ZBYNĚK ŠÍR

FITTING OF PIECEWISE POLYNOMIAL IMPLICIT SURFACES

Abstrakt

In our contribution we discuss the possibility of an efficient fitting of piecewise polynomial implicit surfaces to given point sets. We describe a method which uses simultaneous approximation of given (sampled) data points and (estimated) normals. In particular we analyse the advantages and disadvantages of using tetrahedral and tensor product based piecewise polynomial implicit functions.

Klíčová slova

implicit surfaces, approximate implicitization, B-splines, A-patches, surface fitting

1 Introduction

Traditionally, most CAD systems rely on parametric representations, such as NURBS surfaces, which offer a number of benefits, in particular simple sampling techniques. But the use of implicitly defined surfaces offers also a number of advantages, for example shape constrains (e.g. convexity) can be expressed more easily (see e.g. [7]) and solids can be defined simply by evaluating the sign of the generating real-valued function. Also intersections of an implicit object with parametric ones (for example with straight lines in ray-tracing algorithms) can be found in a particularly efficient way.

In order to exploit the potential of implicit representations, methods for conversion to and from implicit representations are needed. In this paper we discuss the process of approximate implicitization via surface fitting. The implicit surface fitting was addressed first by V. Pratt in [9], since the various methods have been described in the vast literature on the subject - see for example [1, 11]. Most of these methods use the so–called algebraic distance and combine it with a suitable normalization of the coefficients. For instance in [9] the value of one coefficient is kept constant and in [11] the sum of the squared gradients at the data points is constrained.

As an alternative to the standard approach of a 'normalization' in the coefficient space, we use estimated normals [8], as additional information on the shape of the given data. Suppose we are given a set of points $\{\mathbf{p}_i \in \Omega : i = 1, \ldots, N\}$ in a suitable region of interest $\Omega \subset \mathbb{R}^3$. In some applications the normals are also available, but in general the unit normals $\{\mathbf{n}_i : i = 1, \ldots, N'\}$ at *regular* points $\mathbf{p}_i, i = 1, \ldots, N'$ must be estimated by fitting a regression plane to the neighboring points of $\mathbf{n}_p$. The points for which the fitting of regression plane fails are considered as *singular*. The regular points are organized into segments by a simple region growing process and the orientation of the estimated normals is propagated.

As a simple example figure 1(a) shows data points lying on a cube. The points in the interior of each face belongs to the same segment and the normals have corresponding orientation. The points close to the edges of the cube are detected as singular and no normals are estimated.

We define point and normal deviation functionals

$$L(f) = \sum_{i=1}^{N} f(\mathbf{p}_i)^2, \quad K(f) = \sum_{i=1}^{N'} \|\nabla f(\mathbf{p}_i) - \mathbf{n}_i\|^2 \tag{1}$$

and want to minimize the weighted linear combination

$$F(f) = L(f) + w_1 K(f) + w_2 T(f) \to \min, \tag{2}$$

where w_1, w_2 are suitable positive weights and T an additional quadratic tension term, given for example by

$$T(\mathbf{c}) = \iiint\limits_{\Omega} f_{xx}^2 + 2\, f_{xy}^2 + f_{yy}^2 + \ 2\, f_{xz}^2 + 2\, f_{yz}^2 + f_{zz}^2 \; dx\; dy\; dz. \tag{3}$$

It can be shown, that if w_1, w_2 are positive, then F is a positive, strictly convex quadratic functional and therefore restrained to a suitable finite dimensional functional space (polynomials, tensor product polynomials, polynomial spline functions) it has one minimum which can be found solving a system of linear equations. Fitting procedure of polynomial and tensor product polynomial functions the was studied and tested in [12]. The conclusion of this benchmarking reveals, that fitting of one polynomial implicit surface gives relatively good results for simple data sets. On the other hand for large input data sets with a possibly complicated topology the approximation by one algebraic surface is not very efficient and often produced unwanted branches in the region of interest. For this reason we will discuss a possibility of using tetrahedral and tensor product *piecewise* polynomial implicit functions instead of simply polynomial functions.

2 Tetrahedral piecewise polynomials

Two problems arise immediately considering piecewise polynomial functions defined in a union of tetrahedral domains. The first one is the geometry of the segmentation of Ω and the second is the expression of the continuity on the common boundaries of segments.

2.1 Segmentation of the space

The 3D space can not be filled be regular tetrahedra. Therefore there are two possible strategies for the segmentation of the domain of interest Ω into non regular tetrahedra. The first is to make once for ever a choice of the geometry of tessellation, which will be modified only by scaling and Euclidean transformations. The drawback of this approach is that the irregularity of the cells may create conflict with the topology of the data.

The second strategy would be to create a new division of the space for each input data - see e.g. [6]. This preprocessing step is computationally rather expensive and leads to relatively complicated expressions of the boundary continuity of the piecewise polynomials.

2.2 Continuity conditions

Let us suppose, that the region of interest Ω is divided into r tetrahedra. In each tetrahedron we want to construct a trivariate polynomial of degree n described in the Bernstein form. In total we thus get $r\binom{n+3}{3}$ coefficients and the boundary continuity conditions will introduce dependencies among them.

For example the C^0 continuity over the common face of two adjacent tetrahedra is equivalent to the identity of Bézier ordinates corresponding to the coinciding Bézier points. Higher order continuity can be expressed by linear conditions on the coefficients (see for example [5, p. 104]) and even more relaxed geometrical continuity can still be expressed by linear conditions [1].

We must then solve the optimization problem (2) subject to the linear constraints. Such a problem can be solved using Lagrangian multipliers, which leads to a huge system of linear equations involving auxiliary unknowns. Another possibility is to use quadratic programming algorithms, which are comparatively more expensive than the solution of a linear system of equations. The last possibility is to use the linear conditions for to substitute for certain coefficients. The implementation of such substitution process is not straightforward, considering a variable geometry of the triangulation.

The space of piecewise polynomials can also be described using suitable basis functions. This procedure [10], standard for tensor product splines, is unfortunately rather complicated in the case of tetrahedral splines. The continuity of the defined basis functions is ensured by use of auxiliary knots associated with each vertex of the tetrahedralisation. The position of these knots changes the properties of the basis and it is not obvious how these knots should be setup. An inappropriate configuration may create basis functions which have theoretically the required continuity, but which do not look geometrically continuous, because the curvature is too high. Due to these difficulties tetrahedral B-splines are rarely used for implicit representations.

2.3 A-patches

Recently Bajaj, Chen and Xu [2] described a new class of algebraic patches called A-patches. They form a special subclass of surfaces defined implicitly in a tetrahedron by a trivariate polynomial in the Bernstein form. Certain restrictions are imposed on the values of the Bézier ordinates in order to ensure that the surfaces are single sheeted. In several papers (e.g. [3]) the A-patches were used for the reconstruction of curves and surfaces from the scattered data, the main technique being a local interpolation and not an approximation. In fact the inequality conditions associated with A-patches complicates considerably the optimization problem. In addition as A-patches are one-sheeted, they can not describe complicated singularities.

3 Tensor product piecewise polynomials

The space can be divided in a natural and symmetrical way in rectangular boxes, for example in cubes of the same size.

The continuity conditions at the boundaries of the boxes into which the domain of interest is divided can be handled in a direct way, but this approach will suffer drawbacks similar to those described in 2.2 for the tetrahedral piecewise polynomials. On the other hand, for the tensor-product functions the theory of B-splines basis functions, assuring the desired continuity, is simple and efficient - see [5].

In order to obtain the knot vectors, we consider a bounding box of the data, and subdivide a slightly enlarged area in cubic cells of constant size s. This subdivision induces an equidistant grid on the x, y and z axis. In a standard way we obtain tensor-product spline functions which are trivariate polynomials of of degree (n, n, n) in each cubic cell and which are joined with the $n-1$ continuity over the neighboring faces. The domain of interest Ω does not in fact need to contain all the cells within this grid. Obviously it has to contain the cells containing given points. Additionally we consider neighboring cells. The reason for this is that the resulting surface is likely to pass through such a cell, and otherwise might be cut away. Due to this restriction, we only need to take into account the basis functions that do not vanish on Ω.

Clearly, the partition of the space into rectangular boxes, and hence the domain Ω of the tensor–product spline function, depend on the choice of coordinates. A geometrically invariant choice of the coordinate system can be obtained for example with the help of the eigenvectors of the matrix of inertia of the given data. Another disadvantage of tensor product B-splines is a rather high degree $3n$ of the resulting implicit function.

4 Conclusion

From the previous considerations it follows, that many problems occurring in the case of tetrahedral piecewise polynomials does not take place if tensor product polynomials are used. This is due to the existence of basis spline functions, which offer great advantages, including simple implementation, simple conditions for global smoothness and differentiability and simple evaluation of functionals (1), (3).

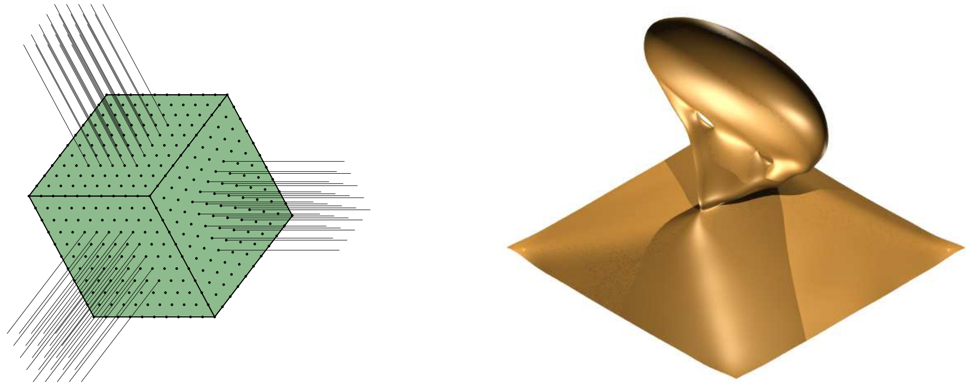
Another advantage of the B-spline representation is that the matrix representing the minimization problem (2) is sparse, which is due to the restricted support of the B-spline basis functions. For this reason both the building of corresponding matrices and solution of the resulting linear system can be greatly accelerated.

For the mentioned reasons the tensor product B-splines were chosen for the test implementations, realized within the EU project GAIA II at the Institute of Applied Geometry of Johannes Kepler University in Linz. Satisfactory results were obtained even for surfaces involving rather complicated singularities. For example the figure 1(b) shows a surface implicitized using a tensor product B-spline of degree $(3, 3, 3)$. The bounding box is divided into $17 \times 17 \times 15$ cells of which 310 form the domain of interest Ω.

In future we plan to use similar methods method for an implicitization of offset surfaces and investigate the possibility of a local refinement using hierarchical B-spline representations (see e.g. [4]), multi degree B-splines or T-splines.

Acknowledgment

The author was supported by the EU project GAIA II, IST-2001-35512.



(a) Segments and estimated normals. (b) Fitted implicit surface.

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