bv(x,t) + av(x-1,t), \\ v(x,0) &= \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0 \end{cases} \end{aligned}$$

(in comparison with (4.1), the coefficients a and c are switched). Hence, according to the first part of the proof, we obtain

$$\sum_{t=0}^{\infty} u(x,t) = \sum_{t=0}^{\infty} v(-x,t) = \frac{(-b-\sqrt{b^2-4ac})^{-x}}{2^{-x} c^{-x} \sqrt{b^2-4ac}}.$$

Finally, if $a+b+c=0$, we get $\sqrt{b^2-4ac} = \sqrt{(a+c)^2-4ac} = \sqrt{(a-c)^2} = |a-c|$, and the desired results follow easily. □

Remark 4.4. When $p,q,r \in [0,1]$ and $p+q+r=1$, we know that the initial-value problem

$$\begin{aligned} u(x,t+1) - u(x,t) &= pu(x+1,t) + (q-1)u(x,t) + ru(x-1,t), \quad x \in \mathbb{Z},\ t \in \mathbb{N}_0, \\ u(x,0) &= \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases} \end{aligned}$$

describes the one-dimensional discrete-time random walk $(X_t)_{t\in\mathbb{N}_0}$ starting from the origin; at every step, the probabilities of going left, standing still, and going right are p, q, and r, respectively. The probability that the random walk visits point x at time t is $P(X_t = x) = u(x,t)$. Now, let us explain the meaning of Theorems 4.2 and 4.3 in this context.

Following the ideas from [7, page 191], we introduce the random variables τ_n^x, $x \in \mathbb{Z}$, $n \in \mathbb{N}_0$:

$$\begin{aligned} \tau_0^x &= \inf\{t \in \mathbb{N}_0; X_t = x\}, \\ \tau_n^x &= \inf\{t > \tau_{n-1}^x; X_t = x\}, \quad n \in \mathbb{N}. \end{aligned}$$

Moreover, let

$$V_x = \sum_{t=0}^{\infty} 1_{(X_t=x)} = \sum_{n=0}^{\infty} 1_{(\tau_n^x<\infty)}$$

be the total time spent at x. Then

$$\mathrm{E}V_x = \sum_{t=0}^{\infty} \mathrm{E}1_{(X_t=x)} = \sum_{t=0}^{\infty} u(x,t),$$

and also

$$\mathrm{E}V_x = \sum_{n=0}^{\infty} \mathrm{E}1_{(\tau_n^x<\infty)} = \sum_{n=0}^{\infty} P(\tau_n^x < \infty).$$

Thus $\sum_{t=0}^{\infty} u(x,t)$ is equal to the expected time spent at x, and

$$\sum_{t=0}^{\infty} u(x,t) = \sum_{n=0}^{\infty} P(\tau_n^x < \infty) = P(\tau_0^x < \infty) + \sum_{n=1}^{\infty} P(\tau_1^x < \infty)^n. \tag{4.7}$$

In the special case when $x = 0$, it follows that $P(\tau_1^0 < \infty) = 1$ if and only if $\sum_{t=0}^{\infty} u(0,t) = \infty$. Hence, Theorem 4.2 can be rephrased in the following way: The random walk returns almost surely to the origin if and only if $p = r$. (In the language of Markov chains, the origin is a recurrent state if $p = r$, and transient if $p \neq r$.)

When $p \neq r$, the sum $\sum_{t=0}^{\infty} u(0,t)$ is finite. Consequently, $P(\tau_1^0 < \infty) < 1$, and

$$\sum_{t=0}^{\infty} u(0,t) = 1 + \sum_{n=1}^{\infty} P(\tau_1^0 < \infty)^n = \frac{1}{1 - P(\tau_1^0 < \infty)}. \tag{4.8}$$

Combining this result with Theorem 4.3, we get

$$1 - P(\tau_1^0 < \infty) = \frac{1}{\sum_{t=0}^{\infty} u(0,t)} = |p - r|,$$

and therefore the probability that the random walk returns to the origin is

$$P(\tau_1^0 < \infty) = 1 - |p - r|.$$

It is likely that this result is already known, but we have been unable to find it in the literature. In the special case when $q = 0$, we get

$$P(\tau_1^0 < \infty) = 1 - |p - r| = p + r - |p - r| = 2\min(p, r),$$

which agrees with the result in [20, paragraph 1.1.4].

For $p \neq r$ and $x \neq 0$, Eq. (4.7) implies

$$\sum_{t=0}^{\infty} u(x,t) = P(\tau_0^x < \infty) + \frac{P(\tau_1^x < \infty)}{1 - P(\tau_1^x < \infty)}.$$

However, the probability $P(\tau_1^x < \infty)$ clearly does not depend on the choice of x. Consequently, we get

$$P(\tau_0^x < \infty) = \sum_{t=0}^{\infty} u(x,t) - \frac{P(\tau_1^0 < \infty)}{1 - P(\tau_1^0 < \infty)} = \sum_{t=0}^{\infty} u(x,t) - \frac{1}{1 - P(\tau_1^0 < \infty)} + 1 = \sum_{t=0}^{\infty} u(x,t) - \sum_{t=0}^{\infty} u(0,t) + 1.$$

Now, consider the case $p < r$ (the analysis for $p > r$ is similar). By Theorem 4.3, the sums $\sum_{t=0}^{\infty} u(x,t)$ have the same values for all $x \geq 0$, which simply means that the probability of visiting any point $x > 0$ is

$$P(\tau_0^x < \infty) = \sum_{t=0}^{\infty} u(x,t) - \sum_{t=0}^{\infty} u(0,t) + 1 = 1.$$

On the other hand, the probability of visiting a point $x < 0$ is less than 1, namely

$$P(\tau_0^x < \infty) = \sum_{t=0}^{\infty} u(x,t) - \sum_{t=0}^{\infty} u(0,t) + 1 = \frac{1}{r-p}\left(\frac{r^x}{p^x} - 1\right) + 1.$$

4.2 Semidiscrete problem

Next, we consider the semidiscrete case $\mathbb{T} = \mathbb{R}$:

$$\begin{aligned} u_t(x,t) &= au(x+1,t) + bu(x,t) + cu(x-1,t), \quad x \in \mathbb{Z},\ t \in [0,\infty), \\ u(x,t_0) &= \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases} \end{aligned} \tag{4.9}$$

From Example 3.1, we know that if $ac \neq 0$, the unique locally bounded solution of (4.9) is

$$u(x,t) = e^{bt} I_x(2t\sqrt{ac}) \left(\sqrt{\frac{c}{a}}\right)^x. \tag{4.10}$$

Again, we examine the asymptotic behavior and summability of the function $t \mapsto u(x,t)$. We will see that the results for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ are quite similar. In a certain sense, the semidiscrete case is even easier to analyze (in the asymptotic analysis, we do not restrict ourselves to a particular value of b).

Recall that if $a, c \geq 0$ and $a + b + c = 0$, then Eq. (4.9) corresponds to the continuous-time random walk on $\mathbb{Z}$, in which the time spent in a given state is given by the exponential distribution. As in the discrete case, one motivation for investigating the summability is the fact that $\int_0^\infty u(x,t)\,\mathrm{d}t$ represents the total expected time spent at x. Also, the origin is a recurrent state if $\int_0^\infty u(0,t)\,\mathrm{d}t = \infty$, and transient otherwise (see [15, Theorem 3.4.2]).

Theorem 4.5. *Let u be the unique locally bounded solution of (4.9) with $a, c > 0$. For every $x \in \mathbb{Z}$, we have*

$$u(x,t) \sim \frac{1}{\sqrt{4\pi t\sqrt{ac}}} \left(\sqrt{\frac{c}{a}}\right)^x e^{(2\sqrt{ac}+b)t}$$

for $t \to \infty$.

Proof. Starting from the explicit solution (4.10) and using the asymptotic estimate $I_x(t) \sim \frac{e^t}{\sqrt{2\pi t}}$ (see [16, formula 10.40.1]), we get

$$u(x,t) \sim e^{bt} \frac{e^{2t\sqrt{ac}}}{\sqrt{4\pi t\sqrt{ac}}} \left(\sqrt{\frac{c}{a}}\right)^x,$$

which proves the statement. ☐

Corollary 4.6. *Let u be the unique locally bounded solution of* (4.9) *with $a, c \geq 0$. Then, for every $x \in \mathbb{Z}$, the integral $\int_0^\infty u(x,t)\,\mathrm{d}t$ is finite if and only if $2\sqrt{ac} + b < 0$. In particular, when $a + b + c = 0$, the integral is finite if and only if $a \neq c$.*

Proof. Using the explicit formulas from the end of Example 3.1, it is easy to verify that the statement holds when $a = 0$ or $c = 0$. For $ac \neq 0$, the asymptotic estimate from Theorem 4.5 implies that $\int_0^\infty u(x,t)\,\mathrm{d}t$ is finite if and only if $2\sqrt{ac} + b < 0$. When $a + b + c = 0$, it follows from the AM-GM inequality that $2\sqrt{ac} + b \leq a + c + b = 0$ with the equality occuring if and only if $a = c$. ☐

Given the complicated nature of explicit solutions, it is surprising that the exact values of time integrals coincide with the discrete case (cf. Theorem 4.3 and Figure 2).

Theorem 4.7. *Let u be the unique locally bounded solution of* (4.9) *with $a, c \geq 0$, $a \neq c$, and $2\sqrt{ac}+b < 0$. Then for every $x \in \mathbb{Z}$, we have the following results:*

- *If $ac \neq 0$, then*

$$\int_0^\infty u(x,t)\,\mathrm{d}t = \begin{cases} \dfrac{(-b - \sqrt{b^2 - 4ac})^x}{2^x a^x \sqrt{b^2 - 4ac}} & \text{for } x \geq 0, \\[2ex] \dfrac{(-b - \sqrt{b^2 - 4ac})^{-x}}{2^{-x} c^{-x} \sqrt{b^2 - 4ac}} & \text{for } x \leq 0. \end{cases}$$

- *If $a = 0$, then*

$$\int_0^\infty u(x,t)\,\mathrm{d}t = \begin{cases} c^x(-b)^{-1-x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

- *If $c = 0$, then*

$$\int_0^\infty u(x,t)\,\mathrm{d}t = \begin{cases} 0 & \text{for } x > 0, \\ a^{-x}(-b)^{-1+x} & \text{for } x \leq 0. \end{cases}$$

In particular, if $a + b + c = 0$, we obtain the following results:

- *If $c > a > 0$, then*

$$\int_0^\infty u(x,t)\,\mathrm{d}t = \begin{cases} \frac{\left(\frac{c}{a}\right)^x}{c-a} & \text{for } x \leq 0, \\ \frac{1}{c-a} & \text{for } x \geq 0. \end{cases}$$

- *If $a > c > 0$, then*

$$\int_0^\infty u(x,t)\,\mathrm{d}t = \begin{cases} \frac{1}{a-c} & \text{for } x \leq 0, \\ \frac{\left(\frac{c}{a}\right)^x}{a-c} & \text{for } x \geq 0. \end{cases}$$

- *If $a = 0$, then*

$$\int_0^\infty u(x,t)\,\mathrm{d}t = \begin{cases} \frac{1}{c} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

- *If $c = 0$, then*

$$\int_0^\infty u(x,t)\,\mathrm{d}t = \begin{cases} 0 & \text{for } x > 0, \\ \frac{1}{a} & \text{for } x \le 0. \end{cases}$$

Proof. Using the explicit formulas from the end of Example 3.1, the reader can easily verify the statements for $a = 0$ and $c = 0$. In the case when $ac \neq 0$, we use (4.10) and the identity

$$\int_0^\infty e^{-\alpha t} I_x(\beta t)\,\mathrm{d}t = \frac{(\alpha - \sqrt{\alpha^2 - \beta^2})^x}{\beta^x \sqrt{\alpha^2 - \beta^2}},$$

which holds for $x > -1$ and $\alpha > |\beta|$ (see [9, formula 6.611.4]). In our case, $\alpha = -b$, $\beta = 2\sqrt{ac}$, and hence $\alpha > \beta = |\beta|$. Also, since $\sqrt{\alpha^2 - \beta^2} = \sqrt{b^2 - 4ac}$, we obtain the following results:
If $x \ge 0$, then

$$\int_0^\infty u(x,t)\,\mathrm{d}t = \left(\sqrt{\frac{c}{a}}\right)^x \int_0^\infty e^{bt} I_x\left(2t\sqrt{ac}\right)\mathrm{d}t = \left(\sqrt{\frac{c}{a}}\right)^x \frac{(-b - \sqrt{b^2 - 4ac})^x}{(2\sqrt{ac})^x \sqrt{b^2 - 4ac}} = \frac{(-b - \sqrt{b^2 - 4ac})^x}{2^x a^x \sqrt{b^2 - 4ac}}.$$

If $x \le 0$, we use the fact that $I_x(t) = I_{-x}(t)$ and obtain

$$\int_0^\infty u(x,t)\,\mathrm{d}t = \left(\sqrt{\frac{c}{a}}\right)^x \int_0^\infty e^{bt} I_{-x}\left(2t\sqrt{ac}\right)\mathrm{d}t = \left(\sqrt{\frac{c}{a}}\right)^x \frac{(-b - \sqrt{b^2 - 4ac})^{-x}}{(2\sqrt{ac})^{-x} \sqrt{b^2 - 4ac}} = \frac{(-b - \sqrt{b^2 - 4ac})^{-x}}{2^{-x} c^{-x} \sqrt{b^2 - 4ac}}.$$

Finally, if $a + b + c = 0$, we get $\sqrt{b^2 - 4ac} = \sqrt{(a+c)^2 - 4ac} = \sqrt{(a-c)^2} = |a - c|$, and the desired results follow easily. ☐

For an arbitrary time scale $\mathbb{T}$, it follows from Theorems 2.3 and 2.4 that if the initial condition is a probability distribution, than so it is for any $t > 0$, provided that $a + b + c = 0$ and $\mu(t) \le -1/b$ for all $t \in \mathbb{T}$. Thus, the solutions of the discrete-space diffusion equation can be interpreted as generalizations of the standard random walk with rich timing possibilities. In Section 3, we presented two methods for finding closed-form solutions to such heterogeneous processes on arbitrary time scales. Unfortunately, it seems that the results are not easily amenable to the analysis of the asymptotic behavior or summability. As before, the question of summability has a probabilistic interpretation because the time integral $\int_0^\infty u(x,t)\Delta t$ represents the total expected time spent at x.

In [19], the connection between the solutions of the discrete-space transport equation and counting stochastic processes is discussed in more details.

5 Multidimensional problems

In this section, we consider N-dimensional diffusion-type equations of the form

$$\begin{aligned} u^\Delta(x_1,\dots,x_N,t) &= a_1 u(x_1+1,x_2,\dots,x_N,t) + \cdots + a_N u(x_1,x_2,\dots,x_N+1,t) + bu(x_1,\dots,x_N,t) \\ &+ c_1 u(x_1-1,x_2,\dots,x_N,t) + \cdots + c_N u(x_1,x_2,\dots,x_N-1,t), \end{aligned}$$

where $t \in \mathbb{T}$, $x_1,\dots,x_N \in \mathbb{Z}$, and $a_1,\dots,a_N,b,c_1,\dots,c_N \in \mathbb{R}$. We use the notation $x = (x_1,\dots,x_N) \in \mathbb{Z}^N$. If $e_1,\dots,e_N$ is the canonical base of $\mathbb{R}^N$, we can write our equation in the more compact form

$$u^\Delta(x,t) = \sum_{i=1}^N a_i u(x+e_i,t) + bu(x,t) + \sum_{i=1}^N c_i u(x-e_i,t), \quad t \in \mathbb{T},\ x \in \mathbb{Z}^N.$$

Obviously, when $a_1 = \cdots = a_N = c_1 = \cdots = c_N = -\frac{b}{2N}$, the problem can be interpreted as the space discretization of the classical N-dimensional diffusion equation. If $\mathbb{T} = \mathbb{Z}$, the values $a_1, \ldots, a_N, c_1, \ldots, c_N, b+1$ are from $[0,1]$ and their sum is 1, the equation corresponds to the N-dimensional random walk; at every step, the values $a_1, \ldots, a_N, c_1, \ldots, c_N$ are the probabilities of going in a direction parallel to one of the coordinate axes, while the probability of standing still is $b+1$.

It turns out that the results obtained in [18] are easily extendable to the N-dimensional case. In the existence-uniqueness theorem, the space $\ell^\infty(\mathbb{Z})$ has to be replaced by $\ell^\infty(\mathbb{Z}^N)$, and the operator $A : \ell^\infty(\mathbb{Z}^N) \to \ell^\infty(\mathbb{Z}^N)$ takes the form

$$A\left(\{u_x\}_{x\in\mathbb{Z}^N}\right) = \left\{\sum_{i=1}^N a_i u_{x+e_i} + b u_x + \sum_{i=1}^N c_i u_{x-e_i}\right\}_{x\in\mathbb{Z}^N}.$$

The norm of A is $\sum_i |a_i| + |b| + \sum_i |c_i|$. Using the same approach as in [18], one can prove the existence of a unique bounded solution on the interval $[T_1, T_2]_\mathbb{T}$ corresponding to a given bounded initial condition at t_0, provided that

$$\mu(t) < \frac{1}{\sum_i |a_i| + |b| + \sum_i |c_i|}, \quad t \in [T_1, t_0)_\mathbb{T}. \tag{5.1}$$

Also, the proof of the superposition principle is completely analoguous to the one-dimensional case.

The N-dimensional analogue of Theorem 2.3 is valid for equations with $\sum_i a_i + b + \sum_i c_i = 0$; moreover, it is necessary to replace the graininess condition by (5.1). To get the N-dimensional minimum and maximum principles (cf. Theorem 2.4), one has to assume that $a_1, \ldots, a_N, c_1, \ldots, c_N \geq 0$, $\mu(t) \leq -1/b$ for $t \in [T_1, T_2)_\mathbb{T}$, and $\sum_i a_i + b + \sum_i c_i \geq 0$ or $\sum_i a_i + b + \sum_i c_i \leq 0$, respectively.

Finally, for equations with symmetric right-hand sides, i.e. those with $a_i = c_i$, $i \in \{1, \ldots, N\}$, we can obtain the N-dimensional versions of Theorems 2.5 and 2.6. Again, the graininess condition from Theorem 2.5 has to be replaced by (5.1).

In the rest of this section, we restrict ourselves for simplicity to equations with symmetric and sum-preserving right-hand sides of the form

$$u^\Delta(x,t) = \sum_{i=1}^N a u(x+e_i,t) - 2aNu(x,t) + \sum_{i=1}^N a u(x-e_i,t), \quad t \in \mathbb{T},\ x \in \mathbb{Z}^N,$$

where $a \in \mathbb{R}$ is a parameter. In particular, we focus on the time scales $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$. In both cases, we obtain explicit solutions, and also investigate the summability and asymptotic behavior.

5.1 Discrete multidimensional problem

First, we turn our attention to the discrete case $\mathbb{T} = \mathbb{Z}$ and the initial-value problem

$$\begin{aligned} u(x,t+1) - u(x,t) &= \textstyle\sum_{i=1}^N a u(x+e_i,t) - 2aNu(x,t) + \sum_{i=1}^N a u(x-e_i,t), \\ u(x,0) &= \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.2}$$

As in Section 3, we can use the generating function method to find the explicit solution. Let

$$F(z_1, \ldots, z_N, t) = \sum_{x_1=-\infty}^{\infty} \cdots \sum_{x_N=-\infty}^{\infty} u(x,t) z_1^{x_1} \ldots z_N^{x_N}.$$

Using (5.2), we obtain

$$F(z, t+1) = (az_1 + \ldots + az_N + a/z_1 + \ldots + a/z_N - 2aN + 1)\, F(z,t),$$

and it follows that

$$F(z,t) = a^t \left(z_1 + \ldots + z_N + 1/z_1 + \ldots + 1/z_N - 2N + 1/a\right)^t, \quad t \in \mathbb{N}_0.$$

Applying the multinomial theorem, we get

$$F(z,t) = a^t \sum_{k_1+\ldots+k_{2N+1}=t} \binom{t}{k_1,\ldots,k_{2N+1}} z_1^{k_1-k_{N+1}} \ldots z_N^{k_N-k_{2N}} (-2N+1/a)^{k_{2N+1}}.$$

Trying to find $u(x,t)$, we seek the coefficient by the term $z_1^{x_1}\ldots z_N^{x_N}$. It must hold that $k_i - k_{N+i} = x_i$, $i \in \{1,\ldots,N\}$. Combining this fact with $k_1 + \ldots + k_{2N+1} = t$, we obtain

$$u(x,t) = a^t \sum_{k_1,\ldots,k_N \in \{0,\ldots t\}} \binom{t}{k_1,\ldots,k_N,k_1-x_1,\ldots,k_N-x_N,t+\sum_i(x_i-2k_i)} (1/a-2N)^{t+\sum\limits_i (x_i-2k_i)}.$$

Our next goal is to investigate the summability of the function $t \mapsto u(x,t)$. We restrict ourselves to the case $a \in (0, \frac{1}{2N}]$, which guarantees that the solution remains nonnegative. First, we focus on $x = 0$, and later extend our analysis to all $x \in \mathbb{Z}^N$.

Lemma 5.1. *Let u be the unique solution of (5.2) with $a \in (0, \frac{1}{2N}]$. Then, the finiteness of the sum $\sum_{t=0}^{\infty} u(0,t)$ does not depend on the value of a.*

Proof. First, we note that

$$\binom{t}{k_1,\ldots,k_N,k_1,\ldots,k_N,t-2\sum_i k_i} = \binom{t}{2\sum_i k_i}\binom{2\sum_i k_i}{k_1,\ldots,k_N,k_1,\ldots,k_N}.$$

All terms in the sum $\sum_{t=0}^{\infty} u(0,t)$ are nonnegative, and we can rearrange them in the following way:

$$\sum_{t=0}^{\infty} u(0,t) = \sum_{t=0}^{\infty} a^t \sum_{k_1,\ldots,k_N \in \{0,\ldots t\}} \binom{t}{2\sum_i k_i}\binom{2\sum_i k_i}{k_1,\ldots,k_N,k_1,\ldots,k_N} (1/a-2N)^{t-2\sum_i k_i} =$$

$$= \sum_{k_1,\ldots,k_N=0}^{\infty} \left(\binom{2\sum_i k_i}{k_1,\ldots,k_N,k_1,\ldots,k_N} a^{2\sum_i k_i} \sum_{t=2\sum_i k_i}^{\infty} \binom{t}{2\sum_i k_i} a^{t-2\sum_i k_i} (1/a-2N)^{t-2\sum_i k_i} \right).$$

We evaluate the inner sum using the identity $\sum_{k=0}^{\infty}\binom{r+k}{r} q^k = \frac{1}{(1-q)^{1+r}}$ (see [16, formula 26.3.4]):

$$\sum_{t=2\sum_i k_i}^{\infty} \binom{t}{2\sum_i k_i} a^{t-2\sum_i k_i}(1/a-2N)^{t-2\sum_i k_i} = \sum_{t=2\sum_i k_i}^{\infty} \binom{t}{2\sum_i k_i}(1-2aN)^{t-2\sum_i k_i} = \frac{1}{(2aN)^{1+2\sum_i k_i}}.$$

Hence, we get

$$\sum_{t=0}^{\infty} u(0,t) = \sum_{k_1,\ldots,k_N=0}^{\infty} \binom{2\sum_i k_i}{k_1,\ldots,k_N,k_1,\ldots,k_N} a^{2\sum_i k_i} \frac{1}{(2aN)^{1+2\sum_i k_i}}$$

$$= \frac{1}{2aN} \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_N=k} \binom{2k}{k_1,\ldots,k_N,k_1,\ldots,k_N} \frac{1}{(2N)^{2k}},$$

and the convergence of the last sum does not depend on the value of a. □

Next, we investigate the finiteness of the sum at $x = 0$.

Theorem 5.2. *Let u be the unique solution of* (5.2) *with $a \in (0, \frac{1}{2N}]$. Then, the sum $\sum_{t=0}^{\infty} u(0,t)$ is finite if and only if $N \geq 3$.*

Proof. For $N = 1$, the statement was already proved in Theorem 4.2. Thanks to the previous lemma, it is enough to examine the case $a = \frac{1}{2N}$, where $N \geq 2$. In this situation, the formula for $u(0,t)$ simplifies to

$$u(0,t) = \frac{1}{(2N)^t} \sum_{k_1+\cdots+k_N=t/2} \binom{t}{k_1,\ldots,k_N,k_1,\ldots,k_N}.$$

Obviously, $u(0,t) = 0$ if t is odd. In the following calculations, t is always even.

For $N = 2$, we have

$$u(0,t) = \frac{1}{4^t} \sum_{k=0}^{t/2} \frac{t!}{k!k!\left(\frac{t}{2}-k\right)!\left(\frac{t}{2}-k\right)!} = \frac{t!}{\left(\frac{t}{2}\right)!\left(\frac{t}{2}\right)!} \frac{1}{4^t} \sum_{k=0}^{t/2} \binom{\frac{t}{2}}{k}\binom{\frac{t}{2}}{k} = \frac{1}{4^t}\binom{t}{\frac{t}{2}}^2 \sim \frac{1}{4^t}\left(\frac{2^t\sqrt{2}}{\sqrt{\pi t}}\right)^2 = \frac{4}{\pi t}$$

as $t \to \infty$, where the asymptotic estimate is a consequence of the Stirling formula. Hence, $\sum_{t=0}^{\infty} u(0,t)$ is infinite.

Now, consider the case $N \geq 3$. For all $k \in \mathbb{N}_0$, the N-dimensional versions of Theorem 2.6 and the maximum principle lead to the estimate

$$u(0, k\cdot 2N + j) \leq u(0, k\cdot 2N), \quad j \in \{0, 2, \ldots, 2N-2\}.$$

Consequently, $\sum_{t=0}^{\infty} u(0,t) \leq N \sum_{k=0}^{\infty} u(0, k\cdot 2N)$, and it remains to prove that the latter sum is finite. We have

$$\begin{aligned} u(0, k\cdot 2N) &= \frac{1}{(2N)^{2kN}} \sum_{k_1+\cdots+k_N=kN} \binom{2kN}{k_1,\ldots,k_N,k_1,\ldots,k_N} \\ &= \binom{2kN}{kN}\frac{1}{(2N)^{2kN}} \sum_{k_1+\cdots+k_N=kN} \binom{kN}{k_1,\ldots,k_N}\binom{kN}{k_1,\ldots,k_N}. \end{aligned}$$

Since $\binom{kN}{k_1,\ldots,k_N}$ has the greatest value when $k_1 = \cdots = k_N = k$, we obtain the estimate

$$u(0, k\cdot 2N) \leq \binom{2kN}{kN}\frac{1}{(2N)^{2kN}}\frac{(kN)!}{k!^N} \sum_{k_1+\cdots+k_N=kN} \binom{kN}{k_1,\ldots,k_N} = \binom{2kN}{kN}\frac{1}{(2N)^{2kN}}\frac{(kN)!}{k!^N} N^{kN}.$$

We simplify the right-hand side and apply the Stirling formula to obtain

$$u(0, k\cdot 2N) \leq \frac{(2kN)!}{(kN)!k!^N(4N)^{kN}} \sim \frac{\sqrt{4\pi kN}}{\sqrt{2\pi kN}(\sqrt{2\pi k})^N}\left(\frac{2kN}{e}\right)^{2kN}\left(\frac{e}{kN}\right)^{kN}\left(\frac{e}{k}\right)^{kN}\frac{1}{(4N)^{kN}} = \frac{\sqrt{2}}{(2\pi k)^{N/2}}$$

as $k \to \infty$. Consequently, $\sum_{k=0}^{\infty} u(0, k\cdot 2N)$ is finite. □

Finally, we extend Theorem 5.2 to arbitrary $x \in \mathbb{Z}^N$.

Theorem 5.3. *Let u be the unique solution of* (5.2) *with $a \in (0, \frac{1}{2N}]$. For every $x \in \mathbb{Z}^N$, the sum $\sum_{t=0}^{\infty} u(x,t)$ is finite if and only if $N \geq 3$.*

Proof. We already know that the statement is true for $x = 0$; it remains to consider the case $x \neq 0$.

- $N \le 2$. Since $a \in (0, \frac{1}{2N}]$, the solution u is nonnegative. Let $t_0 = \min\{t \in \mathbb{Z}; u(x,t) > 0\}$. Using the N-dimensional version of the superposition principle (Theorem 2.2), we conclude that $u(x,t) \ge u(x,t_0)u(0,t-t_0)$ for every $t \ge t_0$, and therefore
$$\sum_{t=0}^{\infty} u(x,t) = \sum_{t=t_0}^{\infty} u(x,t) \ge u(x,t_0) \sum_{t=0}^{\infty} u(0,t) = \infty.$$
- $N \ge 3$. When t is even, we use the N-dimensional version of Theorem 2.6 to see that $u(x,t) \le u(0,t)$ for every $x \in \mathbb{Z}^N$. On the other hand, when t is odd, we use the N-dimensional version of the maximum principle to obtain $u(x,t) \le u(0,t-1)$ for every $x \in \mathbb{Z}^N$. Consequently,
$$\sum_{t=0}^{\infty} u(x,t) = \sum_{k=0}^{\infty} (u(x,2k) + u(x,2k+1)) \le 2 \sum_{k=0}^{\infty} u(0,2k) = 2 \sum_{t=0}^{\infty} u(0,t) < \infty. \qquad \square$$

Remark 5.4. According to the famous result due to G. Pólya [17], the symmetric N-dimensional random walk (which corresponds to our Eq. (5.2) with $a = \frac{1}{2N}$) returns almost surely to the origin if and only if $N \le 2$ (see also [7, Theorem 4.2.3] or [20, Section 1.2]). In the spirit of Eq. (4.8), Theorem 5.2 can be seen as a generalization of this well-known statement to the case when $a \in (0, \frac{1}{2N}]$.

5.2 Semidiscrete multidimensional problem

Finally, we consider the semidiscrete case $\mathbb{T} = \mathbb{R}$ and the initial-value problem
$$\begin{aligned} u_t(x,t) &= \textstyle\sum_{i=1}^N a u(x+e_i,t) - 2aNu(x,t) + \sum_{i=1}^N a u(x-e_i,t), \\ u(x,0) &= \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.3}$$
Again, we find the explicit solution using the generating function method. Let
$$F(z_1,\dots,z_N,t) = \sum_{x_1=-\infty}^{\infty} \cdots \sum_{x_N=-\infty}^{\infty} u(x,t) z_1^{x_1} \dots z_N^{x_N}.$$
Using (5.3), we see that F satisfies
$$\begin{aligned} F_t(z,t) &= a\left(z_1 + \ldots + z_N - 2N + \tfrac{1}{z_1} + \ldots + \tfrac{1}{z_N}\right) F(z,t), \\ F(0,z) &= 1. \end{aligned}$$
One can easily solve this ordinary differential equation and obtain
$$F(z_1,\dots,z_N,t) = e^{at\left(z_1+\ldots+z_N-2N+\frac{1}{z_1}+\ldots+\frac{1}{z_N}\right)} = e^{-2aNt} e^{az_1t} e^{\frac{at}{z_1}} \cdots e^{az_Nt} e^{\frac{at}{z_N}}$$
Using (3.3), we rewrite each pair of exponentials using the modified Bessel function of the first kind:
$$F(z_1,\dots,z_N,t) = e^{-2aNt} \left(\sum_{x_1=-\infty}^{\infty} z_1^{x_1} I_{x_1}(2at) \right) \cdots \left(\sum_{x_N=-\infty}^{\infty} z_N^{x_N} I_{x_N}(2at) \right)$$
The solution of (5.3) corresponds to the coefficient of $z_1^{x_1} \dots z_N^{x_N}$ in this expression. Choosing the relevant term in each sum above, we obtain
$$u(x,t) = e^{-2aNt} \prod_{i=1}^{N} I_{x_i}(2at). \tag{5.4}$$

The next theorem describes the asymptotic behavior and summability of the function $t \mapsto u(x,t)$. As before, it turns out that the case $\mathbb{T} = \mathbb{R}$ is simpler to analyze than $\mathbb{T} = \mathbb{Z}$.

Theorem 5.5. *Let u be the unique bounded solution of (5.3), where $a > 0$. For every $x \in \mathbb{Z}^N$, we have*

$$u(x,t) \sim \frac{1}{(4\pi a t)^{N/2}}$$

for $t \to \infty$. Hence, the integral $\int_0^\infty u(x,t)\,\mathrm{d}t$ is finite if and only if $N \geq 3$.

Proof. We combine Eq. (5.4) with the asymptotic estimate $I_x(t) \sim \frac{e^t}{\sqrt{2\pi t}}$ (see [16, formula 10.40.1]) and obtain

$$u(x,t) \sim e^{-2aNt} \left(\frac{e^{2at}}{\sqrt{4\pi a t}} \right)^N = \frac{1}{(4\pi a t)^{N/2}},$$

which proves the statement. $\square$

6 Conclusion

We conclude the paper with a list of open problems:

- **Other time scales**. In Sections 4 and 5, we studied the asymptotic behavior of nonnegative solutions, their summability and dependence on the dimension when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$; in the latter case, it is easy to generalize all statements to $\mathbb{T} = h\mathbb{Z}$. Is it possible to obtain similar results for general (or at least some other particular) time scales?

- **Time integrals**. Somewhat surprisingly, Theorems 4.3 and 4.7 show that the values of time integrals for $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$ coincide. We leave it up to the reader to verify that the choice $\mathbb{T} = h\mathbb{Z}$ again leads to the same values of integrals. Is this true for arbitrary time scales? Let us offer at least a partial explanation why this might be true: Given the coefficients a, b, c and an arbitrary time scale $\mathbb{T}$, assume that $s(x) = \int_0^\infty u(x,t)\Delta t$ is finite and $\lim_{t\to\infty} u(x,t) = 0$ for all $x \in \mathbb{Z}$. By integrating the equation $u^\Delta(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t)$ with respect to t, we get the recurrence relation

  $$u(x,0) = as(x+1) + bs(x) + cs(x-1), \quad x \in \mathbb{Z}.$$

  Hence, to determine the whole sequence $\{s(x)\}_{x\in\mathbb{Z}}$, it is enough to know the values of two adjacent time integrals $s(x-1)$ and $s(x)$. Taking into account the initial condition $u(0,0) = 1$ and $u(x,0) = 0$ for $x \neq 0$, we obtain the homogenous linear recurrence relation

  $$as(x+1) + bs(x) + cs(x-1) = 0, \quad x > 0,$$

  whose general solution has the form

  $$s(x) = \alpha \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} \right)^x + \beta \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} \right)^x, \quad x > 0.$$

  The solution must be invariant under the change of variables $a \to c$, $c \to a$, $x \to -x$ (see the proof of Theorem 4.3), and therefore we get

  $$s(x) = \alpha \left(\frac{-b - \sqrt{b^2 - 4ac}}{2c} \right)^{-x} + \beta \left(\frac{-b + \sqrt{b^2 - 4ac}}{2c} \right)^{-x}, \quad x < 0.$$

These results agree with Theorems 4.3 and 4.7, where we have $\alpha = \frac{1}{\sqrt{b^2-4ac}}$ and $\beta = 0$. Consequently, our original question whether the time integrals are independent on $\mathbb{T}$ reduces to the problem of deciding whether $\alpha = \frac{1}{\sqrt{b^2-4ac}}$ and $\beta = 0$ for all time scales $\mathbb{T}$.

- **More general equations**. It would be interesting to study the behavior of solutions to discrete-space equations with more general right-hand sides (more than three terms, nonlinearity, time-dependent coefficients, etc.). We remark that various types of nonlinear equations with discrete space and continuous time have been studied in the context of lattice dynamical systems (see e.g. [5, 6]).

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