# Asymptotic behavior of solutions to the semidiscrete diffusion equation

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**Abstract**

We study the asymptotic behavior of bounded solutions to the one-dimensional diffusion (heat) equation with discrete space and continuous time. We show that a bounded solution approaches the average of the initial values if the average exists, and provide estimates in the situation when it does not exist.

**Keywords:** semidiscrete diffusion equation; lattice diffusion equation; modified Bessel function

## 1 Introduction

In this paper, we focus on the semidiscrete diffusion equation (also known as the semidiscrete heat equation or the lattice diffusion equation)

$$\frac{\partial u}{\partial t}(x,t) = a(u(x+1,t) - 2u(x,t) + u(x-1,t)), \quad x \in \mathbb{Z}, \quad t \geq 0, \tag{1.1}$$

where $a > 0$ is the strength of the diffusion. One way to obtain Eq. (1.1) is to begin with the classical one-dimensional diffusion equation $\frac{\partial u}{\partial t}(x,t) = a\frac{\partial^2 u}{\partial x^2}(x,t)$, and discretize the spatial variable. However, the semidiscrete equation is of independent interest, and it arises in several applications:

- Eq. (1.1) describes a continuous-time symmetric random walk on $\mathbb{Z}$, where $a$ is the intensity of transitions between two neighboring integers. The value $u(x,t)$ is the probability that the random walk visits point $x$ at time $t$; cf. [6, Section 4].
- Eq. (1.1) describes the flow of a chemical in an infinite system of tanks arranged in a row, where each two neighbors are connected by pipes. The value $u(x,t)$ is the amount of the chemical in tank $x$ at time $t$; cf. [10, Section 3].
- Eq. (1.1) describes the dynamics of an infinite chain of cars, each of them being coupled to its two neighbors. The value $u(x,t)$ is the displacement of car $x$ at time $t$ from its equilibrium position; cf. [5, Example 1].

In general, initial-value problems for Eq. (1.1) do not have unique solutions (see [13, pp. 531–532]). However, for bounded initial data, there exists a unique globally bounded solution (see [13, Theorem 3.5]), which can be expressed by as follows. The fundamental solution $v(x,t) = e^{-2at}I_x(2at)$, where $I_x$ is the modified Bessel function of the first kind of order $x$, satisfies $v(x,0) = 1$ if $x = 0$, and $v(x,0) = 0$ otherwise (see [14, Example 3.1]). Now, given a bounded sequence $\{c_k\}_{k\in\mathbb{Z}}$, the unique bounded solution of Eq. (1.1) satisfying $u(x,0) = c_x$ for all $x \in \mathbb{Z}$ is given by the superposition formula

$$u(x,t) = e^{-2at}\sum_{k\in\mathbb{Z}} c_k I_{x-k}(2at), \quad x \in \mathbb{Z}, \quad t \geq 0, \tag{1.2}$$

where the infinite series is absolutely convergent (see [13, Corollary 3.8 and the proof of Theorem 3.7]).

We are interested in the asymptotic behavior of $u(x,t)$ as $t \to \infty$. We will show that if the average of the initial values $\lim_{k\to\infty} \frac{1}{2k+1}\sum_{l=-k}^{k} c_l$ exists and equals $d$, then $\lim_{t\to\infty} u(x,t) = d$ for each $x \in \mathbb{Z}$. Under additional assumptions, the limit is uniform with respect to $x$. On the other hand, our main result is more general and provides information on the limit superior and limit inferior of $u(x,t)$ as $t \to \infty$ even in the case when the average of the initial values does not exist.

The corresponding results for the classical one-dimensional diffusion equation $\frac{\partial u}{\partial t}(x,t) = a\frac{\partial^2 u}{\partial x^2}(x,t)$ are well known; see e.g. [15, 16] and the references therein. Interestingly, a detailed analysis of the asymptotic behavior in the semidiscrete case is missing, and the goal of our paper is to fill this gap. We are aware of the following existing results: 1) Section 4 in [14] provides information on the asymptotic behavior of the fundamental solution. 2) Theorem 4 in [5] shows that for initial data in $\ell^2(\mathbb{Z})$, the solution always tends to zero uniformly with respect to $x$, and provides a sufficient condition for initial data in $\ell^\infty(\mathbb{Z})$ guaranteeing that the solution tends uniformly to zero. On the other hand, our results apply to arbitrary initial data in $\ell^p(\mathbb{Z})$ as well as in $\ell^\infty(\mathbb{Z})$, which is a natural phase space for the diffusion equation (for example, the simple case of a constant initial condition, which leads to a constant solution, is included in $\ell^\infty(\mathbb{Z})$).

## 2 Main results

We begin with a few lemmas that will be needed in the proof of our main result. Most importantly, we use summation by parts to derive an alternative formula for the solution (1.2).

Throughout this section, we use the fact that the modified Bessel functions of the first kind satisfy $I_{-k}(t) = I_k(t)$ for each $k \in \mathbb{Z}$ and $t \geq 0$ (see formula 10.27.1 in [9]).

**Lemma 2.1.** *Let $\{c_k\}_{k\in\mathbb{Z}}$ be an arbitrary real sequence. Then for each $N \in \mathbb{N}$ and $t \geq 0$, we have*

$$\sum_{k=-N+1}^{N} c_k I_k(t) = \sum_{k=0}^{N-1} (I_k(t) - I_{k+1}(t)) \sum_{l=-k}^{k} c_l + I_N(t) \sum_{k=-N+1}^{N} c_k.$$

*Proof.* Consider the sequence $\{d_k\}_{k\in\mathbb{Z}}$ such that $d_0 = c_0$ and $d_k = d_{k-1} + c_k$ for each $k \in \mathbb{Z}$. Note that for each pair $i, j \in \mathbb{Z}$ with $i > j$, we have $d_i - d_j = \sum_{k=j+1}^{i}(d_k - d_{k-1}) = \sum_{k=j+1}^{i} c_k$.

Using the summation by parts formula and performing some manipulations, we get

$$\begin{aligned}
\sum_{k=-N+1}^{N} c_k I_k(t) &= \sum_{k=-N+1}^{N} (d_k - d_{k-1}) I_k(t) = d_N I_{N+1}(t) - d_{-N} I_{-N}(t) - \sum_{k=-N}^{N} d_k (I_{k+1}(t) - I_k(t)) \\
&= d_N I_{N+1}(t) - d_{-N} I_N(t) - d_N (I_{N+1}(t) - I_N(t)) - \sum_{k=-N}^{N-1} d_k (I_{k+1}(t) - I_k(t)) \\
&= I_N(t)(d_N - d_{-N}) - \sum_{k=-N}^{N-1} d_k (I_{k+1}(t) - I_k(t)) \\
&= I_N(t) \sum_{k=-N+1}^{N} c_k - \sum_{k=-N}^{-1} d_k (I_{k+1}(t) - I_k(t)) + \sum_{k=0}^{N-1} d_k (I_k(t) - I_{k+1}(t)) \\
&= I_N(t) \sum_{k=-N+1}^{N} c_k - \sum_{k=1}^{N} d_{-k} (I_{-k+1}(t) - I_{-k}(t)) + \sum_{k=0}^{N-1} d_k (I_k(t) - I_{k+1}(t)) \\
&= I_N(t) \sum_{k=-N+1}^{N} c_k - \sum_{k=1}^{N} d_{-k} (I_{k-1}(t) - I_k(t)) + \sum_{k=0}^{N-1} d_k (I_k(t) - I_{k+1}(t)) \\
&= I_N(t) \sum_{k=-N+1}^{N} c_k - \sum_{k=0}^{N-1} d_{-k-1} (I_k(t) - I_{k+1}(t)) + \sum_{k=0}^{N-1} d_k (I_k(t) - I_{k+1}(t)) \\
&= I_N(t) \sum_{k=-N+1}^{N} c_k + \sum_{k=0}^{N-1} (d_k - d_{-k-1})(I_k(t) - I_{k+1}(t)),
\end{aligned}$$

which completes the proof since $d_k - d_{-k-1} = \sum_{l=-k}^{k} c_l$. $\square$

The next lemma provides the promised alternative formula for the solution (1.2), and shows the explicit dependence of the solution on the sums (or, equivalently, averages) of the initial values.

**Lemma 2.2.** *Let $\{c_k\}_{k\in\mathbb{Z}}$ be a bounded real sequence. Then the unique bounded solution to the initial-value problem*

$$\frac{\partial u}{\partial t}(x,t) = a(u(x+1,t) - 2u(x,t) + u(x-1,t)), \quad x \in \mathbb{Z}, \quad t \geq 0, \tag{2.1}$$

$$u(x,0) = c_x, \quad x \in \mathbb{Z}, \tag{2.2}$$

*is given by the formula*

$$u(x,t) = e^{-2at}\sum_{k=0}^{\infty}(I_k(2at) - I_{k+1}(2at)) \sum_{l=x-k}^{x+k} c_l, \quad t \geq 0.$$

*Proof.* Let us verify the result for $x = 0$. Beginning with the formula (1.2) and using Lemma 2.1, we get

$$\begin{aligned} u(0,t) &= e^{-2at}\sum_{k\in\mathbb{Z}} c_k I_k(2at) = e^{-2at}\lim_{N\to\infty}\sum_{k=-N+1}^{N} c_k I_k(2at) \\ &= e^{-2at}\lim_{N\to\infty}\left(\sum_{k=0}^{N-1}(I_k(2at) - I_{k+1}(2at))\sum_{l=-k}^{k} c_l + I_N(2at)\sum_{k=-N+1}^{N} c_k\right). \end{aligned}$$

Let $M \geq 0$ be such that $|c_k| \leq M$ for all $k \in \mathbb{Z}$. Then $|I_N(2at)\sum_{k=-N+1}^{N} c_k| \leq I_N(2at)2NM$. Using the asymptotic formula (see formula 10.41.1 in [9])

$$I_n(x) \sim \frac{1}{\sqrt{2\pi n}}\left(\frac{ex}{2n}\right)^n \quad \text{for } n \to \infty,$$

we see that $\lim_{N\to\infty} I_N(2at)N = 0$, and therefore $\lim_{N\to\infty} I_N(2at)\sum_{k=-N+1}^{N} c_k = 0$, which completes the proof for $x = 0$.

For an arbitrary fixed $x \in \mathbb{Z}$, note that the function $\tilde{u}(y,t) = u(y+x,t)$ is a solution of the semidiscrete diffusion equation and satisfies $\tilde{u}(y,0) = \tilde{c}_y$, $y \in \mathbb{Z}$, with $\tilde{c}_y = u(y+x,0) = c_{y+x}$. Thus, we get

$$u(x,t) = \tilde{u}(0,t) = e^{-2at}\sum_{k=0}^{\infty}(I_k(2at) - I_{k+1}(2at))\sum_{l=-k}^{k}\tilde{c}_l = e^{-2at}\sum_{k=0}^{\infty}(I_k(2at) - I_{k+1}(2at))\sum_{l=x-k}^{x+k} c_l. \quad \square$$

We need the following two results involving the modified Bessel functions of the first kind. The first one is straightforward, but the second seems to be of independent interest; we were unable to find it in the literature dealing with Bessel functions.

**Lemma 2.3.** *The following statements hold for each $a > 0$:*

1. *For each $k_0 \in \mathbb{N}$, we have $\lim_{t\to\infty} e^{-2at}\sum_{k=0}^{k_0-1}(I_k(2at) - I_{k+1}(2at))(2k+1) = 0$.*
2. *For each $t \geq 0$, we have $e^{-2at}\sum_{k=0}^{\infty}(I_k(2at) - I_{k+1}(2at))(2k+1) = 1$.*

*Proof.* For the proof of the first part, we use the asymptotic formula (see formula 10.30.4 in [9])

$$I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{for } x \to \infty,$$

which implies $\lim_{x\to\infty} e^{-x}I_n(x) = 0$ for each $n \in \mathbb{Z}$. It follows that for each fixed $k \in \mathbb{Z}$, we have

$$\lim_{t\to\infty} e^{-2at}(I_k(2at) - I_{k+1}(2at))(2k+1) = 0,$$

and the proof of the first statement is complete.

To prove the second statement, observe that the unique bounded solution of Eq. (2.1) with the initial condition $u(x,0) = 1$ for all $x \in \mathbb{Z}$ is the constant solution $u(x,t) = 1$ for all $x \in \mathbb{Z}$, $t \geq 0$. Thus, by Lemma 2.2, we get

$$1 = u(0,t) = e^{-2at}\sum_{k=0}^{\infty}(I_k(2at) - I_{k+1}(2at))(2k+1), \quad t \geq 0. \qquad \square$$

We are now ready for the main result.

**Theorem 2.4.** *For each bounded real sequence $\{c_k\}_{k\in\mathbb{Z}}$, the unique bounded solution to the initial-value problem (2.1)–(2.2) has the following properties:*

1. *For every $x \in \mathbb{Z}$,*

$$\liminf_{k\to\infty}\frac{1}{2k+1}\sum_{l=x-k}^{x+k} c_l \leq \liminf_{t\to\infty} u(x,t) \leq \limsup_{t\to\infty} u(x,t) \leq \limsup_{k\to\infty}\frac{1}{2k+1}\sum_{l=x-k}^{x+k} c_l.$$

2. *If $x \in \mathbb{Z}$ and $\lim_{k\to\infty}\frac{1}{2k+1}\sum_{l=x-k}^{x+k} c_l = d$, then $\lim_{t\to\infty} u(x,t) = d$.*

3. *If $\lim_{k\to\infty}\frac{1}{2k+1}\sum_{l=x-k}^{x+k} c_l = d$ holds uniformly for all $x \in \mathbb{Z}$, then $\lim_{t\to\infty} u(x,t) = d$ uniformly with respect to $x \in \mathbb{Z}$.*

*Proof.* Let $M > 0$ be such that $|c_k| \leq M$ for all $k \in \mathbb{Z}$. Choose an arbitrary $x \in \mathbb{Z}$, and denote

$$d_k = \frac{1}{2k+1}\sum_{l=x-k}^{x+k} c_l, \quad k \in \mathbb{N},$$
$$\underline{d} = \liminf_{k\to\infty} d_k, \quad \overline{d} = \limsup_{k\to\infty} d_k.$$

Using Lemma 2.2, we have

$$u(x,t) = e^{-2at}\sum_{k=0}^{\infty}(I_k(2at) - I_{k+1}(2at))(2k+1)d_k, \quad t \geq 0.$$

Given an $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that $\underline{d} - \varepsilon < d_k < \overline{d} + \varepsilon$ for all $k \geq k_0$.

Using the fact that $\{I_k\}_{k=0}^{\infty}$ is a decreasing sequence on $(0,\infty)$ (see [9, Section 10.37]) together with the first part of Lemma 2.3, we see there exists a $t_0 > 0$ such that

$$0 < e^{-2at}\sum_{k=0}^{k_0-1}(I_k(2at) - I_{k+1}(2at))(2k+1) < \varepsilon, \quad t \geq t_0.$$

Thus, according to the second part of the same lemma, we get

$$1 - \varepsilon < e^{-2at}\sum_{k=k_0}^{\infty}(I_k(2at) - I_{k+1}(2at))(2k+1) < 1, \quad t \geq t_0.$$

Observing that $|d_k| \leq M$ for every $k \in \mathbb{N}$, we obtain

$$u(x,t) < \varepsilon M + (\overline{d} + \varepsilon)e^{-2at}\sum_{k=k_0}^{\infty}(I_k(2at) - I_{k+1}(2at))(2k+1), \quad t \geq t_0.$$

Depending on whether $\overline{d} + \varepsilon$ is nonnegative or nonpositive, the second term on the right-hand side is majorized either by $\overline{d} + \varepsilon$, or by $(\overline{d} + \varepsilon)(1 - \varepsilon) = \overline{d} + \varepsilon - \varepsilon\overline{d} - \varepsilon^2$, yielding the estimate

$$u(x,t) < \varepsilon M + \max(\overline{d} + \varepsilon, \overline{d} + \varepsilon - \varepsilon\overline{d} - \varepsilon^2) = \overline{d} + \varepsilon M + \varepsilon + \varepsilon\max(0, -\overline{d} - \varepsilon), \quad t \geq t_0.$$

This proves that $\limsup_{t\to\infty} u(x,t) \le \overline{d}$. To prove the remaining part of the first statement, we consider the lower bound

$$u(x,t) > -\varepsilon M + (\underline{d} - \varepsilon)e^{-2at} \sum_{k=k_0}^{\infty} (I_k(2at) - I_{k+1}(2at))(2k+1), \quad t \ge t_0.$$

Depending on whether $\underline{d} - \varepsilon$ is nonnegative or nonpositive, the second term on the right-hand side is minorized either by $(\underline{d} - \varepsilon)(1-\varepsilon) = \underline{d} - \varepsilon - \varepsilon\underline{d} + \varepsilon^2$, or by $\underline{d} - \varepsilon$, yielding the estimate

$$u(x,t) > -\varepsilon M + \min(\underline{d} - \varepsilon - \varepsilon\underline{d} + \varepsilon^2, \underline{d} - \varepsilon) = \underline{d} - \varepsilon M - \varepsilon + \varepsilon \min(-\underline{d} + \varepsilon, 0), \quad t \ge t_0.$$

This proves that $\liminf_{t\to\infty} u(x,t) \ge \underline{d}$.

The second statement of the theorem is an immediate consequence of the first one.

The third statement follows from the fact that all the previous estimates are independent of $x$ if $\lim_{k\to\infty} \frac{1}{2k+1}\sum_{l=x-k}^{x+k} c_l = d$ holds uniformly for all $x$. □

Finally, we derive two useful corollaries of Theorem 2.4.

**Corollary 2.5.** *If $\{c_k\}_{k\in\mathbb{Z}}$ is a bounded real sequence such that $\lim_{k\to\infty} \frac{1}{2k+1}\sum_{l=-k}^{k} c_l = d$, then the unique bounded solution to the initial-value problem* (2.1)–(2.2) *satisfies $\lim_{t\to\infty} u(x,t) = d$ for each $x \in \mathbb{Z}$.*

*Proof.* It suffices to verify that for each nonzero $x \in \mathbb{Z}$, we have $\lim_{k\to\infty} \frac{1}{2k+1}\sum_{l=x-k}^{x+k} c_l = d$. Suppose that $x$ is positive; the proof for a negative $x$ is similar. Then

$$\lim_{k\to\infty} \frac{1}{2k+1}\sum_{l=x-k}^{x+k} c_l = \lim_{k\to\infty} \frac{2(k-x)+1}{2k+1} \cdot \lim_{k\to\infty} \frac{1}{2(k-x)+1}\left(\sum_{l=x-k}^{-x+k} c_l + \sum_{l=-x+k+1}^{x+k} c_l\right).$$

Let $M > 0$ be such that $|c_k| \le M$ for all $k \in \mathbb{Z}$. Then $|\sum_{l=-x+k+1}^{x+k} c_l| \le 2xM$ and consequently

$$\lim_{k\to\infty} \frac{1}{2(k-x)+1}\sum_{l=-x+k+1}^{x+k} c_l = 0,$$

$$\lim_{k\to\infty} \frac{1}{2k+1}\sum_{l=x-k}^{x+k} c_l = \lim_{k\to\infty} \frac{1}{2(k-x)+1}\sum_{l=x-k}^{-x+k} c_l = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{l=-N}^{N} c_l = d. \quad □$$

**Corollary 2.6.** *If $\{c_k\}_{k\in\mathbb{Z}}$ is a real sequence such that $\lim_{k\to\pm\infty} c_k = d$, then the unique bounded solution to the initial-value problem* (2.1)–(2.2) *satisfies $\lim_{t\to\infty} u(x,t) = d$ uniformly with respect to $x$.*

*Proof.* According to the third part of Theorem 2.4, it suffices to show that $\lim_{k\to\infty} \frac{1}{2k+1}\sum_{l=x-k}^{x+k} c_l = d$ uniformly with respect to $x$.

Since $\lim_{k\to\infty} c_k = d$, we get $\lim_{k\to\infty} \frac{1}{k+1}\sum_{l=x}^{x+k} c_l = d$ uniformly with respect to $x$; this follows from the fact that each convergent sequence is almost convergent to the same limit (see [8, Theorem 1]). But since $\lim_{k\to\infty} \frac{k+1}{2k+1} = \frac{1}{2}$, we get $\lim_{k\to\infty} \frac{1}{2k+1}\sum_{l=x}^{x+k} c_l = \frac{d}{2}$ uniformly with respect to $x$.

Similarly, since $\lim_{k\to-\infty} c_k = d$, we get $\lim_{k\to\infty} \frac{1}{k}\sum_{l=x-k}^{x-1} c_l = d$ uniformly with respect to $x$. But since $\lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2}$, we get $\lim_{k\to\infty} \frac{1}{2k+1}\sum_{l=x-k}^{x-1} c_l = \frac{d}{2}$ uniformly with respect to $x$.

Consequently,

$$\lim_{k\to\infty} \frac{1}{2k+1}\sum_{l=x-k}^{x+k} c_l = \lim_{k\to\infty} \frac{1}{2k+1}\sum_{l=x}^{x+k} c_l + \lim_{k\to\infty} \frac{1}{2k+1}\sum_{l=x-k}^{x-1} c_l = \frac{d}{2} + \frac{d}{2} = d$$

uniformly with respect to $x$. □

**Remark 2.7.** Inspecting the proof of Corollary 2.6, we can obtain the following more general result: If $\{c_k\}_{k\in\mathbb{Z}}$ is almost convergent to $d_1$ for $k \to \infty$ and to $d_2$ for $k \to -\infty$, then the unique bounded solution to the initial-value problem (2.1)–(2.2) satisfies $\lim_{t\to\infty} u(x,t) = (d_1 + d_2)/2$ uniformly with respect to $x$.

**Remark 2.8.** If $\{c_k\}_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$ for an arbitrary $p \in [1,\infty)$, then $\lim_{k\to\pm\infty} c_k = 0$, and Corollary 2.6 yields $\lim_{t\to\infty} u(x,t) = 0$ uniformly with respect to $x$. This generalizes a result from [5, Theorem 4] dealing with the case $p = 2$.

## 3 Conclusion

We conclude the paper with a few suggestions for further research:

- Investigate the asymptotic behavior of solutions to the $N$-dimensional semidiscrete diffusion equation $\frac{\partial u}{\partial t}(x,t) = \sum_{i=1}^N au(x+e_i,t) - 2aNu(x,t) + \sum_{i=1}^N au(x-e_i,t)$, $x \in \mathbb{Z}^N$, $t \geq 0$, where $e_1, \dots, e_N$ is the canonical basis of $\mathbb{R}^N$. For any bounded initial data $\{c_k\}_{k\in\mathbb{Z}^N}$, the unique bounded solution satisfying $u(x,0) = c_x$ for all $x \in \mathbb{Z}^N$ is $u(x,t) = \sum_{k\in\mathbb{Z}^N} c_k v(x-k,t)$, $x \in \mathbb{Z}^N$, $t \geq 0$ (see [12, Theorem 2.5]), where $v(x,t) = e^{-2aNt}\prod_{i=1}^N I_{x_i}(2at)$ is the fundamental solution (see [14, Section 5.2]). For the asymptotic behavior of solutions to the $N$-dimensional diffusion equation with continuous time and space, see [2, 7] and the references therein.

- Investigate the asymptotic behavior of solutions to the one-dimensional purely discrete equation $\Delta u(x,t) = a(u(x+1,t) - 2u(x,t) + u(x-1,t))$, $x \in \mathbb{Z}$, $t \in \mathbb{N}_0$, where $\Delta u(x,t) = u(x,t+1) - u(x,t)$. For any bounded initial data $\{c_k\}_{k\in\mathbb{Z}}$, the unique solution satisfying $u(x,0) = c_x$ for all $x \in \mathbb{Z}$ is $u(x,t) = \sum_{k=x-t}^{x+t} c_k \cdot w(x-k,t)$, where $w$ is the fundamental solution (see [11, Section 4]). If $a \neq 1/2$, then $w(x,t) = (1-2a)^t \mathcal{I}_{|x|}^{2a/(1-2a)}(t)$, where $\mathcal{I}_x$ is the discrete modified Bessel function of the first kind of order $x$ (see [11, Theorem 4.1] for this formula, as well as [1] for more information on discrete Bessel functions).

- Investigate the asymptotic behavior of solutions to the one-dimensional semidiscrete diffusion equation with the discrete fractional Laplacian. For more details and explicit solutions to initial-value problems with bounded initial data, see [4]. Additional information on the discrete Laplacian and its fractional counterpart may be found in [3].

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