ISF representation of hypersurfaces and rational convolutions computation

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Abstract. Recently, the support function representation of hypersurfaces has been introduced into CAGD. This representation is very useful in computation of convolutions of hypersurfaces. However, for a given hypersurface (represented parametrically or implicitly) it is not always possible to represent it via support function. In this paper we introduce the so-called implicit support function (ISF) representation which removes this main disadvantage. We show how ISF representation can be used in computation of rational convolutions of hypersurfaces, i.e., how to identify that the convolution is rational and how to find its rational parameterization.

Keywords: Convolution hypersurfaces, support function representation, implicit support function representation, rational parameterizations.

1 Introduction

Convolution of hypersurfaces is a fundamental operation in Computer Aided Geometric Design (CAGD) which provides one of the main chalenging problems. Since convolutions with spheres correspond to classical offsets and convolutions with arbitrary surfaces represent the so-called general offsets, operation of convolution is especially useful in machining – they provide paths for cutting tools.

Since CAGD most often uses a parametric description of hypersurfaces (typically curves and surfaces), one of the tasks is to find a rational parameterization of the convolution hypersurface, if it exists. In such a case, the convolution curve/surface can be represented again in the standard (NURBS) form. The classical offsets have been studied intensively in recent years (see e.g. [2, 3, 9]), there are also some papers related to general offsets (see e.g. [6, 7, 10]).

Nevertheless, a parametric or implicit representation of hypersurfaces seems to be not very suitable for the operation of convolution and that is why “better” representations w.r.t the convolution are looked for. Recently, the support function representation of hypersurfaces has been introduced into CAGD (see [4, 11]). This representaiton is very suitable for describing convolutions of hypersurfaces as this operation corresponds to the sum of the associated support functions. The main drawback of this representation is that not all hypersurfaces (given parametrically or implicitly) can be represented via support function.

This paper is devoted to the representation of hypersurfaces which is based on the support function representation and removes the main disadvantage of the SF representation, i.e., it is available for all algebraic hypersurfaces with non-degenerated Gauss image. We call it *implicit support function representation* as it is closely related to the SF representation but, in fact, this is a dual representation of a given hypersurface. We show how to find ISF representation for parametrically or implicitly given hypersurface and how this representation can be used for deciding rationality of the convolution and for finding rational parametrizations of the convolutions.

The remainder of the paper is organized as follows: Section 2 is devoted to the convolution of hypersurfaces, Section 3 introduces implicit support function representation, gives some basic properties and also algorithms for computing ISF for parametric and implicit hypersurfaces. Section 4 uses ISF representation for computation of rational parameterizations of convolution curves/surfaces. Finally, Section 5 concludes the paper.

2 Convolutions of hypersurfaces

The notion of convolution hypersurfaces (especially curves or surfaces) are used in various fields of mathematics and there are also many applications in the technical praxis. Let us start with the definition.

Definition 2.1 *Let A and B be smooth hypersurfaces in the affine space $\mathbb{R}^{n+1}$. The* convolution hypersurface *$C = A \star B$ is defined*

$$C = \{\mathbf{a} + \mathbf{b} \,|\, \mathbf{a} \in A, \mathbf{b} \in B \textit{ and } \mathbf{n_a} \parallel \mathbf{n_b}\}, \tag{1}$$

where $\mathbf{n_a}$ and $\mathbf{n_b}$ are the normal vectors of A and B at points $\mathbf{a} \in A$ and $\mathbf{b} \in B$. The points $\mathbf{a}$, $\mathbf{b}$ are called corresponding points.

Let us look at this problem from the point of view of parameterizations of hypersurfaces A and B. Let $\mathbf{a}(u_1, \ldots, u_n)$ and $\mathbf{b}(s_1, \ldots, s_n)$ be parameterizations of hypersurfaces A and B, respectively. For the sake of brevity, we denote $\bar{u} = (u_1, \ldots, u_n)$ etc.

There are two main tasks concerning the computation of parameterizations of convolution hypersurfaces. First, we want to decide if the convolution $C = A \star B$ is rational, and second, we want to compute the rational parameterization of the convolution hypersurface using the parameterizations $\mathbf{a}(\bar{u})$ and $\mathbf{b}(\bar{s})$ of A and B, respectively, if it is possible. Basically, we have two possibilities how to solve these problems. We can

1. find a rational reparameterization $\bar{u} \mapsto \phi(\bar{s})$ such that the parameterizations $\mathbf{a}(\phi(\bar{s})), \mathbf{b}(\bar{s})$ fulfill *the convolution condition* (for a given $\bar{s}_0$ normal vectors at $\mathbf{a}(\phi(\bar{s}_0)), \mathbf{b}(\bar{s}_0)$ are parallel), or
2. find suitable rational parameterizations $\tilde{\mathbf{a}}(\bar{t}), \tilde{\mathbf{b}}(\bar{t})$ of both hypersurfaces which fulfill the convolution condition described above, or directly find rational parametrization $\mathbf{c}(t)$ without rational ϕ.

Since the first approach was thorougly studied in [7], we focus on the second approach in the rest of this paper. This approach, which is based on the *implicit support function representation*, is useful especially in case when we are not able to decide rationality of a convolution and/or find a rational reparameterization ϕ. In curve/surface case, we can always give an answer for the first question – is the convolution rational? Moreover, in the curve case (and sometime also in the surface case) we can find the rational parameterization of the convolution curve/surface.

3 Implicit support function representation

In this part, we use a hypersurface representation outgoing from the so-called *support function representation of hypersurfaces* (cf. [11, 4]). The support function representation is a certain kind of a dual representation, most widely used as a tool in the convex geometry for the representation of convex bodies, see e.g. [5]. Recently, this concept has been extended to the so-called quasi-convex hypersurfaces [11].

A hypersurface is described as the envelope of its tangent hyperplanes

$$T_{\mathbf{n}} := \{\mathbf{x} : \mathbf{n} \cdot \mathbf{x} = h(\mathbf{n})\}, \tag{2}$$

where the *support function (SF)* $h(\mathbf{n})$ is a function defined on the unit sphere S_n (or its suitable subset). This representation is very suitable for describing convolutions of hypersurfaces as this operation corresponds to the sum of the associated support functions, i.e.,

$$h_C = h_A + h_B. \tag{3}$$

However, given a parametric or implicit representation of a hypersurface, it is not always possible to represent it via SF – mainly due to the fact, that for each vector $\mathbf{n}$ only one value of h is possible.

Further, we use a hypersurface representation which removes this main drawback of the support function representation, i.e., it is available for all algebraic hypersurfaces with non-degenerated Gauss image. A hypersurface is here represented as an envelope of tangent hyperplanes (2) where $\mathbf{n}$ and h satisfy the implicit polynomial equation

$$D(\mathbf{n}, h) = 0, \tag{4}$$

i.e., from now $h(\mathbf{n})$ does not have to be a function defined only on the unit sphere S_n.

To emphasize the connection between the standard one-valued SF representation and (generally) multi-valued SF representation (4), we will call $D(\mathbf{n}, h) = 0$ the *implicit support function* (or shortly *ISF*) *representation* of a hypersurface. Since $D(\mathbf{n}, h) = 0$ is a dual representation of a given hypersurface we immediately obtain:

Lemma 3.1 *For the implicit support function $D(\mathbf{n}, h) = 0$ of a hypesurface A, it holds:*

1. *$D(\mathbf{n}, h)$ is a homogeneous polynomial in $n_1, \ldots, n_n, h$.*
2. *A is irreducible iff $D(\mathbf{n}, h)$ is irreducible.*

The connection between the rationality of a given hypersurface and the associated ISF representation is expressed as follows:

Lemma 3.2 *There exists a rational representation of a hypersurface if and only if the zero locus of the corresponding implicit support function $D = 0$ is rational.*

Proof. See [8]. □

Implicit support function can be easily obtained from a parametric or implicit representation of an arbitrary hypersurface just by applying suitable elimination method, e.g. Gröbner basis method (cf. [1]) – see Algorithm 1 and Algorithm 2.

Input: Parameterization
$\mathbf{x} : \mathbb{R}^n \to \mathbb{R}^{n+1} : (u_1, \ldots, u_n) \to (x_1, \ldots, x_{n+1})$
Output: ISF $D(\mathbf{n}, h) = 0$
begin
  $I \longleftarrow \langle \mathbf{n} \cdot \frac{\partial \mathbf{x}}{\partial u_1}, \ldots, \mathbf{n} \cdot \frac{\partial \mathbf{x}}{\partial u_n}, \mathbf{n} \cdot \mathbf{x} - h, 1 - wh||\mathbf{a}_u \times \mathbf{a}_v||^2 \rangle$;
  $\prec \longleftarrow$ a term order such that w and each u_i is greater than any n_i and h;
  $G \longleftarrow$ a Gröbner basis of I w.r.t $\prec$;
  $D \longleftarrow G \cap k[n_1, \ldots, n_{n+1}, h]$;
  return D
end

Algorithm 1: Finds ISF for a hypersurface given parametrically.

Input: Polynomial $F(x_1, \ldots, x_{n+1})$
Output: ISF $D(\mathbf{n}, h) = 0$
begin
  $I \longleftarrow \langle F, \frac{\partial F}{\partial x_1} - \lambda n_1, \ldots, \frac{\partial F}{\partial x_{n+1}} - \lambda n_{n+1}, \mathbf{n} \cdot \mathbf{x} - h, 1 - wh \rangle$;
  $\prec \longleftarrow$ a term order such that w, λ and each x_i is greater than any n_i and h;
  $G \longleftarrow$ a Gröbner basis of I w.r.t $\prec$;
  $D \longleftarrow G \cap k[n_1, \ldots, n_{n+1}, h]$;
  return D
end

Algorithm 2: Finds ISF for a hypersurface given implicitly.

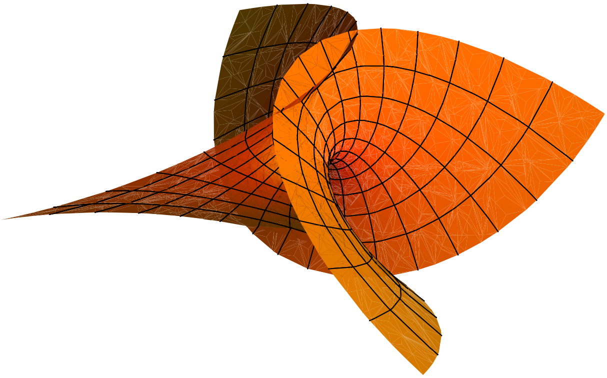

Figure 1: The Enneper surface.

Example 3.3 *The Enneper surface (see Fig. 1) can be given either in the implicit form*

$$\begin{aligned} F(x,y,z) \quad = \quad & -64z^9+1152z^7+432x^2z^6-432y^2z^6+3888x^2z^5+ \\ & 3888y^2z^5-5184z^5+6480x^2z^4-6480y^2z^4+1215x^4z^3+ \\ & 1215y^4z^3-3888x^2z^3+6318x^2y^2z^3-3888y^2z^3+ \\ & 4374x^4z^2-4374y^4z^2-729x^4z-729y^4z+1458x^2y^2z+ \\ & 729x^6-729y^6+2187x^2y^4-2187x^4y^2, \end{aligned} \tag{5}$$

or in the parametric form

$$\mathbf{a}(u,v)=\left(u-\frac{u^3}{3}+uv^2,-v-u^2v+\frac{v^3}{3},u^2-v^2\right). \tag{6}$$

Using Algorithm 1 for the parametric form (6) or Algorithm 2 for the implicit form (5), we can find the implicit support function representation of the Enneper surface in the form

$$\begin{aligned} D(n_1,n_2,n_3,h) \quad = \quad & -4n_1^6+9h^2n_1^4+4n_2^2n_1^4-3n_3^2n_1^4-18hn_3n_1^4+ \\ & 4n_2^4n_1^2-12hn_3^3n_1^2+18h^2n_2^2n_1^2+6n_2^2n_3^2n_1^2- \\ & 4n_2^6+9h^2n_2^4+12hn_2^2n_3^3-3n_2^4n_3^2+18hn_2^4n_3. \end{aligned}$$

We can also easily switch from ISF representation to the parametric one. If $D(\mathbf{n},h)=0$ (ISF of a hypersurface A) is parameterized by rational functions in $\bar{u}$

$$n_1(\bar{u}),\ldots,n_{n+1}(\bar{u}),h(\bar{u}),$$

then the tangent hyperplanes of A can be written in the form

$$n_1(\bar{u})x_1+\ldots+n_{n+1}(\bar{u})x_{n+1}-h(\bar{u})=0. \tag{7}$$

Differentiating (7) with respect to u_i, $i = 1, \ldots, n$, gives along with (7) the system of $n+1$ linear equations in variables x_j, $j = 1, \ldots, n+1$. Using Cramer's rule, we obtain a rational parameterization of A.

4 Rational convolutions computation using ISF

In what follows, we want to use ISF representation of hypersurfaces for computing rational parameterizations of convolution hypersurfaces $C = A \star B$ (mainly curves and surfaces, here), especially in case when we are not able to find a direct rational reparameterization of one hypersurface w.r.t the other one in order to fulfill the convolution condition. To obtain ISF of a convolution hypersurface $C = A \star B$, we have to sum points with parallel normal vectors on hypersurfaces A and B. Hence, it is enough to add the equation (3) and eliminate variables h_A, h_B from the system of equations

$$D_A(\mathbf{n}, h_A) = 0,\; D_B(\mathbf{n}, h_B) = 0,\; h_C - h_A - h_B = 0, \tag{8}$$

for more details see Algorithm 3.

Input: ISFs $D_A(n_1, \ldots, n_{n+1}, h_A)$ and $D_B(n_1, \ldots, n_{n+1}, h_B)$ representing A and B
Output: ISF $D_C(n_1, \ldots, n_{n+1}, h_C)$ of the convolution hypersurface $C = A \star B$
begin
  $I \longleftarrow \langle D_A, D_B, h_C - h_A - h_B \rangle$;
  $\prec \longleftarrow$ a term order such that h_A and h_B are greater than h_C and any n_i;
  $G \longleftarrow$ a Gröbner basis of I w.r.t $\prec$;
  $D_C \longleftarrow G \cap k[h_C, n_1, \ldots, n_{n+1}]$;
  return D_C
end

Algorithm 3: Computes ISF of a convolution hypersurface of two hypersurfaces.

Example 4.1 *Let A be the cardioid parameterized by*

$$\mathbf{a}(u) = \left(\frac{-2u^4 + 2u^2}{u^4 + 2u^2 + 1}, \frac{-4u^3}{u^4 + 2u^2 + 1} \right)^\top$$

and B be the Tschirhausen cubic parameterized by

$$\mathbf{b}(s) = \left(s^2, s - \frac{1}{3}s^3 \right)^\top,$$

see Fig. 2. Applying the first approach mentioned in Section 2 we are not able to find rational reparametrizations $u \mapsto u(s)$ and $s \mapsto s(u)$ and rational parameterization of the convolution curve $C = A \star B$. However, we know that C is rational because the cardiod is a PH curve (see [3] for more details on PH curves).

Thus, this is the case when ISF representation is useful. Algorithm 1 yields ISFs of the cardiod and Tschirhausen cubic in the form

$$\begin{aligned} D_A(n_1,n_2,h_A) &= 16h_A^3+24n_1h_A^2-27n_2^2h_A-15n_1^2h_A+2n_1^3, && (9)\\ D_B(n_1,n_2,h_B) &= 9h_B^2n_2^2-12h_Bn_1^3-18h_Bn_1n_2^2- && (10)\\ &\quad -3n_1^2n_2^2-4n_2^4. \end{aligned}$$

Next, applying Algorithm 3 we obtain the ISF representation of $C = A\star B$ in the form

$$\begin{gathered} D_C(n_1,n_2,h_C) = (186624n_2^6)h_C^6+(-559872n_1n_2^6-746496n_1^3n_2^4)h_C^5+\\ +(-878688n_2^8-676512n_2^6n_1^2+1119744n_2^4n_1^4+995328n_2^2n_1^6)h_C^4+\\ +(-442368n_1^9+2072304n_2^8n_1+4944240n_2^6n_1^3+3297024_n2^4n_1^5)h_C^3+\\ +(-663552n_1^{10}-3504384n_1^8n_2^2-4316247n_1^4n_2^6-6453216n_1^6n_2^4-\\ -63990n_1^2n_2^8+642033n_2^{10})h_C^2+(414720n_1^{11}++1938816n_1^9n_2^2-\\ -753570n_1n_2^{10}+772362n_1^5n_2^6+2832948n_1^7n_2^4-1290204n_1^3n_2^8)h_C+\\ +(289224n_1^8n_2^4+458010n_1^4n_2^8-108387n_1^2n_2^{10}-82944n_1^{10}n_2^2+\\ +755109n_1^6n_2^6--55296n_1^{12}-128164n_2^{12}). \end{gathered}$$

It can be shown that $\operatorname{genus}(D_C) = 0$ *and thus D_C (and also C) is a rational curve. We use a parameterization algorithm (see e.g. [12]) and obtain*

$$\begin{gathered} n_1(t) = \tfrac{6t-20t^3+6t^5}{t^6+3t^4+16t^3+3t^2+1}, \quad n_2(t) = \tfrac{1-15t^2+15t^4-t^6}{t^6+3t^4+16t^3+3t^2+1},\\ h_C(t) = -\tfrac{2}{3}\cdot\tfrac{1-3t-54t^5-23t^9-54t^7-3t^{11}+264t^6-23t^3-9t^4-12t^2-9t^8-12t^{10}+t^{12}}{1+14t^3+t^{12}+14t^9+6t^{11}+6t-300t^6+6t^{10}+111t^8+12t^7+111t^4+12t^5+6t^2}. \end{gathered} \tag{11}$$

Then, we can compute an envelope of tangents

$$n_1(t)x+n_2(t)y = h_C(t)$$

to get the rational parameterization $\mathbf{c}(t)$ of the convolution curve C (which is too long to include it into the paper).

Moreover, we can find suitable parameterizations $\tilde{\mathbf{a}}(t)$ and $\tilde{\mathbf{b}}(t)$ of A and B, respectively, which fulfill the convolution condition, i.e., for a given t_0 the corresponding normal vectors $\mathbf{n}_{\tilde{\mathbf{a}}}(t_0)$ and $\mathbf{n}_{\tilde{\mathbf{b}}}(t_0)$ are parallel.

Substituting $n_1(t)$, $n_2(t)$ into (9) and (10) we arrive at

$$\begin{aligned} h_A(t) &= \tfrac{16t^3}{t^6+3t^4+16t^3+3t^2+1},\\ h_B(t) &= \tfrac{2\left(t^3+3t^2-3t-1\right)^2\left(t^6-3t^5+3t^4+10t^3+3t^2-3t+1\right)}{3(t^3-3t^2-3t+1)^2(t^6+3t^4+16t^3+3t^2+1)}. \end{aligned} \tag{12}$$

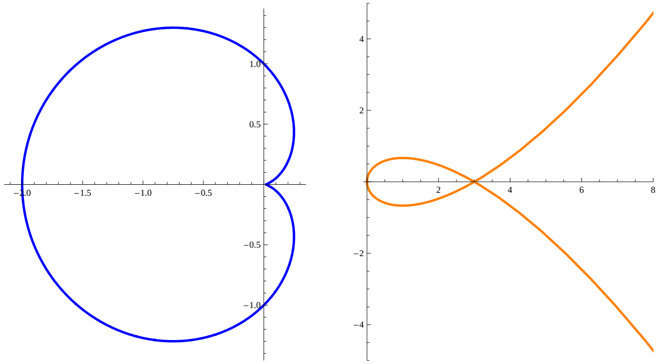

Figure 2: The cardioid (left) and the Tschirnhausen cubic (right).

Then, we apply the algorithm for computing an envelope of the system of tangents (7) and find parameterizations

$$\begin{aligned}\tilde{\mathbf{a}}(t) &= \left(\tfrac{8t^2(-6t^2+t^4+1)}{(t^2+1)^4}, \tfrac{32(t^2-1)t^3}{(t^2+1)^4}\right)^\top,\\ \tilde{\mathbf{b}}(t) &= \left(\tfrac{(t^3+3t^2-3t-1)^2}{(t^3-3t^2-3t+1)^2}, \tfrac{-2(t^3+3t^2-3t-1)(t^6-12t^5+3t^4+40t^3+3t^2-12t+1)}{3(t^3-3t^2-3t+1)^3}\right)^\top.\end{aligned}$$

These parameterizations can be achieved from the original parameterizations $\mathbf{a}(u)$, $\mathbf{b}(s)$ *by rational reparameterizations* $\phi : \Omega \to D_{\mathbf{a}}$, $\psi : \Omega \to D_{\mathbf{b}}$

$$\phi \quad : \qquad u = \frac{2t}{1-t^2}, \tag{13}$$

$$\psi \quad : \qquad s = \frac{t^3+3t^2-3t-1}{t^3-3t^2-3t+1}, \tag{14}$$

which are defined for a certain $\Omega \subseteq D_{\mathbf{a}} \cap D_{\mathbf{b}}$.

5 Conclusion

The paper has been devoted to the problem of finding rational parametrizations of convolution hypersurfaces (mainly curves and surfaces). The implicit support function representation which is very suitable for describing the convolutions has been introduced. We have also presented algorithms for computing ISF of hypersurfaces given parametrically or implicitly which are based on the Gröbner basis method. Finally, the algorithm for deciding the rationality of a convolution curve/surface and (eventually) finding its rational parameterization has been shown.

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