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HERMITE INTERPOLATION BY PLANAR BIARCS

Abstract

G^1 Hermite interpolation by planar biarcs is shortly outlined. A geometrical interpretation of the main characterization property of the biarc interpolants leads directly to a new interpolation method. This method is tested on several examples and is compared to the standard approaches.

Keywords

Biarc, Hermite interpolation, G-code, CNC manufacturing

1 Introduction

It is very important in the Computer Numerically Controlled (CNC) manufacturing, to control precisely the speed of the tool along its path. Also offsets of the curve are exploited in the CNC machinig, since in many cases some part of the machine must move at a given constant distance from the manufactured shape. For this reason the curves with simple (analyticaly expressible) arc-length function (implying simple offsets) are very suitable for CNC manufacturing. The traditional approach is to use curves composed of linear and circular segments, for which the arc-length function can be easily expressed. The industrial description of such circular splices is called G-code.

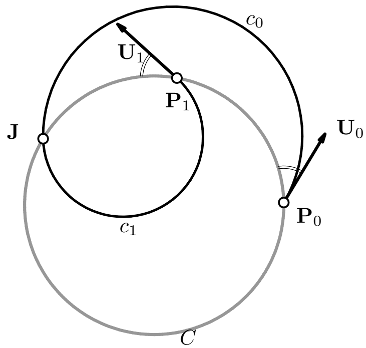
Several techniques for generating suitable G-code curves were developed. Among them the biarc interpolation is one of the main techniques - see e.g. [1, 2, 3]. In this paper we present a new interpolation method (section 2) which we test on examples and compare with standard methods (section 3).

2 G^1 Hermite interpolation by biarcs

Suppose that two circular arcs c_0, c_1 are given in the plane. We say, that they form a *biarc interpolating given oriented* G^1 *data* (i.e. end points $\mathbf{P}_0$, $\mathbf{P}_1$ and unit tangent vectors $\mathbf{U}_0$, $\mathbf{U}_1$) if and only if the two circular arcs share one common end point $\mathbf{J}$ called *joint* and satisfy the following properties: The arc c_0 has end points $\mathbf{P}_0$, $\mathbf{J}$ and $\mathbf{U}_0$ is tangent to c_0 and points toward the interior of this arc. The arc c_1 has end points $\mathbf{J}$ and $\mathbf{P}_1$ and $\mathbf{U}_1$ is tangent to c_1 and points toward the exterior of this arc. The two arcs have a common tangent vector at $\mathbf{P}$, pointing toward the exterior of c_0 and toward the interior of c_1. This means that an interpolating biarc represents a G^1 smooth path from the data $\mathbf{P}_0, \mathbf{U}_0$ to the data $\mathbf{P}_1, \mathbf{U}_1$.

It is a known fact [2] that there is one dimensional parametric system of interpolating biarcs to general planar data and that the locus of all possible joints $\mathbf{J}$ is a circle passing through $\mathbf{P}_0$ and $\mathbf{P}_1$. A simple geometric proof of the following characterization theorem can be found in [4].

Proposition 1 *Consider the family of biarcs interpolating given oriented G^1 data $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{U}_0$, $\mathbf{U}_1$. Then the locus of all possible joints $\mathbf{J}$ is the circle C passing through the points $\mathbf{P}_0$, $\mathbf{P}_1$ and having the same oriented angles with the vectors $\mathbf{U}_0$ and $\mathbf{U}_1$.*



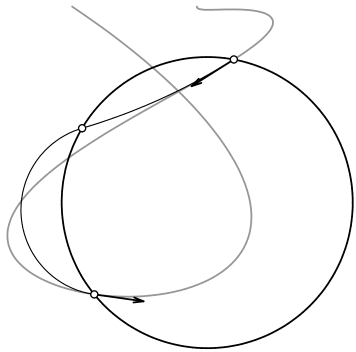
Note, that for any data there is precisely one such circle C (possible degenerated into a line). It represents the unique rotation transforming the data $\mathbf{P}_0, \mathbf{U}_0$ to the data $\mathbf{P}_1, \mathbf{U}_1$.

Various biarc interpolation schemes were proposed which are distinguished by the choice of the joint $\mathbf{J}$. The two most important are the "equal chord" biarc and the "parallel tangent" biarc. The former is constructed so that the two segments $\mathbf{P}_0\mathbf{J}$ and $\mathbf{J}\mathbf{P}_1$ has the same length and the latter so that the tangent at the point $\mathbf{J}$ is parallel to the segment $\mathbf{P}_0\mathbf{P}_1$.

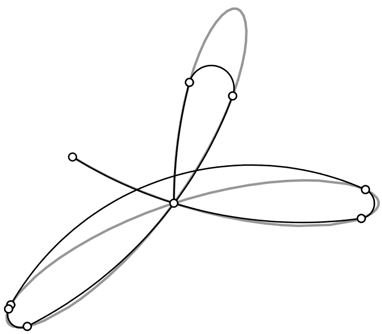
We propose a new choice of the joint $\mathbf{J}$ which is based on the following simple observation. Suppose, that the G^1 data are taken from a C^1 continuous curve - see the next figure. Then by the construction of the circle C the two boundary vectors are both pointing outside or inside the circle C. The curve therefore intersects the circle C in at least one more intermediate point, which we take as the joint $\mathbf{J}$.

3 Examples and comparison

As a first example we consider G^1 data taken from a continuous curve (grey line). On both figures the charasteristic circle C is shown together with the interpolating biarc constructed by the classical "equal chord" method (left figure) and by our new method (right figure).



As a next example we convert a parametric curve (grey line) which we convert into biarc spline using the "equal chord" method and the new method. In both cases we interpolate data taken from $2, 4, ...128$ segments of the curve. The figure shows the curve together with its arc conversion (using the new method) based on 4 biarc segments. In the table the conversion errors for both methods are shown.



	Error	
Parts	Eq. ch. m.	New m.
2	1.51	1.37
4	$3.35\ 10^{-1}$	$2.97\ 10^{-1}$
8	$3.52\ 10^{-2}$	$2.14\ 10^{-2}$
16	$1.47\ 10^{-2}$	$1.35\ 10^{-2}$
32	$9.80\ 10^{-4}$	$5.14\ 10^{-4}$
64	$1.12\ 10^{-4}$	$6.82\ 10^{-5}$
128	$1.04\ 10^{-5}$	$8.64\ 10^{-6}$

These two examples shows general quality of the new method, which we have observed on many other data. Due to the additional point **J** taken from the curve, it typically produces biarcs which are closer to the original curve.

In addition to the higher precision, the new method has following advantages comparing to standard methods:

- This biarc conversion is in fact an *arc conversion*. All the end points of the arcs lie on the curve and it is therefore clear, which arc matches which part of the curve. This makes it very easy to evaluate the distance between the arc-spline and the curve.
- The construction reproduce arc-splines, i.e. it has the *arc-splines precision*.
- The construction is invariant under the group of Möbius transformations.

4 Conclusion

The proposed method can be used for conversion of (piecewise) C^1 continuous splines into arc splines and can thus find interresting applications in the context of CNC manufacturing. In our future researche we want to investigate the space biarc approximation and compare the biarc interpolation schemes to the interpolation by Pythagorean Hodograph curves.

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References

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