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Penrose Transform for homogeneous spaces of SL(4,C)

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Diploma Thesis

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I certify that this thesis is all my own work. It is freely available for all who can use it. I used only the cited literature.

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Contents

Chapter 1 - Introduction

The object of this work is the Penrose transform. This transform was developed as a part of twistor theory. The twistor description of solutions of massless field equations was one of first motivations for the twistor theory developed by R. Penrose and his school starting from 60's. It can be understood as a generalization of the Radon transform for functions to holomorphic setting. The basic geometry underlying the transform coincides with the geometry in the Helgason's formulation of the generalized Radon transform which is given by a double fibration of homogeneous spaces. In the case of the Penrose transform, homogeneous spaces involved are homogeneous spaces of complex simple Lie groups.

The first versions of the Penrose transform were formulated in physical case, it means in dimension 4 [Pe]. The corresponding double fibration was formed by homogeneous spaces of the group $G = SL(4, \mathbb{C})$:

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ \mathbb{P}_3 & & \mathbb{M} \end{tikzcd}$$

See 5.2. for definitions. Let V be a power of hyperline-section bundle on $\mathbb{P}_3$. $\mathbb{P}_x := \eta(\tau^{-1}(x)) \simeq \mathbb{P}_1$ for all $x \in \mathbb{M}$. We can understand $\mathbb{M}$ as a space of 1-lines in $\mathbb{P}_3$. If ω is a 1-form on $\mathbb{P}_3$ with values in V, then we can integrate it over each $\mathbb{P}_x$ and we obtain a function on $\mathbb{M}$. It was shown, that this function is a solution of Maxwell equations. On the other hand, $[\omega]$ is an element of $H^1(\mathbb{P}^I, \mathcal{O}(V))$. There is in fact a correspondence:

$$\mathcal{P} : H^1(\mathbb{P}^I, \mathcal{O}(V)) \to \ \{\text{solutions of Maxwell equations on } \mathbb{M}^I\ \}$$

[Pe] [Ea]

The most general situation for which the Penrose Transform was studied is the following one:

$$\begin{tikzcd} & F \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ Z & & X \end{tikzcd}$$

where F, Z, X are smooth manifolds, η, τ smooth submersions and $F \xrightarrow{\eta\times\tau} Z \times X$ an embedding. In addition there is required that:

(1) Z is a complex manifold and $\eta(\tau^{-1}(x))$ is a compact complex submanifold of Z for each $x \in X$
(2) η has contractible fibres.

Under this conditions, there is a mechanism, the Penrose transform, which relates cohomology $H^r(Z, \mathcal{O}(V))$ of a holomorphic vector bundle V on Z to solutions of systems of linear partial differential equations on X [Bai].

Our work will use method formulated in the book [B-E] by Baston and Eastwood, where the principle of the transform was formulated for a quite general double fibration:

$$\begin{tikzcd} & G/Q \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ G/R & & G/P \end{tikzcd}$$

where G is a semisimple complex Lie group and P, R, Q are parabolic subgroups of G, $P \cap R = Q$.

The authors described there a mechanism which relates cohomology of holomorphic vector bundle V on G/R to solutions of certain local linear operators on G/P. They developed certain machinery (see chapter 4 for brief description of this method) how to interpret the cohomology groups on G/R by objects on G/P using suitable spectral sequences. This is a very efficient procedure, nevertheless it loses character of an integral transformation and identification of objects on X needs further work. There are also analytical methods developed for certain cases, even in higher dimensions which give more explicit description of the transform as an integral transform, including an explicit description of the inverse transform (which is an additional and valuable information), see e.g. [G-H], [B-S], [Wo].

Here we are going to study here only four-dimensional situation where the double fibration is given by homogeneous spaces of the group $G = SL(4, \mathbb{C})$. The original Penrose transform gave interpretation to cohomology groups with values in sections of holomorphic line bundles. Here we shall concentrate on the four-dimensional situation and the case of general vector bundles and we discuss also the interpretation of conformally invariant equations on so called ambitwistor space.

We shall use methods developed in [B-E] and we shall describe the transform for two geometric situations, for which $G/P = \mathbb{M}$ - the complexified compactified Minkowski space.

In the chapter 2 we review basic definitions and tools : direct and inverse images, spectral sequences, Cartan B-theorem, Leray theorem, definition of jet bundle and of local and differential operator.

In chapter 3 we talk about homogeneous spaces G/P, associated bundles, action of G and $\mathfrak{g}$ on sections and cohomological groups, about parabolic subgroups, their representations, Haas diagram, BGG resolution and BBW theorem for computation of direct images.

In chapter 4 we can already give the general principle of computation and see which conditions must be proved.

The chapter 5 is the longest one. The section 5.1. explains why we restrict our investigations to two cases only. The twistor geometry of the corresponding double diagrams is described in coordinates in 5.2. and 5.3. The section 5.4. contains concrete computations of BGG and of direct images for both cases. In 5.5. we classify all the G-equivariant differential operators on $\mathbb{M}$.

The chapter 6 contains results of procedure: Theorem 6.1. is the interpretation of the cohomology groups with value in sections of any homogeneous vector bundle on the twistor space in terms of kernels of conformally invariant differential operators on $\mathbb{M}$. The theorem 6.2. gives correspondence for cohomologies for all homogeneous vector bundles on $\times\!\!-\!\!\!-\!\!\!\bullet\!\!\!-\!\!\!-\!\!\times$ -ambitwistor space. Like image of the correspondence we obtain solutions of G-equvariant differential operators on $\mathbb{M}$. In 6.4. we show, that we obtain all the solutions of all the differential operators.

The case of the twistor space $\mathbb{P}_3$ was described in [Ea1] and the main strategy of the proof was outlined there. We added here the complete proofs and discussion of types of differential operators kernels are obtained. The ambitwistor case was discussed on the level of examples in [Ea2]. A full discussion and all but one proofs were added. Some subcases need a verification of nontriviality of certain operators d in the spectral sequence.

We tried to expose clearly all the procedure of Penrose Transform, beginning by necessary tools, setting the general method and finally applying this method to concrete situations. We tried to present the results completely and systematically. The main original contribution consists in investigation of the problem of identification of operators, which occur in the transform. We proved carefully, that these operators are local and equivariant (5.4.6., 6.3.) On the other hand we classified all the equivariant local operators on $\mathbb{M}$. We was able to identify, which operators occurred. We showed in 6.4., that kernels of all the differential operators on $\mathbb{M}$ were obtained by Penrose Transform.

We described solutions of all G-equivariant operators on complexified, compactified Minkowski space. The action of G corresponds to the action of the Lorentz group on classic Minkowski space. We described all the possible physical laws by this way, even those laws not yet discovered by physics.

Chapter 2 - Generalities

First let us give several definitions and facts, which will be useful later. We suppose elementary knowledge about sheaves, presheaves, complex manifolds, and algebraic topology.

2.1. Definition. Let X, Y be two complex manifolds and $\mathcal{O}_X, \mathcal{O}_Y$ their structural sheaves. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$ modules and $\mathcal{G}$ a sheaf of $\mathcal{O}_Y$ modules. Let $f : X \to Y$ be holomorphic mapping. Then we define:

(1) sheaf $f^{-1}\mathcal{G}$ on X by the presheaf

$$W \to \mathcal{G}(f(W))$$

for W open subset of X. This sheaf is called **topological pullback** of $\mathcal{G}$.

(2) If the mapping f is proper, we define the sheaves $f^q_*\mathcal{F}$ on Y for all $q \in \mathbb{N}$ by the presheaves:

$$U \to H^q(f^{-1}(U), F)$$

for U open subset of Y with the natural restrictions. The sheaf $f^q_*\mathcal{F}$ is called q^{th} **direct image** of the sheaf $\mathcal{F}$.

For details see [W-W, 3.6.6. and 7.1.].

2.2. Theorem. *Let X, Y be complex manifolds, $f : X \to Y$ a surjective holomorphic mapping of maximal rank, and B a holomorphic vector bundle on X. If, for some $N \geq 0$, $H^p(f^{-1}(x), \mathbb{C}) = 0$ for all $p = 0, 1, \ldots, N$ and $x \in X$, then there is a canonical isomorphism*

$$H^q(X, \mathcal{O}(B)) \to H^q(Y, f^{-1}\mathcal{O}(B))$$

for $q = 0, 1, \ldots, N$.

[Bu]

Now we will review some information about spectral sequences which are powerful tools for computing cohomology groups. We will restrict ourselves to the description of bigraded spectral sequences. For general definition, all proofs and details see [B-T, 14], [W-W, 3.6.]

2.3. Definition. A triple (K, d', d'') is called a **bicomplex**, if K is a module of the form

$$K = \bigoplus_{p,q \in \mathbb{Z}}^{\infty} K^{p,q},$$

where $K^{p,q}$ are modules over ring R, $K^{p,q} = 0$ for p or q negative and d', d'' are homomorphisms:

$$d' : K^{p,q} \to K^{p+1,q}$$
$$d'' : K^{p,q} \to K^{p,q+1}$$

satisfying $(d')^2 = (d'')^2 = 0$, and $d'd'' + d''d' = 0$. Given a bicomplex , we associate the standard complex to it in the following way. We set

$$K^r = \bigoplus_{p+q=r} K^{p,q}$$

and

$$d = d' + d''$$

then clearly $d^2 = 0$, hence (K^*, d) is a complex.

2.4. Definition. **Bigraded spectral sequence** is a sequence of two-dimensional arrays (possibly infinite) of modules $E_r, r = 0, 1, 2, \ldots$ over ring R. The groups of the array are labeled

$$\{E_r^{p,q}\}, p, q \in \mathbb{Z}.$$

Each array E_r is equipped with a differential d_r, whereby d_r is a homomorphism which maps the array to itself in a specific fashion and satisfies $d_r^2 = 0$. Namely, d_r will map elements in the following manner:

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

We shall only be concerned with spectral sequences which have $E^{p,q} = 0$ for p or q negative. The first two terms of a spectral sequence are illustrated for p and q below:

$$\begin{array}{|ccccccccccc}
\cdots & & \cdots & & \cdots & & & & \cdots & \\
E_1^{0,q} & \to & E_1^{1,q} & \to & E_1^{2,q} & \to & \cdots & \to & E_1^{p,q} & \to \\
\cdots & & \cdots & & \cdots & & & & \cdots & \\
\cdots & & \cdots & & \cdots & & & & \cdots & \\
E_1^{0,1} & \to & E_1^{1,1} & \to & E_1^{2,1} & \to & \cdots & \to & E_1^{p,1} & \to \\
E_1^{0,0} & \to & E_1^{1,0} & \to & E_1^{2,0} & \to & \cdots & \to & E_1^{p,0} & \to \\
\hline
\end{array}$$

In the diagram above, the horizontal arrows represent the mappings

$$d_1 : E_1^{p,q} \to E_1^{p+1,q}.$$

For $r = 2$ we have

$$\begin{array}{|ccccc}
\cdots & \cdots & \cdots & & \cdots \\
E_2^{0,q} & E_2^{1,q} & E_1^{2,q} & \cdots & E_2^{p,q} \\
\cdots & \cdots & \cdots & & \cdots \\
\cdots & \cdots & \cdots & & \cdots \\
E_2^{0,1} & E_2^{1,1} & E_2^{2,1} & \cdots & E_2^{p,1} \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & \cdots & E_2^{p,0} \\
\hline
\end{array}$$

and we have $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$, which we recognize as the familiar knight's move from chess (two over and one down). The next condition for a spectral sequence is that the arrays are linked cohomologically from one order to the next. Namely, if we let $H(E_r)$ be the cohomology of the differential module (E_r, d_r), then one must have

$$H(E_r) \cong E_{r+1}.$$

Thus we see that each term of the array is inductively determined from the previous one with its differential.

Because $E^{p,q} = 0$ for p or q negative, both

$$d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}$$

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

are trivial for $r > r_0 = \max(p,q) + 1$. So

$$E_r^{p,q} = E_{r_0}^{p,q} \text{ for } r \geq r_0$$

and we define

$$E_\infty^{p,q} = E_{r_0}^{p,q}$$

This summarizes the basic aspects of spectral sequence.

Now we want to show spectral sequences associated with a bicomplex:

2.5. Theorem. *Let (K, d', d'') be a bicomplex. Then there are two bigraded spectral sequences ${}'E_r, {}''E_r$ which converge to global cohomology of K, i.e.:*

$${}'E^{p,q} \Rightarrow H_d(K^*)$$

$${}''E^{p,q} \Rightarrow H_d(K^*)$$

The convergence means, that there is a filtration of $H^i(K^*)$:

$$0 = H_{-1}^i \subset H_0^i \subset H_1^i \subset \cdots \subset H_i^i = H^i(K^*)$$

such that

$$H_k^i / H_{k-1}^i = E_\infty^{i-k,k}.$$

The first three arrows of these sequences are:

(1) ${}'E_0^{p,q} = K^{p,q}$
(2) ${}'E_1^{p,q} = H_{d''}^q(K^{p,*})$
(3) ${}'E_2^{p,q} = H_{d'}^p(H_{d''}^q(K^{*,*}))$

(1) ${}''E_0^{p,q} = K^{q,p}$
(2) ${}''E_1^{p,q} = H_{d'}^q(K^{*,p})$
(3) ${}''E_2^{p,q} = H_{d''}^p(H_{d'}^q(K^{*,*}))$

2.6. Lemma. *Let*

$$0 \to \mathcal{S} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \cdots \to \mathcal{A}^n \ldots \qquad (*)$$

be an exact sequence of sheaves over a topological space X . Let M be an open subset of X. Then there is a spectral sequence whose first term is given by

$$E_1^{p,q} = H^q(M, \mathcal{A}^p), \qquad (**)$$

the operator d_1 is induced from the mappings of (*) and

$$E^{p,q} \Rightarrow H^i(M, \mathcal{S})$$

Proof. We construct bicomplex $K^{**} = C^*(M, \mathcal{A}^*)$. There are two standard spectral sequences (2.5.). The first of them gives $H^p(K) = H^p(M, \mathcal{S})$ and the second is exactly (**). For details see [WW, example 3.6.5.] □

2.7. Theorem (Leray spectral sequence)**.** *Let X, Y be two complex manifolds and $\mathcal{O}_X, \mathcal{O}_Y$ their structural sheaves. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$ modules and let $f : X \to Y$ a proper holomorphic map. Then there exists a spectral sequence E , whose second term is:*

$$E_2^{p,q} = H^p(Y, f_*^q \mathcal{F})$$

and its limit is $H^i(X, \mathcal{F})$, i.e.

$$E^{p,q} \Rightarrow H^i(X, \mathcal{F})$$

[W-W,3.6.]

2.8. Theorem (Cartan B)**.** *Let X be a complex manifold, X' a Stein subset of X and $\mathcal{F}$ a coherent sheaf over X. Then*

$$H^q(X', \mathcal{F}) = 0 \text{ for all } q \geq 1.$$

[G-R, 8.14.]

2.9. Lemma. *Let $f : X \to Y$ be as in 2.7. Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$ modules. Let $Y' \subset Y$ be open and Stein and $X' = f^{-1}Y'$. Then*

$$H^q(X', \mathcal{F}) = \Gamma(Y', f_*^q \mathcal{F})$$

Proof. The sheaf $f_*^q \mathcal{F}$ is coherent. By 2.8. we have that $H^p(Y, f_*^q \mathcal{F}) = 0$ for $p \geq 1$. Thus the spectral sequence from 2.7. satisfies:

$$E_2^{p,q} = E_\infty^{p,q} = \begin{cases} \Gamma(Y', f_*^q \mathcal{F}) & \text{for } p = 0 \\ 0 & \text{for } p \geq 1 \end{cases}$$

and this gives our Lemma. □

Following theorem is important in proof of equivariance of the Penrose transform.

2.10. Lemma. *Let $K = \bigoplus_{p,q} K^{p,q}$ be a bicomplex. Suppose that there is an action of a Lie group or Lie algebra S on each $K^{p,q}$. Let both horizontal and vertical operators be S-equivariant. Then the following two properties hold for both canonical spectral sequences (2.5.) of K :*

(1) *there is the action of S on $E_k^{p,q}$ induced from the action on $K^{p,q}$.*
(2) *all operators d_k of the spectral sequence are equivariant with respect to the action from (1).*

Proof. Denote the horizontal operator by d and the vertical operator by δ. We will prove the lemma for the first spectral sequence:

$$E_1^{p,q} = \text{ Ker } \delta / \text{ Im } d$$

the proof for the second one is similar. The elements of $E_k^{p,q}$ are in fact the classes:$\underbrace{[[\ldots[}_{k} u] \ldots]]$, where:

$$u \in \ker \delta$$
$$[u] \in \ker d_1$$
$$\vdots$$
$$\underbrace{[[\ldots[}_{k-1} u] \ldots]] \in \ker d_{k-1}$$

For simplicity we will denote $\underbrace{[[\ldots[}_{l} u] \ldots]]$ by $[u]$.

The proof will be done by induction.

(1) δ is S-equivariant, and so the action of S on $E_1^{p,q}$ is induced by quotient.
(2) $i \geq 1$ Suppose that on $E_i^{p,q}$ is the action of S induced by quotient. We prove first that $d_i : E_i^{p,q} \to E_i^{p+i,q-i+1}$ is S-equivariant. By the [BT] $d_i([b]) = [e]$ iff there exist $c_j \in K^{p+j,q-j}$ for $j = 1 \ldots i-1$ so that:

$$\begin{aligned} \delta(c_1) &= d(b) \\ \delta(c_j) &= d(c_{j-1}) \text{ for } j = 1, \ldots, i-2 \\ d(c_{i-1}) &= e \end{aligned}$$

Let $X \in S$. Because the action on $E^{p,q}$ is given by quotient, we have $X[b] = [Xb]$. Let us define $c_j^{'} = Xc_j$ and this gives us $d_i([Xb]) = [Xe]$. Because $X[e] = [Xe]$ we have

$$d_i(X[b]) = X d_i([b]),$$

which is exactly the invariance of d_i. Now we can define the action of S on $E_{i+1}^{p,q}$ by quotient. $\square$

2.11. Theorem. *Let M be a complex manifold and* $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ *an open covering of M. Let* $\mathcal{F}$ *be a sheaf on M. If*

$$H^p(\cap_{i\in I} U_i, \mathcal{F}) = 0 \textit{ for all } I \subset \mathcal{A}, \mid I \mid < \infty, p \geq 1$$

then

$$H^p(M, \mathcal{F}) = H^p(\mathcal{U}, \mathcal{F}) \textit{ for all } p \geq 1.$$

[G-R, 6.D.4.]

Now we will define local and differential operator.

2.12. Definition. Let M be a complex manifold, E and F two holomorphic vector bundles over M. A linear mapping of sheaves

$$L : \mathcal{O}(E) \to \mathcal{O}(F)$$

is called a **local operator** from E to F iff for any open subset U of M the linear mapping of sections $L_U : \Gamma(U, E) \to \Gamma(U, F)$ is continuous in the following sense: For any sequence $s_n \in \Gamma(U, E)$ and $s_n \stackrel{d}{\to} 0$ on U, the sequence $L_U(s_n)$ converges uniformly to 0 on any V, $\bar{V} \subset U$. We note, that $\stackrel{d}{\to}$ means uniform convergence for sections and all their derivations.

Suppose that actions α, β of Lie group or Lie algebra S on $\mathcal{O}(M, E)$ and $\mathcal{O}(M, F)$ are given. Then the local operator L is called S-equivariant iff

$$\forall s \in S : L \circ \alpha(s) = \beta(s) \circ L$$

2.13. Definition. Let $E \to M$ be a holomorphic bundle over a complex manifold. Let $\mathcal{O}(E)$ be the sheaf of holomorphic sections. Let $z_1, ..., z_m$ be holomorphic coordinates for a neighbourhood U of a point $x \in M$. For $k = 0, 1, ...$ we define the equivalences $\sim_k$ on $\mathcal{O}_x(E)$ by

$$[f] \sim_k [g],$$

where f,g are sections defined in a neighbourhood of $x \in M$ if and only if

$$f(x) = g(x) \text{ and } d^I f(x) = d^I g(x) \text{ for } |I| \leq k$$

where $I \in \mathbb{N}^m$ is a multi-index and d^I is a I-partial derivative in coordinates $z_1, ..., z_m$. This definition does not depend on coordinates [Pom, 9.1.]. We define the space of **k-jets** as:

$$J^k_x E = \mathcal{O}_x(E)/\sim_k$$

If we set $J^k E = \bigcup_{x \in M} J^k_x E$, we obtain a holomorphic vector bundle on M and evidently $J^0 E = E$, see [Pom,9.7.].

There are natural projections

$$\pi_k : \mathcal{O}(E) \to J^k E$$

$$\pi^{k+r}_k : J^{r+k} E \to J^k E$$

2.14. Definition. Suppose that M' is a smooth real manifold and E' a smooth vector bundle over M'. The definition 2.13. can be repeated. Starting with the sheaf $\mathcal{E}(E')$ of smooth section, taking smooth coordinates, we get a smooth vector bundle $J^k(E')$. See [Pom].

2.15. Remark. Differential operators of order k can be identified with homomorphisms $D : J^kE \to F$, which transform holomorphic sections of J^kE to holomorphic sections of F, see [Po, 2]. They are examples of local operators in the above sense, as is shown in the next lemma.

2.16. Lemma. *For each homomorphism $D : J^kE \to F$ transforming holomorphic sections of J^kE to holomorphic sections of F, there is just one local operator L such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{O}(E) & \xrightarrow{L} & \mathcal{O}(F) \\ {\scriptstyle \pi_k}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ J^kE & \xrightarrow[D]{} & F \end{array}$$

Such a local operator will also be called a **differential operator** *of order k.*

Proof. We have only one possibility to define L: If U is an open subset of M, and $s \in \Gamma(U, E)$ then we set

$$L_U(s) \equiv D\pi_k s \in \Gamma(U, F).$$

The operator L is clearly well-defined and continuous. $\square$

Chapter 3 - Generalities about homogeneous case

In this chapter we give the basic facts about homogeneous spaces of Lie group. We define homogeneous bundle and actions of Lie group and algebra on homogeneous space and bundle. Parabolic subalgebras and subgroups are defined and classified. In the end of this chapter, we cite BGG-resolution and computation of direct images, which are basic tools in computing the Penrose Transform.

3.1. In this chapter let G be a complex Lie group, $\mathfrak{g}$ its Lie algebra, P a closed complex Lie subgroup of G and $\mathfrak{p}$ the Lie algebra of P. Then G/P is a homogeneous space of G (action given by $g[h] = [gh]$), it has a natural structure of a complex manifold and the action of G is holomorphic. We have surjective holomorphic mapping $\pi : G \to G/P$, given by the projection.

3.2. If we have the homogeneous space G/P and an irreducible representation of P,

$$\varphi : P \to GL(\mathbb{V})$$

where $\mathbb{V}$ is a finite dimensional complex vector space, then we can construct the **associated homogeneous vector bundle** $V = G \times_\varphi \mathbb{V} := G \times \mathbb{V}/\sim$ on G/P. where $\sim$ is given by: $(g, v) \sim (gp, \varphi(p^{-1})v)$ for all $g \in G, v \in \mathbb{V}, p \in P$. There is an action of G on V given by $h[(g, v)] := [(hg, v)]$.

3.3. Definition. Let us define **equivariant sections** of the trivial bundle $G^U \times \mathbb{V}$, where U is an open subset of G/P and $G^U := \pi^{-1}(U)$:

$$\mathit{\Gamma}(G^U, \mathbb{V})^P := \{f : G^U \to \mathbb{V} \text{ holomorphic} : \forall g \in G^U, p \in P : f(gp) = \varphi(p^{-1})f(g)\}$$

3.4. Lemma. *We identify*

$$\mathit{\Gamma}(G^U, \mathbb{V})^P \simeq \mathit{\Gamma}(U, V) \qquad (*)$$

via $f \simeq f'$, where $f'([g]) := [(g, f(g)]$. If $U = G/P$ then this is an identification of G-modules.

For proofs of 3.2.,3.3.,3.4. see in [Sl, 2.10.].

3.5. Definition. The action of G on V induces the **action of** G **on** $\mathcal{O}(V)$: If $[f]$ is a germ in $x \in U$, f an equivariant section defined in G^U, then $[hf]$ is a germ in hx, hf defined in G^{hU} by $hf(y) := f(h^{-1}y)$.

This action preserves the equivalence relations $\sim_k$ from 2.13. and so we get **action of** G **on** J^kV and π_k, π_k^{k+r} are G-equivariant. Clearly a differential operator L is G-equivariant, if and only if the mapping D from 2.16. is G-equivariant.

3.6. Definition. An action of G on $\mathit{\Gamma}(U, G \times_\varphi V)$ can be defined only in the case $U = G/P$. For a general open set $U \subset G/P$, we can define the action Ψ of the algebra $\mathfrak{g}$ on the space of sections $\mathit{\Gamma}(U, G \times_\varphi V)$ - so called **infinitesimal action.**

For this we use the identification 3.4.: Let $X \in \mathfrak{g}, s \in \Gamma(G^U, \mathbb{V})^P, m \in G^U$ then we define

$$[\Psi(X)s](m) := \frac{d}{dt} s(exp(tX)m)(0),$$

where $s(exp(tX)m)$ is defined for $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. The section $\Psi(X)s$ is again an element of $\Gamma(G^U, \mathbb{V})^P$ and it is easy to see that Ψ is an action of $\mathfrak{g}$.

3.7. Let us recall the definition of cohomology groups with coefficients in sheaves: If E is a vector bundle over X, then there is an exact sequence of sheaves:

$$\mathcal{E}^{0,0}(E) \xrightarrow{\bar{\partial}^0} \mathcal{E}^{0,1}(E) \xrightarrow{\bar{\partial}^1} \dots \xrightarrow{\bar{\partial}^{i-1}} \mathcal{E}^{0,i}(E) \xrightarrow{\bar{\partial}^i} \dots \qquad (*)$$

called the Dolbeault resolution. ($\mathcal{E}^{0,i}(E) := \mathcal{O}(E \otimes \wedge^i T^{0,1})$). We define

$$H^i(Y, E) := \text{ Ker } \bar{\partial}^i / \text{ Im } \bar{\partial}^{i-1} \qquad (**)$$

for $Y \subset X$ and $\bar{\partial}^{i-1}, \bar{\partial}^i$ considered on sections over Y.

If E is a homogeneous vector bundle on G/P, then $E \otimes \wedge^i T^{0,1}$ is a homogeneous vector bundle, too. Thus by 3.5. we define action of G on all sheaves in $(*)$ and this sequence is G-equivariant.

3.8. Definition. By 3.6. we can define the action of $\mathfrak{g}$ on $\mathcal{E}^{0,i}(U, E)$. By $(**)$ we get the **action of $\mathfrak{g}$ on** $H^i(U, E)$.

3.9. Definition. Let $Q \subset R \subset G$ be three Lie groups. Then there is a natural mapping $\eta : G/Q \to G/R$. If E is a homogeneous vector bundle on G/R then we define using 2.2. the **action of G on** $\eta^{-1}\mathcal{O}(E)$ and **action of $\mathfrak{g}$ on** $H^i(U, \eta^{-1}\mathcal{O}(E))$.

Now we will define distinguished subgroups of G, which will be of our interest so called parabolic subgroups. We will also describe their representations. For basic facts about classification and representations of semisimple Lie groups, see [F-H]. For more details about parabolic subgroups, see [B-E].

3.10. Definition. Let us fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. It is a maximal solvable subalgebra. Each subalgebra $\mathfrak{p}$ containing $\mathfrak{b}$, i.e. $\mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g}$ is called a **parabolic subalgebra**. There is only a finite number of parabolic subalgebras containing a fixed Borel algebra. All parabolic subalgebras (up to conjugation) are constructed by a simple procedure:

Let us write $\mathfrak{n}^{\pm}$ for the subalgebras generated by the positive or negative root elements respectively, i.e. $\mathfrak{n}^+ = [\mathfrak{b}, \mathfrak{b}]$. The whole algebra is a sum

$$\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha) = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^- = \mathfrak{n}^- \oplus \mathfrak{b}.$$

Let us fix a set $\Sigma \subset \Delta_0^+$ of simple roots and write Δ_Σ for its span in the set of all roots. Now we define the subalgebras

$$\mathfrak{l} = \mathfrak{h} \oplus \big(\bigoplus_{\alpha\in\Delta_\Sigma} \mathfrak{g}_\alpha\big), \quad \mathfrak{n} = \bigoplus_{\alpha\in\Delta^+\setminus\Delta_\Sigma} \mathfrak{g}_\alpha, \quad \mathfrak{p} = \mathfrak{l}\oplus\mathfrak{n}$$

By the definition, $\mathfrak{p}$ contains the whole Borel algebra $\mathfrak{b}$ and the algebra $\mathfrak{g}$ splits as a vector space direct sum of Lie subalgebras $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{p}$. The subalgebra $\mathfrak{l}$ is reductive, $\mathfrak{n}$ is nilpotent. Hence $\mathfrak{l}$ is the reductive Levi factor of the parabolic subalgebra $\mathfrak{p}$. The semisimple factor is $[\mathfrak{l},\mathfrak{l}] = \bigoplus_{\alpha\in\Delta_\Sigma} \mathfrak{g}_\alpha$ and $\mathfrak{l} = \mathfrak{h}_\Sigma \oplus (\bigoplus_{\alpha\in\Delta_\Sigma} \mathfrak{g}_\alpha)$ where $\mathfrak{h}_\Sigma$ is the linear subspace in $\mathfrak{h}$ corresponding to $\Sigma \subset \mathfrak{h}^*$.

If $\mathfrak{g}$ is a semisimple Lie algebra and G a simply connected Lie group with $\mathfrak{g}$ as Lie algebra. Then $\mathfrak{p}$ gives rise to a subgroup of G which will be denoted P. The subgroups which arise this way are called the **parabolic subgroups** of G.

The parabolic subalgebras in semisimple complex algebras can be effectively denoted by means of the Dynkin diagrams if we replace by a cross the nodes corresponding to the simple roots which are not in Σ. The corresponding Lie subgroups will be denoted by the same modified Dynkin diagram.

3.11. Representations of parabolic subalgebras. In general, the representations of the parabolic subalgebras of semisimple algebras need not be completely reducible. But we shall still restrict ourselves to the irreducible ones. Let us fix a parabolic algebra $\mathfrak{p} \subset \mathfrak{g}$ and its Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ corresponding to a subset $\Sigma \subset \Delta_0^+$ as above. If V is a finite dimensional irreducible representation space of $\mathfrak{p}$, then $\mathfrak{n}$ acts by nilpotent endomorphisms by Engel's theorem, and so $\mathfrak{n}$ acts trivially. The reductive part $\mathfrak{l}$ decomposes into the semisimple factor $\mathfrak{s} = [\mathfrak{l},\mathfrak{l}]$ and the center $\mathfrak{z}$. We can always arrange $\mathfrak{h} = (\mathfrak{h}\cap\mathfrak{s})\oplus\mathfrak{z}$. An irreducible representation of $\mathfrak{p}$ is determined by a dominant weight for $\mathfrak{s}$ and an element from $\mathfrak{z}^*$ and so the representation is specified by a weight λ for $\mathfrak{g}$ such that $\lambda(H_\alpha)$ is a non-negative integer for all $\alpha \in \Sigma$. Such a weight is called **dominant for** $\mathfrak{p}$. More precisely, λ decomposes into a dominant weight $\lambda_\mathfrak{s}$ for $\mathfrak{s}$ and an element from $\mathfrak{z}^*$.

For semisimple $\mathfrak{g}$, the classifications of its representations coincides with classification of representations of group G. However, if the parabolic $\mathfrak{p} \subset \mathfrak{g}$ corresponds to $P \subset G$, then in order for an irreducible representation of $\mathfrak{p}$ to "exponentiate" to one for P it is necessary and sufficient that the dominant weight λ be integral for $\mathfrak{g}$ and not just for $\mathfrak{p}$.

Notation. We shall denote by $\mathbb{V}_\lambda$ the irreducible P module corresponding to a $\mathfrak{p}$ module with the highest weight λ -integral for $\mathfrak{g}$ and dominant for $\mathfrak{p}$. The associated vector bundle over G/P will be denoted by V_λ and the sheaf of sections of this bundle by $\mathcal{O}_\mathfrak{p}(\lambda)$.

Let us denote by δ the weight defined by

$$\delta(H_\alpha) = 1 \text{ for all } \alpha$$

and $\delta_{\mathfrak{p}}$ the weight defined by

$$\delta_{\mathfrak{p}}(H_\alpha) = \begin{cases} 1 & \text{for } \alpha \in \Sigma \\ 0 & \text{for } \alpha \in \mathfrak{h} \setminus \Sigma \end{cases}$$

We shall express the representation determined by a dominant weight λ for $\mathfrak{p}$ by inscribing the values $(\lambda+\delta)(H_\alpha)$ on the fundamental coroots over the corresponding nodes.

3.12. Definition. Let W be the Weyl group of $\mathfrak{g}$. Let $Q \subseteq R$ be two parabolic subgroups of G and $\mathfrak{q}, \mathfrak{r}$ their Lie algebras. Then we define

i) the Haas diagram $W^{\mathfrak{q}}$ attached to $\mathfrak{q}$ as the set of all elements $w \in W$ which send weights dominant for $\mathfrak{g}$ by affine action to weight dominant for $\mathfrak{q}$.
ii) the relative Haas diagram $W^{\mathfrak{q}}_{\mathfrak{r}}$ attached to the fibration $G/Q \to G/R$ as the set of all elements $w \in W$ which send weights dominant for $\mathfrak{r}$ by affine action to weight dominant for $\mathfrak{q}$.

[B-E,4.3.]

3.13. Theorem (relative Bernstein-Gelfand-Gelfand resolution)**.** *Let G be a semisimple Lie group , $\mathfrak{g}$ its Lie algebra, $Q \subseteq R$ two parabolic subgroups of G and $\mathfrak{q}, \mathfrak{r}$ their Lie algebras. If λ is a dominant integral weight for R, then there is an exact G-equivariant resolution*

$$0 \to \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda) \to \Delta^*(\lambda),$$

where

$$\Delta^p(\lambda) = \bigoplus_{w \in W^q_r, l(w)=p} \mathcal{O}_{\mathfrak{q}}(w.\lambda)$$

See [BE,theorem 8.4.1,8.7.], for equivariance [Ro],[Sl,8.3.]. We recall, that $l(w)$ denotes the minimal number of simple reflections necessar to generate $w \in W$.

3.14. Recipe. (The direct images) We will need following recipe for computing direct images of sheaves of sections of homogeneous bundles. Let $P \subseteq Q$ be subgroups of G. There is natural projection $\tau : G/Q \to G/P$. Let λ be a weight dominant and integral for $\mathfrak{q}$. Then we have following theorem-recipe for direct images of $\mathcal{O}_{\mathfrak{q}}(\lambda)$.

Step one: Determine the Haas diagram $W^{\mathfrak{q}}_{\mathfrak{p}}$ by allowing $W_{\mathfrak{p}}$ to act on $\delta_{\mathfrak{q}}$. It is only necessary to record simple reflection.

Step two: To compute direct images of $\mathcal{O}_{\mathfrak{q}}(\lambda)$, act on λ with the graph of simple reflections constructed in step one.

Step three: If any element of the resulting orbit is repeated, all the direct images vanish. Otherwise ...

Step four: Precisely one element of the orbit has positive entries over all nodes not crossed through in $\mathfrak{p}$ - it gives a weight dominant integral for $\mathfrak{p}$ which will be denoted μ. If l is the number of simple reflections required to produce μ then

$$\tau_*^l \mathcal{O}_{\mathfrak{q}}(\lambda) \simeq \mathcal{O}_{\mathfrak{p}}(\mu)$$

and all other direct images vanish. [BE, 5.1.] This computation is $\mathfrak{g}$-equivariant, it means, that the isomorphisms:

$$\Gamma(U, \tau_*^l \mathcal{O}_{\mathfrak{q}}(\lambda)) = H^l(\tau^{-1}(U), E_\lambda) \simeq \Gamma(U, \mathcal{O}_{\mathfrak{p}}(\mu))$$

are $\mathfrak{g}$-equivariant with respect to actions defined in 3.6. and 3.8.

For concrete applications of recipe and BGG resolution, see Chapter 5.

Chapter 4 - Principle of Penrose Transform

In this chapter we will describe the general principle of our computations.

Let $G = SL(4,\mathbb{C})$ and P, Q, R be parabolic subgroups of G, $Q = P \cap R$. We denote $A = G/R, B = G/Q, C = G/P$ and have double fibration:

$$\begin{tikzcd} & B \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ A & & C \end{tikzcd} \tag{FIB}$$

We take an open subset C' of C and define $B' = \tau^{-1}(C'), A' = \eta(B')$. We obtain following restriction of FIB:

$$\begin{tikzcd} & B' \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ A' & & C' \end{tikzcd}$$

Let E_λ be a homogeneous irreducible vector bundle on A. We proceed by the following steps:

(1) Using 2.2. we know, that

$$H^r(A', \mathcal{O}(E_\lambda)) = H^r(B', \eta^{-1}\mathcal{O}(E_\lambda))$$

on condition that fibres of η are topologically trivial. This condition is proved in 5.2.10. and 5.3.6. for a large class of C', which induces topology of C.

(2) There is an exact G-equivariant sequence:

$$0 \to \eta^{-1}\mathcal{O}(E_\lambda) \to \Delta^*(\lambda),$$

see 3.13.

(3) Following 2.6. we construct the double complex

$$K^{**} = C^*(B', \Delta^*(\lambda))$$

and by 2.6. there is such a spectral sequence that:

$$E_1^{pq} = H^q(B', \Delta^q(\lambda))$$

$$E^{pq} \Rightarrow H^r(B', \eta^{-1}\mathcal{O}(E_\lambda))$$

(4) By lemma 2.9. we obtain

$$H^q(B', \Delta^p(\lambda)) = \mathit{\Gamma}(C', \tau_*^q \Delta^p(\lambda))$$

because fibration τ is proper and C' will be chosen to be Stein, and sheaves Δ are free.

(5) So we have the spectral sequence:

$$E_1^{pq} = \mathit{\Gamma}(C', \tau_*^q \Delta^p(\lambda)) \tag{SEQ}$$

and

$$E^{pq} \Rightarrow H^r(B', \eta^{-1}\mathcal{O}(E_\lambda)) = H^r(A', \mathcal{O}(E_\lambda))$$

.

(6) We will compute explicitly $\tau_*^q \Delta^p(\lambda)$, see 5.4.2., 5.4.3.,5.4.4.,5.4.5.

(7) Now we would like to determine E_∞^{pq}. For this we must identify the operators d_1 in this spectral sequence which are local and G-equivariant by 6.3. We will use the classification of G-equivariant local operators 5.5.3.

(8) We obtain the identification between $H^r(A', \mathcal{O}(E_\lambda))$ and Ker D -solution of some equivariant differential operator , as described later (6.1., 6.2.).

(9) The identification from (8) is $\mathfrak{g}$-equivariant, because the resolution Δ 3.13., and the computation of direct images 3.14. are $\mathfrak{g}$-equivariant.

(10) We will discuss in 6.3., that solutions of all G-equivariant differential operators on $\bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet$ are obtained (see chapter 5 for definition).

Chapter 5 - Objects and Computations

In this chapter we will describe concrete objects, which will be used in Penrose Transform, and prove conditions necessar for the working of the Penrose Transform. We will define and explore two double fibrations, which make the base of Penrose Transform. We apply theorems 3.13. and 3.14. for computing resolutions and direct images for the both fibrations. We are interested in Minkowski space $\bullet\!-\!\!-\!\times\!-\!\!-\!\bullet$ and so we classify all the G-equivariant differential operators on it and show, that any local G-equivariant operator is differential.

5.1. Two double fibrations.

In this and the following chapter, G will denote the group $SL(4,\mathbb{C}) = \bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet$. If we consider the basic representation of G on $T = \mathbb{C}^4$ and fix a basis $\{t_1, t_2, t_3, t_4\}$ of T, then G is represented by 4×4 matrices with determinant equal to 1.

$$G = \{M = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}, \det(M) = 1\}$$

where stars denote arbitrary complex numbers.

5.1.1. We will work with compact homogeneous spaces G/P. For this it is necessary, that P be parabolic. In the notation from 3.10. there are 8 parabolic subgroups of $\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet$ up to conjugation :

$$\begin{array}{ll} P_1 = \bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet & \quad P_5 = \times\!-\!\!-\!\times\!-\!\!-\!\bullet \\ P_2 = \times\!-\!\!-\!\bullet\!-\!\!-\!\bullet & \quad P_6 = \bullet\!-\!\!-\!\times\!-\!\!-\!\times \\ P_3 = \bullet\!-\!\!-\!\bullet\!-\!\!-\!\times & \quad P_7 = \times\!-\!\!-\!\bullet\!-\!\!-\!\times \\ P_4 = \bullet\!-\!\!-\!\times\!-\!\!-\!\bullet & \quad P_8 = \times\!-\!\!-\!\times\!-\!\!-\!\times \end{array}$$

We will use the same notation for the corresponding homogeneous space. So for example $\times\!-\!\!-\!\bullet\!-\!\!-\!\bullet$ will denote G/P_2 ,too.[B-E,2]

5.1.2. We are interested in such cases of the Penrose transformation which have on the right the space $G/P_4 = \bullet\!-\!\!-\!\times\!-\!\!-\!\bullet$. On the left we must take such G/P that $P \not\subseteq P_4$ and $P \not\supseteq P_4$ otherwise one of fibrations τ, η would be trivial and computation of the Penrose transform, too. So there are three possibilities: $P = P_2, P = P_3$ and $P = P_7$. Cases $P = P_2$ and $P = P_3$ are symmetric, so we will restrict ourselves to two cases.

5.2. Twistor space on left.

Let us explore the double fibration:

$$\begin{tikzcd} & G/P_5 \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ G/P_2 & & G/P_4 \end{tikzcd}$$

This is

$$\begin{tikzcd} & \times\!-\!\times\!-\!\bullet \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ \times\!-\!\bullet\!-\!\bullet & & \bullet\!-\!\times\!-\!\bullet \end{tikzcd}$$

5.2.1. The basic action of G on T induces an action of G on the projective space $\mathbb{P}(T)$. This action is transitive. We ask what is the stabilizer of the element $[(1,0,0,0)]$. It is clearly a matrix of form:

$$A = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}, \det(A) = 1$$

These matrices form exactly the group $\times\!-\!\bullet\!-\!\bullet$. So we see that the homogeneous space $\times\!-\!\bullet\!-\!\bullet$ is in fact $\mathbb{P}_3$. The dimension of this complex manifold is 3. There are homogeneous coordinates on $\mathbb{P}(T)$ with respect to the basis $\{t\}$ from 5.1.

5.2.2. Now let us define the set $\mathbb{M} = \mathbb{G}_2(T) = \{S \subset T; \dim S = 2\}$ so called Grassmann manifold. This is a complex manifold. We define $\mathbb{M}^I \subset \mathbb{M}$ by:

$$\mathbb{M}^I = \{S \subset T; S = \mathrm{span}\{(1,0,v_{00},v_{10}),(0,1,v_{01},v_{11})\}\}$$

where $v_{ij} \in \mathbb{C}$ for $i,j \in \{0,1\}$. This is a dense open submanifold of $\mathbb{M}$ and there is holomorphic bijection:

$$\begin{aligned} \chi : \mathbb{C}^{2\times 2} &\to \mathbb{M}^I \\ z_{ij} &\to \mathrm{span}\{(1,0,z_{00},z_{10}),(0,1,z_{01},z_{11})\} \end{aligned}$$

which gives coordinates on $\mathbb{M}$ and so the dimension of $\mathbb{M}$ is 4. There is the induced action of G on $\mathbb{M}$ as well. This action is clearly transitive and the stabilizer of $\mathrm{span}\{(1,0,0,0)(0,1,0,0)\}$ are matrices

$$A = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \det(A) = 1,$$

which form exactly the group $\bullet\!-\!\times\!-\!\bullet$. So the space $\bullet\!-\!\times\!-\!\bullet$ is the Grassmanian $\mathbb{G}_2^4$ and is called the complexified compactified Minkowski space.

5.2.3. There is a subgroup O in $P_4 = \bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet$:

$$O = \{o_t = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix}, t \in \mathbb{C}^* \}.$$

In fact the group O is the centre of the L-reductive factor of P_4 and so in any representation of P_4, elements of O must be represented by non-zero multiples of identity (see 3.11.) Let us compute the action of o_t on $\mathbb{M}$ in coordinates z.

$$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix} \text{span} \begin{pmatrix} I \\ z \end{pmatrix} = \text{span} \begin{pmatrix} t.I \\ t^{-1}z \end{pmatrix} = \text{span} \begin{pmatrix} I \\ t^{-1}zt^{-1} \end{pmatrix}$$

and so o_t acts on z by multiplication by t^{-2}.

5.2.4. There is a real subgroup $G' \subset G$ formed by all real matrices. Define $P_4' = P_4 \cap G'$. There is a real subspace $T_{\mathbb{R}} = \text{span}_{\mathbb{R}}\{t_1, t_2, t_3, t_4\}$ of T . The space G'/P' is real 2-Grassmanian of $T_{\mathbb{R}}$ and there are real coordinates

$$\chi : \mathbb{R}^{2\times 2} \to \mathbb{M}^{I,}$$
$$x_{ij} \to \text{span}\{(1, 0, x_{00}, x_{10}), (0, 1, x_{01}, x_{11})\}$$

The action of element

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in P_4'$$

where $A, B, C, 0$ are 2×2 matrices, is the same on z and x and is given by:

$$Mx = Cx(A + Bx)^{-1}$$

$$Mz = Cz(A + Bz)^{-1}$$

(The inverses of parentheses are defined in a neighbourhood of $z = 0$.)

5.2.5. We define manifold:

$$\mathbb{F} = \mathbb{F}_{1,2}(T) = \{(S_2, S_1); S_1 \subset S_2 \subset T; \text{dim}S_1 = 1, \text{dim}S_2 = 2\}$$

This is called the flag manifold and has the dimension 5. As in 5.2.1. and 5.2.2. we can see, that $\times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet = \mathbb{F}$. Our diagram is so :

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ \mathbb{P} & & \mathbb{M} \end{tikzcd}$$

5.2.6. Let us describe this geometric situation in a little more detail. We have already defined $\mathbb{M}^I \subset \mathbb{M}$. We define

$$\mathbb{F}^I = \tau^{-1}(\mathbb{M}^I), \mathbb{P}^I = \eta(\mathbb{F}^I)$$

and we can restrict the diagram to

$$\begin{tikzcd} & \mathbb{F}^I \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ \mathbb{P}^I & & \mathbb{M}^I \end{tikzcd}$$

5.2.7. Lemma. *There is a holomorphic bijection* $\varphi : \mathbb{M}^I \times \mathbb{P}_1 \to \mathbb{F}^I$.

Proof:. Let $z_{ij} \in \mathbb{C}^{2\times 2}$ be coordinates for $\mathbb{M}^I$ as above and $[v_0, v_1]$ homogeneous coordinates for $\mathbb{P}_1$. We define $\varphi : \mathbb{M}^I \times \mathbb{P}_1 \to \mathbb{F}$ by:

$$\varphi(z, [v]) = (S_2^{z,[v]}, S_1^{z,[v]}) \in \mathbb{F}$$

where

$$\begin{aligned} S_2^{z,[v]} &= \operatorname{span}\{(1, 0, z_{00}, z_{10}), (0, 1, z_{01}, z_{11})\} \\ S_1^{z,[v]} &= \operatorname{span}\{(v_0, v_1, z_{00}.v_0 + z_{01}.v_1, z_{10}.v_0 + z_{11}.v_1)\} \end{aligned}$$

This definition is correct and φ is holomorphic bijection between $\mathbb{M}^I \times \mathbb{P}_1$ and $\mathbb{F}^I$. $\square$

5.2.8. The proof of the lemma 5.2.7. enables us to describe η and τ in coordinates: For $\mathbb{F}^I$ we have "semi-homogeneous" coordinates

$$(z, [v]), z \in \mathbb{C}^{2\times 2}, v \in \mathbb{C}^2, v \neq 0$$

given by isomorphism in lemma 5.2.7. On $\mathbb{P}^I$ we have homogeneous coordinates

$$[u], u \in \mathbb{C}^4, u \neq 0$$

and on $\mathbb{M}^I$ are simply coordinates

$$z, z \in \mathbb{C}^{2\times 2}$$

Then we can write

$$\begin{aligned} \eta(z_{00}, z_{10}, z_{01}, z_{11}, [v_0, v_1]) &= [(v_0, v_1, z_{00}.v_0 + z_{01}.v_1, z_{10}.v_0 + z_{11}.v_1)] \\ \tau(z_{00}, z_{10}, z_{01}, z_{11}, [v_0, v_1]) &= (z_{00}, z_{10}, z_{01}, z_{11}) \end{aligned}$$

5.2.9. Lemma. *The fibres of the fibration $\eta : \mathbb{F}^I \to \mathbb{P}^I$ are topologically trivial.*

Proof. We shall describe fibres in coordinates. Let $u = [(u_1, u_2, u_3, u_4)]$ be a point in $\mathbb{P}^I$. This means, that $(u_1, u_2) \neq (0, 0)$. Then $\eta^{-1}(u) \cap \mathbb{F}^I$ are exactly points $(z, [v])$ such that:

$$\begin{aligned} v_0 &= u_1 \\ v_1 &= u_2 \\ z_{00}.v_0 + z_{01}.v_1 &= u_3 \\ z_{10}.v_0 + z_{11}.v_1 &= u_4. \end{aligned}$$

Hence $[v]$ can be computed from the first two equations and we obtain the following equations for z:

$$\begin{aligned} z_{00}.u_1 + z_{01}.u_2 &= u_3 \\ z_{10}.u_1 + z_{11}.u_2 &= u_4 \end{aligned}$$

We can solve each equation separately and because $(u_1, u_2) \neq (0, 0)$, we see that all z satisfying both equations form a 2-dimensional linear set in $\mathbb{C}^4$ which is a topologically trivial set. $\square$

5.2.10. Remark. Let a subset $\mathbb{M}'$ of $\mathbb{M}^I$ be convex in coordinates and define:

$$\mathbb{F}' = \tau^{-1}(\mathbb{M}'), \mathbb{P}' = \eta(\mathbb{F}'),$$

then $\eta : \mathbb{F}' \to \mathbb{P}'$ has topologically trivial fibres, because they are obtained as the intersection of two convex sets. Because we can choose any basis $\{t\}$, the sets $\mathbb{M}'$ form the basis of topology of all $\mathbb{M}$ (if we change $\{t\}$, then we change the position of $\mathbb{M}^I$ in $\mathbb{M}$.

5.2.11. Lemma. *Let E be a homogeneous vector bundle over $\times\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet$. then cohomology groups.*

$$H^i(\mathbb{P}^I, \mathcal{O}(E)) = 0, \mathit{for}\ i \geq 2.$$

Proof. $\mathbb{P}^I$ is covered by two Stein sets A, B described in coordinates as follows:

$$A = \{[(t, 1, z_{00}.t + z_{01}, z_{10}.t + z_{11})]; t \in \mathbb{C}, z \in \mathbb{M}^I\}$$

$$B = \{[(1, t, z_{00} + z_{01}.t, z_{10} + z_{11}.t)]; t \in \mathbb{C}, z \in \mathbb{M}^I\}$$

By [G-R,7.A. ex.9] an intersection of two Stein subsets is a Stein subset. So $\{A, B\}$ is a covering of $\mathbb{P}^I$ satisfying the condition from 2.11. and we get

$$H^i(\mathbb{P}^I, \mathcal{O}(E)) = H^i(\{A, B\}, \mathcal{O}(E)).$$

This equals to 0 for $i \geq 2$ because this is a two set covering. $\square$

5.3. Ambitwistor space on left.

Now we will study the second case:

$$\begin{tikzcd} & G/P_8 \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ G/P_7 & & G/P_4 \end{tikzcd}$$

This is

$$\begin{tikzcd} & \times\!-\!\!-\!\times\!-\!\!-\!\times \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ \times\!-\!\!-\!\bullet\!-\!\!-\!\times & & \bullet\!-\!\!-\!\times\!-\!\!-\!\bullet \end{tikzcd}$$

5.3.1. The basic action of G on T defines the action of G on every flag manifold of T. It is easy to show [B-E,2.3.], that

$$\times\!-\!\!-\!\times\!-\!\!-\!\times = \mathbb{B} = \mathbb{F}^4_{1,2,3}$$
$$\times\!-\!\!-\!\bullet\!-\!\!-\!\times = \mathbb{A} = \mathbb{F}^4_{1,3}.$$

We define

$$\mathbb{B}^I = \tau^{-1}(\mathbb{M}^I), \mathbb{A}^I = \eta(\mathbb{B}^I)$$

and we can restrict our diagram to

$$\begin{tikzcd} & \mathbb{B}^I \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ \mathbb{A}^I & & \mathbb{M}^I \end{tikzcd}$$

5.3.2. Lemma. *There is a holomorphic bijection* $\varphi : \mathbb{M}^I \times \mathbb{P}_1 \times \mathbb{P}_1 \to \mathbb{B}^I$.

Proof:. Let $z_{ij} \in \mathbb{C}^{2\times 2}$ be coordinates for $\mathbb{M}^I$ as above and $[v_0, v_1], [w_0, w_1]$ homogeneous coordinates for the both copies of $\mathbb{P}_1$. We define $\varphi : \mathbb{M}^I \times \mathbb{P}_1 \times \mathbb{P}_1 \to \mathbb{B}$ by:

$$\varphi(z, [v], [w]) = (S_3^{z,[v],[w]}, S_2^{z,[v],[w]}, S_1^{z,[v],[w]})$$

where

$S_3^{z,[v],[w]}$ is the 3-space orthogonal to $(-z_{00}.w_0 - z_{01}w_1, -z_{10}w_0 - z_{11}w_1, w_0, w_1)$

$S_2^{z,[v],[w]} = \text{span}\{(1, 0, z_{00}, z_{10}), (0, 1, z_{01}, z_{11})\}$

$S_1^{z,[v],[w]} = \text{span}\{(v_0, v_1, z_{00}.v_0 + z_{01}.v_1, z_{10}.v_0 + z_{11}.v_1)\}$

This definition is correct and φ is holomorphic bijection between $\mathbb{M}^I \times \mathbb{P}_1 \times \mathbb{P}_1$ and $\mathbb{B}^I$. $\square$

So we have the "semi-homogeneous" coordinations $(z, [v], [w])$ on $\mathbb{B}$.

5.3.3. $\mathbb{A}$ can be viewed as a subset of $\mathbb{P}(T)\times\mathbb{P}(T)$ formed by such elements $([a],[b])$, that $< a,b >= 0$. We identify $([a],[b])$ with $(S_3^{a,b}, S_1^{a,b})$, where

$$S_3^{a,b} \text{ is the 3-space orthogonal to } b$$
$$S_1^{a,b} = \mathrm{span}\{a\}$$

In such a way we have homogeneous coordinates for $\mathbb{A}$.

5.3.4. We can describe η and τ in coordinates:

$$\eta(z,[v_0,v_1],[w_0,w_1]) = ((v_0, v_1, z_{00}.v_0 + z_{01}.v_1, z_{10}.v_0 + z_{11}.v_1), \\ (-z_{00}w_0 - z_{01}w_1, -z_{10}w_0 - z_{11}w_1, w_0, w_1))$$

$$\tau(z_{00}, z_{10}, z_{01}, z_{11}, [v_0,v_1],[w_0,w_1]) = (z_{00}, z_{10}, z_{01}, z_{11}).$$

5.3.5. Lemma. *The fibration $\eta : \mathbb{B}^I \to \mathbb{A}^I$ has topologically trivial fibres.*

Proof. We can calculate in coordinates. Let $(a,b) = ([a_1,a_2,a_3,a_4]],[b_1,b_2,b_3,b_4])$ be a point in $\mathbb{A}^I$. This means, that $(a_1,a_2) \neq (0,0)$ and $(b_3,b_4) \neq (0,0)$. The fibre of the point (a,b) it is $\eta^{-1}(a,b)\cap\mathbb{B}^I$ are exactly points $(z,[v],[w])$ such that:

$$\begin{aligned} v_0 &= a_1 \\ v_1 &= a_2 \\ z_{00}.v_0 + z_{01}.v_1 &= a_3 \\ z_{10}.v_0 + z_{11}.v_1 &= a_4 \\ -z_{00}.w_0 - z_{01}.w_1 &= b_1 \\ -z_{10}.w_0 - z_{11}.w_1 &= b_2 \\ w_0 &= b_3 \\ w_1 &= b_4 \end{aligned}$$

$[v],[w]$ is determined and we obtain following equations for z:

$$\begin{aligned} z_{00}.a_1 + z_{01}.a_2 &= a_3 \\ z_{10}.a_1 + z_{11}.a_2 &= a_4 \\ -z_{00}.b_3 - z_{01}.b_4 &= b_1 \\ -z_{10}.b_3 - z_{11}.b_4 &= b_2 \end{aligned}$$

We can solve each equation separately and see that all such z, form a linear set in $\mathbb{C}^4$ and this is topologically trivial set. $\square$

5.3.6. Remark. Let a subset $\mathbb{M}'$of $\mathbb{M}^I$ be convex in coordinates and define:

$$\mathbb{B}' = \tau^{-1}(\mathbb{M}'), \mathbb{A}' = \eta(\mathbb{B}'),$$

then $\eta : \mathbb{B}' \to \mathbb{A}'$ has fibres topologically trivial, because they are obtained as intersection of two convex sets.

5.3.7. Lemma. *Let E be a homogeneous vector bundle over* $\times\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\times$. *then cohomologies*

$$H^i(\mathbb{A}^I, \mathcal{O}(E)) = 0, \textit{for } i \geq 3$$

Proof:. $\mathbb{A}^I$ is covered by three Stein sets A,B,C in coordinates:

$$A = \{([t, 1, z_{00}.t + z_{01}, z_{10}.t + z_{11}], [-z_{00}s - z_{01}, -z_{10}s - z_{11}, s, 1]); t, s \in \mathbb{C}, z \in \mathbb{M}^I\}$$

$$B = \{([1, t, z_{00} + z_{01}.t, z_{10} + z_{11}.t], [-z_{00} - z_{01}s, -z_{10} - z_{11}s, 1, s])\}; t, s \in \mathbb{C}, z \in \mathbb{M}^I\}$$

$$C = \{([p, r, z_{00}p + z_{01}.r, z_{10}p + z_{11}r], [-z_{00}s - z_{01}t, -z_{10}s - z_{11}t, s, t])\};$$
$$p, r, t, s \in \mathbb{C}, r \neq p, s \neq t, z \in \mathbb{M}^I\}$$

and the rest of proof is analogous to the proof of 5.2.11. □

5.4. BGG-resolution and direct images.

In the section 3.11. it was shown how to classify all the irreducible representations of parabolic groups. Now we will apply this procedure to $G = SL(4, \mathbb{C})$ and compute BGG-resolution and direct images 3.13. and 3.14. for both double fibrations.

5.4.1. The irreducible finite dimensional representations of a parabolic subgroup of G are classified by three integer numbers written over the nodes of the Dynkin diagram. The integers over the uncrossed nodes must be positive, over the crossed nodes just integer.

For example: All the representations of $\times\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet$ are denoted by $\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}$ where a is an integer and b, c are positive integers (≥ 1). We will use the same notation for the corresponding associated bundle over G/P and in the spectral sequence even for the section of this bundle over a given set. It will always be clear from the context in which sense the notation is used.

If $(1), (2), (3)$ are three simple reflections generating the Weyl group of G, then we can compute the affine action of these elements on the weight $\overset{a}{\bullet}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet} \in \mathfrak{h}^*$ in the following way:

$$(1).\ \overset{a}{\bullet}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet} = \overset{-a}{\bullet}\!\!-\!\!\!-\!\!\overset{a+b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}$$
$$(2).\ \overset{a}{\bullet}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet} = \overset{a+b}{\bullet}\!\!-\!\!\!-\!\!\overset{-b}{\bullet}\!\!-\!\!\!-\!\!\overset{b+c}{\bullet}$$
$$(3).\ \overset{a}{\bullet}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet} = \overset{a}{\bullet}\!\!-\!\!\!-\!\!\overset{b+c}{\bullet}\!\!-\!\!\!-\!\!\overset{-c}{\bullet}$$

[Sl,10.20.]

5.4.2. Lemma (BGG relative to η). *For a,b,c integers $b, c \geqq 1$ there is the following exact sequence of sheaves on the space* $\times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet$ *:*

$$0 \to \eta^{-1}(\mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet})) \to \mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\times}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) \to \mathcal{O}(\overset{a+b}{\times}\!\!-\!\!\!-\!\!\overset{-b}{\times}\!\!-\!\!\!-\!\!\overset{b+c}{\bullet}) \to \mathcal{O}(\overset{a+b+c}{\times}\!\!-\!\!\!-\!\!\underset{-b-c}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}) \to 0$$

This sequence is the relative BGG resolution of the sheaf $\eta^{-1}(\mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}))$ *and is G-equivariant.*

Proof. The construction of relative BGG-resolution and of its relative version is described in 3.13. We will just apply the recipe for the computation of BGG for mapping

$$\eta : \times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet \;\to\; \times\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet$$

The relative Haas diagram of this mapping is computed in [BE] and is like this:

$$id \to (2) \to (23),$$

where the integer numbers denote simple reflections. We act on the weight $\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\times}\!\!-\!\!\!-\!\!\overset{c}{\bullet}$ by the elements of the relative Haas diagram and get the required result . $\square$

5.4.3. Lemma(BGG relative to η). *For a,b,c integers $b \geqq 1$ we have the following exact sequence of sheaves on the space* $\times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\times$ *:*

$$0 \to \eta^{-1}(\mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\times})) \to \mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\times}\!\!-\!\!\!-\!\!\overset{c}{\times}) \to \mathcal{O}(\overset{a+b}{\times}\!\!-\!\!\!-\!\!\overset{-b}{\times}\!\!-\!\!\!-\!\!\overset{b+c}{\times}) \to 0$$

This sequence is the relative BGG resolution of the sheaf $\eta^{-1}(\mathcal{O}(\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\times}))$ *and is G-equivariant.*

Proof.
The relative Hass diagram for mapping

$$\eta : \times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet \;\to\; \times\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet$$

is :

$$id \to (2)$$

We act on the weight $\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\times}\!\!-\!\!\!-\!\!\overset{c}{\times}$ by the elements of the relative Haas diagram and we get the required result . $\square$

5.4.4. Lemma. *For the mapping* $\tau : \times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet \;\to\; \bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet$ *, we can compute the direct images of sheaves of sections of homogeneous vector bundles as follows:*

(1)

$$\tau_*^0\mathcal{O}(\overset{k}{\times}\!\!-\!\!\!-\!\!\overset{l}{\times}\!\!-\!\!\!-\!\!\overset{m}{\bullet}) = \begin{cases} \mathcal{O}(\overset{k}{\bullet}\!\!-\!\!\!-\!\!\overset{l}{\times}\!\!-\!\!\!-\!\!\overset{m}{\bullet}), & \textit{for } k > 0 \\ 0 & \textit{for } k \leq 0 \end{cases}$$

(2)
$$\tau_*^1\mathcal{O}(\overset{k}{\times}\!-\!\overset{l}{\times}\!-\!\overset{m}{\bullet}) = \begin{cases} \mathcal{O}(\overset{-k}{\bullet}\!-\!\overset{k+l}{\times}\!-\!\overset{m}{\bullet}) & \textit{for } k<0 \\ 0 & \textit{for } k\geq 0 \end{cases}$$

(3) *all the higher direct images vanish*

Proof. We will apply the Bott-Borel-Weyl theorem following the recipe 3.14. First we must compute the relative Haas diagram for $\tau : \times\!-\!\times\!-\!\bullet \to \bullet\!-\!\times\!-\!\bullet$, which is $\tau : G/P_5 \to G/P_4$. For this we act on the weight $\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$ by the Weyl group $W_{\mathfrak{p}_4}$. $W_{\mathfrak{p}_4}$ is the Weyl group of the Levi reductive factor of algebra $\mathfrak{p}_4$ and is generated by reflection (1),(3). The orbit of $\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$ is $\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \xrightarrow{(1)} \overset{-1}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}$ and

$$W_{\mathfrak{p}_4}^{\mathfrak{p}_5} = \{id, (1)\}.$$

The action of the $W_{\mathfrak{p}_4}^{\mathfrak{p}_5}$ on bundle $\overset{k}{\bullet}\!-\!\overset{l}{\times}\!-\!\overset{m}{\bullet}$ gives the relative Haas diagram:

$$\{id, (3)\}$$

and using directly the point [4] from the recipe 3.14. we obtain the statement of the lemma. $\square$

5.4.5. Lemma. *For the mapping $\tau : \times\!-\!\times\!-\!\times \to \bullet\!-\!\times\!-\!\bullet$, we can compute the direct images of sections of homogeneous vector bundles as follows:*

(1)
$$\tau_*^0\mathcal{O}(\overset{k}{\times}\!-\!\overset{l}{\times}\!-\!\overset{m}{\times}) = \begin{cases} \mathcal{O}(\overset{k}{\bullet}\!-\!\overset{l}{\times}\!-\!\overset{m}{\bullet}) & \textit{for } k,m>0 \\ 0 & \textit{otherwise} \end{cases}$$

(2)
$$\tau_*^1\mathcal{O}(\overset{k}{\times}\!-\!\overset{l}{\times}\!-\!\overset{m}{\bullet}) = \begin{cases} \mathcal{O}(\overset{-k}{\bullet}\!-\!\overset{k+l}{\times}\!-\!\overset{m}{\bullet}) & \textit{for } k<0, m>0 \\ \mathcal{O}(\overset{k}{\bullet}\!-\!\overset{l+m}{\times}\!-\!\overset{-m}{\bullet}) & \textit{for } k>0, m<0 \\ 0 & \textit{otherwise} \end{cases}$$

(3)
$$\tau_*^2\mathcal{O}(\overset{k}{\times}\!-\!\overset{l}{\times}\!-\!\overset{m}{\times}) = \begin{cases} \mathcal{O}(\overset{-k}{\bullet}\!-\!\underset{k+l+m}{\times}\!-\!\overset{-m}{\bullet}) & \textit{for } k,m<0 \\ 0 & \textit{otherwise} \end{cases}$$

(4) *all the higher direct images vanish*

Proof. We will apply the Bott-Borel-Weyl theorem following the recipe 3.14. The relative Haas diagram is:

$$\{id, (1), (3), (13)\}$$

and using directly the point [4] from the recipe 3.10. we obtain the statement of the lemma. $\square$

5.4.6. We will need some topological conditions for identifications 5.4.4. In fact there are two identifications:

$$H^0(B', \overset{k}{\times}\!\!-\!\!-\!\!\overset{l}{\times}\!\!-\!\!-\!\!\overset{m}{\bullet}) \simeq \Gamma(C', \overset{k}{\bullet}\!\!-\!\!-\!\!\overset{l}{\times}\!\!-\!\!-\!\!\overset{m}{\bullet})$$

$$H^1(B', \overset{k}{\times}\!\!-\!\!-\!\!\overset{l}{\times}\!\!-\!\!-\!\!\overset{m}{\bullet}) \simeq \Gamma(C', \overset{-k}{\bullet}\!\!-\!\!-\!\!\underset{k+l}{\times}\!\!-\!\!-\!\!\overset{m}{\bullet}),$$

where C' is a Stein open subset of $\mathbb{M}^I$ and $B' = \tau^{-1}(C') = C' \times \mathbb{P}^1$. We denote $E := \overset{k}{\times}\!\!-\!\!-\!\!\overset{l}{\times}\!\!-\!\!-\!\!\overset{m}{\bullet}$ and $\mathbb{F}_0 := \overset{k}{\bullet}\!\!-\!\!-\!\!\overset{l}{\times}\!\!-\!\!-\!\!\overset{m}{\bullet}$, $\mathbb{F}_1 := \overset{-k}{\bullet}\!\!-\!\!-\!\!\underset{k+l}{\times}\!\!-\!\!-\!\!\overset{m}{\bullet}$ or 0 if the coefficients does not make sense.

Let us recall that

$$H^i(B', \overset{k}{\times}\!\!-\!\!-\!\!\overset{l}{\times}\!\!-\!\!-\!\!\overset{m}{\bullet}) = \text{ Ker } \bar{\partial}^i / \text{ Im } \bar{\partial}^{i-1}$$

and so we have mappings

$$P: \text{ Ker } \bar{\partial}^i \to \Gamma(C', \mathbb{F}_i).$$

There is a topology of uniform convergence on $\mathcal{E}^{0,i}(B', E)$ - see [H-L]. There are coordinates 5.2.8. on B'. The coordinates for $\mathbb{C} \simeq \{[v_0, v_1] \in \mathbb{P}^1; v_1 \neq 0\} \subset \mathbb{P}^1$ will be denoted by $v = v_0/v_1$. The forms on $\mathbb{P}^1$ are determined by their restrictions to $\mathbb{C}$.

There is a space of harmonic forms

$$\mathcal{H}^i(\mathbb{P}_1, E) \subset \mathcal{E}^{0,i}(\mathbb{P}^1, E)$$

(E restricted to $\mathbb{P}^1$) see [We, chap.5, ex.5.5.] These forms are $\bar{\partial}$-closed and

$$\mathcal{H}^i(\mathbb{P}_1, E) \simeq H^i(\mathbb{P}^1, \mathcal{O}(E)),$$

for $i = 0$ it is:

$$\mathcal{H}^0(\mathbb{P}^1, E) = \text{ Ker } \bar{\partial}^0.$$

Let us denote by $\mathcal{H}^1(B', E)$ the subspace of $\mathcal{E}^{0,1}(B', E)$ of forms, restriction of which to $C' \times \mathbb{C}$ can be written as

$$\omega_h = f(z, v) d\bar{v},$$

where $f \in \Gamma(B', E)$, and $f(z, v) d\bar{v} \in \mathcal{H}^i(\mathbb{P}_1, E)$ for each $z \in C'$ fixed. The form ω_h is $\bar{\partial}$-closed, hence $f(z, v)$ is holomorphic for each $v \in \mathbb{C}$ fixed.

Lemma. *Let E be a homogeneous vector bundle on* $\times\!-\!\!-\!\times\!-\!\!-\!\bullet$ *then,*

(1)
$$\mathbb{F}_i \simeq \mathcal{H}^i(\mathbb{P}_1, E).$$

(2) *If $\omega \in$ Ker $\bar\partial^1$, then there is unique form $\omega_h \in \mathcal{H}^1(B', E)$ such that*
$$[\omega_h] = [\omega].$$

(3) *The mapping*
$$P: \ Ker\ \bar\partial^0 \to \Gamma(C', \mathcal{H}^0(\mathbb{P}^1, E))$$
is given in the following way: If $\omega \in$ Ker $\bar\partial^0$ then $\bar\partial_z\omega = \bar\partial_v\omega = 0$ and so there is a unique $s \in \Gamma(C', \mathcal{H}^0(\mathbb{P}^1, E))$ such that $\omega(z,v) = (s(z))(v)$,then $P(\omega) = s$.

(4) *The mapping*
$$P: \ Ker\ \bar\partial^1 \to \Gamma(C', \mathcal{H}^1(\mathbb{P}^1, E))$$
is given in the following way: If $\omega \in$ Ker $\bar\partial^1$ then by (2) there is a unique ω_h and a unique $s \in \Gamma(C', \mathcal{H}^i(\mathbb{P}_1, E))$ such that $\omega_h(z,v) = (s(z))(v)$, then $P(\omega) = s$.

(5) *P are continuous in the topology of uniform convergence.*

Proof. The identification 5.4.4. is based on the Leray spectral sequence 2.7. In our case we can see explicitly how this sequence is constructed and how it induces the identification. See [GH,3.5.] for more details.

Let us define double complex:
$$K^{p,q} := \{\omega_1 = \sum_{\substack{|I|=q \\ |J|=p}} F_I(z,v)d\bar v_I \wedge d\bar z_J; F_I \in \mathcal{C}^\infty(B', E)\},$$
the horizontal operator is $\bar\partial_z$ and vertical one is $\bar\partial_v$. The associated complex 2.3. is exactly the Dolbeault resolution:
$$K^n = \bigoplus_{p+q=n} K^{p,q} = \mathcal{E}^{0,n}(B', E).$$
By 2.5. there is a spectral sequence E converging to total cohomology whose first term is:
$$E_1^{p,q} = \text{ Ker } \bar\partial_v / \text{ Im } \bar\partial_v = \{\omega_2 = \sum_{|J|=p} G_J d\bar z_J; G_J \in C^\infty(C', H^q(\mathbb{P}^1, \mathcal{O}(E))\}.$$

The space $H^q(\mathbb{P}^1, \mathcal{O}(E))$ is a finite dimensional vector space. In such a way,

$$E_1^{p,q} = \mathcal{E}^{0,p}(C', H^q(\mathbb{P}^1, \mathcal{O}(E)))$$

and $d_1 = \bar{\partial}_z$. It follows, that

$$E_2^{p,q} = H^p(C', \mathcal{O}(H^q(\mathbb{P}^1, \mathcal{O}(E)))) = \begin{cases} \Gamma(C', H^q(\mathbb{P}^1, \mathcal{O}(E))) & \text{for } p = 0 \\ 0 & \text{for } p \geq 1 \end{cases}$$

Evidently $E_\infty = E_2$ and because E converges to the cohomology of K^n, we obtain identities:

$$H^q(B', E) = \Gamma(C', H^q(\mathbb{P}^1, \mathcal{O}(E))). \tag{*}$$

Now let us prove the lemma:

(1) The (*) was obtained by the same procedure as the results 5.4.4. (construction in this proof is only an explicitation) and so

$$\mathbb{F}_q = H^q(\mathbb{P}^1, \mathcal{O}(E)) = \mathcal{H}^q(\mathbb{P}^1, E)$$

If $\omega \in$ Ker $\bar{\partial}_1 = K^{0,1} \oplus K^{1,0}$, then

$$P(\omega) = [[\omega]] \in E_2^{0,1} \oplus E_2^{1,0}.$$

It is clear, that P restricted to $\mathcal{H}^1(B', E)$ is a bijection. From this follows (2) and (4).

(3) is evident.

For to prove (4), we can use the fact

$$\text{Ker } \bar{\partial}^i = \text{ Im } \bar{\partial}^{i-1} \oplus \mathcal{H}^i(B', E).$$

and see that P is simply projection to $\mathcal{H}^i(B', E)$ and it is continuous. $\square$

5.5. Operators on the Minkowski space.

Now we will give the classification of all local G-equivariant operators between homogeneous bundles over $\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet$. (See 2.12. for definition)

5.5.1. Lemma. *Let E_λ, F_ρ be two irreducible homogeneous bundles over the space $\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet$. There is one-to-one correspondence between differential operators from E_λ to F_ρ and P_4-equivariant homomorphisms $V : J^k_x E_\lambda \to F_{\rho,x}$ where $k \in \mathbb{N}$ and $x = [e] = 0$ in coordinates 5.2.2.*

Proof. By remark 2.15., a differential operator is given by a homomorphism of bundles, transforming holomorphic sections to holomorphic sections:

$$L : J^k E_\lambda \to F_\rho.$$

We define mapping Z by restriction:

$$Z : \mathrm{Hom}(J^k E_\lambda, F_\rho)^G \to \mathrm{Hom}(J^k_x E_\lambda, F_{\rho,x})^P$$

$$Z(L) = L_x$$

(1) Z is injective. Let $L_x = L'_x$, $g \in G$ and $A \in J^k_{[g]}$. Then $L(A) = gL(g^{-1}A) = gL_x(g^{-1}A) = gL'_x(g^{-1}A) = gL'(g^{-1}A) = L'(A)$ and so $L = L'$.

(2) Z is surjective. Bundle $J^k E_\lambda$ is a homogeneous bundle associated to some representation $\mathbb{V}_\omega$. Let $C : J^k_x E_\lambda \to F_{\rho,x}$ be P-equivariant.

Define $L \in \mathrm{Hom}(J^k E_\lambda, F_\rho)^G$ by:

$$L([g,j]) = gC([e,j]) \text{ for } g \in G, j \in \mathbb{V}_\omega.$$

a) The map L is well-defined: $L([gp, \lambda(p^{-1})j]) = gpC([e, \lambda(p^{-1})j] = gC([p, \lambda(p^{-1})j]) = gC([e,j]) = L([g,j])$. See 3.2. for definition of associated vector bundle and action of G on it.

b) L is evidently linear and G-equivariant.

c) L transform holomorphic sections of $J^k E_\lambda$ to holomorphic sections of F_ρ. For this we will use identification 3.6. Let $s : G^U \to \mathbb{V}_\omega$ be a holomorphic section. Define a mapping $C' : \mathbb{V}_\omega \to \mathbb{F}_\rho$ by setting for $C'(j)$ the only $f \in \mathbb{F}_\rho$ that $C[e,j] = [e,f]$. We claim that $L(s) = C' \circ s : G^U \to \mathbb{F}_\rho$. Really $L([g, s(g)]) = gC[e, s(g)] = g[e, C's(g)] = [g, C's(g)]$ and so $L(s)(g) = C'(s(g))$. But s is holomorphic, C' is linear and so $L(s)$ is holomorphic.

d) $L_x = C$ evidently.

□

5.5.2. Lemma. *Let $P = P_4$, let $G', P' = P'_4$ be as in 5.2.4. We denote $M = G/P = \bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet$ and $M' = G'/P'$. Let $\mathbb{E}_\lambda$ be a representation space for P and by restriction for P' as well. There are associated complex vector bundles E_λ over M and E'_λ over M'. Then as P' modules:*

$$J^k_x E_\lambda \simeq J^k_x E'_\lambda$$

where $x = [e]$. See 2.13. and 2.14. for definition.

Proof. We have

$$\pi_k : \mathcal{O}_x(E_\lambda) \to J^k_x E_\lambda$$
$$\pi'_k : \mathcal{E}_x(E_\lambda) \to J^k_x E'_\lambda$$

In $\mathcal{E}_x(E_\lambda)$ there is a submodule $\mathcal{A}_x(E_\lambda)$ of all analytic germs in x. It is really a submodule because the action of G' on M' is analytic. The image $\pi'_k(\mathcal{A}_x(E_\lambda))$ is evidently all $J^k_x E_\lambda$. If we consider the coordinates z on M and x on M' from 5.2.2. and 5.2.4., we have:

$$\mathcal{O}_x(E_\lambda) \simeq S(z) \otimes_{\mathbb{C}} \mathbb{E}_\lambda$$

$$\mathcal{A}_x(E_\lambda) \simeq S'(x) \otimes_{\mathbb{R}} \mathbb{E}_\lambda$$

where

$$S(z) = \{\sum_I \alpha_I z^I; \alpha_I \in \mathbb{C}; \text{nonzero radius of convergence}\}$$

$$S'(x) = \{\sum_I \beta_I x^I; \beta_I \in \mathbb{R}; \text{nonzero radius of convergence}\}$$

The action of P' on $\mathbb{E}_\lambda$ is given by λ and on $S(z)$ and $S'(x)$ by transformation of coordinates 5.2.4. Let us denote by $S(x)$ the module $S'(x) \otimes_{\mathbb{R}} \mathbb{C}$. Evidently $S(x) = \{\sum_I \gamma_I x^I; \gamma_I \in \mathbb{C}; \text{non zero radius of convergence}\}$ and $\mathcal{A}_x(E_\lambda) = S(x) \otimes_{\mathbb{C}} \mathbb{E}_\lambda$.

We define

$$F : S(x) \to S(z)$$

$$\sum_I \gamma_I x^I \to \sum_I \gamma_I z^I.$$

This is an isomorphism of P' modules because P' transform z in the same way as x. Thus we have an isomorphism of P' modules:

$$F \otimes Id : \mathcal{A}_x(E_\lambda) \to \mathcal{O}_x(E_\lambda).$$

This isomorphism evidently induces an isomorphism

$$F : J^k_x E'_\lambda \to J^k_x E_\lambda.$$

□

5.5.3. Theorem. *Let E_λ, F_ρ be two homogeneous irreducible bundles over the space* $\bullet\!\!-\!\!\!\times\!\!\!-\!\!\bullet$ *corresponding to representations λ, ρ. Then each G-equivariant local operator from E_λ to F_ρ is differential.*

Proof. Let L be such an operator. Then L_x is a P-equivariant mapping from $\mathcal{O}_x(E_\lambda)$ into $\mathcal{O}_x(F_\rho)$. We compose it with evaluation of germ in x and this mapping must thus be O-equivariant. By 5.2.3. ,the element

$$o_2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix} \in O$$

acts by multiplication by 0.25 on coordinates from 5.2.5. and so by multiplication by $4k$ on homogeneous polynomials of degree k in $S(z)$. It must also act by multiplication by some number a_λ, a_ρ on $\mathbb{E}_\lambda, \mathbb{F}_\rho$. So if p_k is a homogeneous polynomial in z of degree k we have by equivariance:

$$4ka_\lambda L(p_k \otimes h) = L(o_2(p_k \otimes h)) = o_2 L(p_k \otimes h) = a_\rho L(p_k \otimes h)$$

and so either $4ka_\lambda = a_\rho$ or $L(p_k \otimes h) = 0$. The second equation can hold for only one $k = k_0$. If f is defined in a neighbourhood U of 0, and

$$f = \sum_0^\infty P_n$$

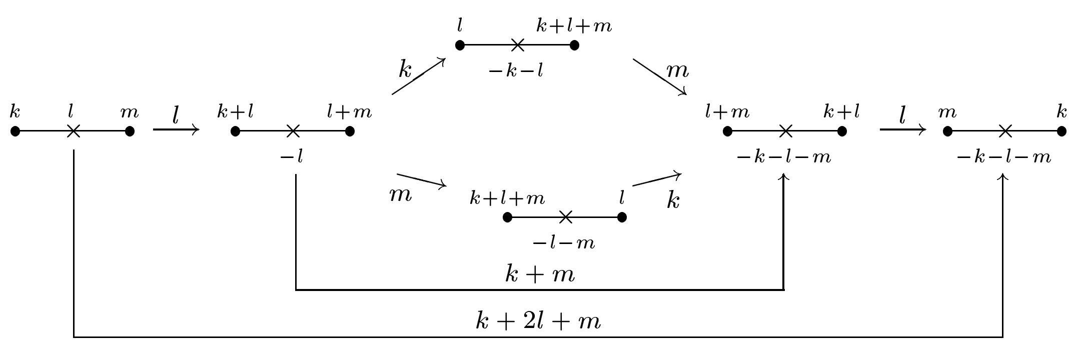
is the Taylor serie for f in U, then $S_m = \sum_0^m P_n \rightrightarrows f$ in U with all derivatives. Hence $L(S_m) = L(P_{k_0})$ for $m \geq k_0$ and by continuity of a local operator

$$L(f) = L(P_{k_0}).$$

So it follows, that L_x depends only on k-th derivatives. By equivariance we have the same for $L_{[g]}$ generally and so L factorizes through $\pi_k : \mathcal{O}(E_\lambda) \to \mathcal{O}(F_\rho)$ and is a differential operator. $\square$

5.5.4. Theorem (list of operators). *Let E_λ and F_ρ be two irreducible homogeneous vector bundles over* $\bullet\!-\!\times\!-\!\bullet$ *. We consider the vector space D^{E_λ, F_ρ} of all G -equivariant local operators from E_λ to F_ρ.*

The dimension of D^{E_λ, F_ρ} is 0 (there is only a zero operator) or 1. The dimension is one if and only if there exist three integers $k, l, m \geq 0$ so that E_λ and F_ρ occur in the following diagram and they are joined by some arrow:



In this case D^{E_λ, F_ρ} is generated by a differential operator d^{E_λ, F_ρ}. The order of this operator is given by the number written over the arrow in the diagram.

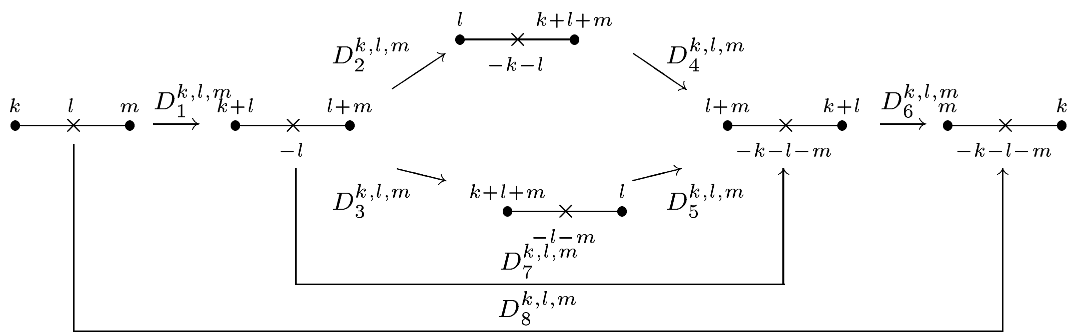
Proof. By 5.5.3. every local equivariant operator must be differential. By 5.5.1. we must classify all the homomorphisms of P-modules $J^k_x E_\lambda$ and $F_{\rho,x}$. Thus it must be a homomorphism of P'-modules as well. By 5.5.2. $J^k_x E_\lambda = J^k_x E'_\lambda$ as P'-modules.

All homomorphisms of P'-modules $J^k_x E'_\lambda \to F_{\rho,x}$ are classified in [Sl,8] and by [Sl, 2.6.] (analogy of 5.5.1. for real homogeneous space) correspond to G'-equivariant operators from E'_λ to F'_ρ. In addition the obtained homomorphisms of P' modules $J^k_x E_\lambda$ and $F_{\rho,x}$ must be homomorphisms of P-modules as well, because they are $\mathbb{C}$-linear and $P = P' \otimes_{\mathbb{R}} \mathbb{C}$.

It follows that differential G-invariant operators between E_λ and F_ρ are in one-to-one correspondence with G' operators from E'_λ to F'_ρ on M'. The latter are well classified in [Sl,8.13 and 8.14.] and (DIG) is the same as the diagram in [Sl, 8.14.]. $\square$

5.5.5. Remark. The coefficients in the diagram above have a property that each homogeneous bundle appears for just one choice of k,l,m.

5.5.6. Notation. If there are several operators between two bundles, then they differ only by a multiplication. Because we will be interested by Ker and Coker of these operators, we will choose one of them and denote it in dependance on coefficients k, l, m and on its position in diagram as follows:



(DIG)

We omit the sheaves and write just bundles.

So for example for $k = 3, l = 2, c = 4$ we have

$$D_3^{2,3,4} : \mathcal{O}(\overset{2}{\bullet}\!-\!\overset{-5}{\times}\!-\!\overset{9}{\bullet}) \to \mathcal{O}(\overset{6}{\bullet}\!-\!\overset{-9}{\times}\!-\!\overset{5}{\bullet})$$

and the order of this operator is 4. We say that operator $D_k^{a,b,c}$ is an operator of type $[k]$. Operators from diagram (DIG) are G-equivariant and if we disregard bottom arrows, we get an exact sequence. Exactly:

$$\begin{aligned} \text{Im } D_1 &= \text{Ker } (D_2 \oplus D_3) \\ \text{Im } (D_2 \oplus D_3) &= \text{Ker } (D_4 \oplus D_5) \\ \text{Im } (D_4 \oplus D_5) &= \text{Ker } D_6 \\ D_6 &\text{ is surjective.} \end{aligned}$$

These conditions follows from the fact, that (DIG) is in fact the BGG resolution. [B-E]

Now we will give several properties of those diagrams (DIG).

5.5.7. Lemma. *There is an isomorphism between Ker* $D_3^{k,l,m}/Im\ D_1^{k,l,m}$ *and Ker* $D_4^{k,l,m}$ *for all allowed coefficients* k, l, m. *This isomorphism is induced by operator* $D_2^{k,l,m}$ *and is* G *equivariant.*

Proof. We define mapping:

$$I^{k,l,m} : \text{Ker } D_3^{k,l,m}/Im\ D_1^{k,l,m} \to \text{Ker } D_4^{k,l,m}$$

$$I^{k,l,m}([s]) = D_2^{k,l,m}(s)$$

for $s \in \text{Ker } D_3^{k,l,m}$.

(1) This definition is correct, because if $[s] = [t]$, then $s = t + D_1^{k,l,m}(u)$ and $I^{k,l,m}([s]) = D_2^{k,l,m}(t + D_1^{k,l,m}(u)) = D_2^{k,l,m}(t) = I^{k,l,m}([t])$.

(2) The image of $I^{k,l,m}$ is just Ker $D_4^{k,l,m}$. For $s \in \text{Ker } D_3^{k,l,m}$ is $D_5^{k,l,m} D_3^{k,l,m} = 0 = D_4^{k,l,m} D_2^{k,l,m}$ and so $Im\ I^{k,l,m} \subseteqq \text{Ker } D_4^{k,l,m}$. On the other hand for $v \in \text{Ker } D_4^{k,l,m}$: $v \oplus 0 \in$ Ker $[D_4^{k,l,m} \oplus D_5^{k,l,m}]$ and from the exactness of (DIG) it follows that there exists $s \in \overset{k+l}{\bullet}\!-\!\underset{-l}{\times}\!-\!\overset{l+m}{\bullet}$ such that $s \in \text{Ker } D_3^{k,l,m}$ and $D_2^{k,l,m}(s) = v$.

(3) $I^{k,l,m}$ is injective. Let $D_2^{k,l,m}(s) = D_3^{k,l,m}(s) = 0$, then $s \in \text{Im } D_1^{k,l,m}$ and $[s] = 0$.

(4) The linearity and $G-$ equivariance of $I^{k,l,m}$ follows from the linearity and $G-$ equivariance of the $D_2^{k,l,m}$ $\square$

5.5.8. Lemma. *The sequences of sheaves*

$$\mathcal{O}(\overset{l}{\bullet}\!-\!\underset{-k-l}{\times}\!-\!\overset{k+l+m}{\bullet}) \xrightarrow{D_4^{k,l,m}} \mathcal{O}(\overset{l+m}{\bullet}\!-\!\underset{-k-l-m}{\times}\!-\!\overset{k+l}{\bullet}) \xrightarrow{D_6^{k,l,m}} \mathcal{O}(\overset{m}{\bullet}\!-\!\underset{-k-l-m}{\times}\!-\!\overset{k}{\bullet}) \to 0 \qquad (1)$$

$$\mathcal{O}(\overset{k+l+m}{\bullet}\!-\!\underset{-l-m}{\times}\!-\!\overset{l}{\bullet}) \xrightarrow{D_5^{k,l,m}} \mathcal{O}(\overset{l+m}{\bullet}\!-\!\underset{-k-l-m}{\times}\!-\!\overset{k+l}{\bullet}) \xrightarrow{D_6^{k,l,m}} \mathcal{O}(\overset{m}{\bullet}\!-\!\underset{-k-l-m}{\times}\!-\!\overset{k}{\bullet}) \to 0 \qquad (2)$$

from [DIG] are exact.

Proof. From the proof of the theorem 6.1. it follows (without using this lemma) that the operator $D_2^{k,l,m}$ is surjective on section over $\mathbb{M}^I$. We will prove the surjectivity of this operator on sheaves. Let (f) be a germ in $0 \in \mathbb{M}^I$, f local section of $\overset{k+l}{\bullet}\!-\!\underset{-l}{\times}\!-\!\overset{l+m}{\bullet}$. Let f be defined in polydisk $B_r(0)$. We can consider the mapping:

$$D_2 : C_k^B(B_{r_1}, \overset{k+l}{\bullet}\!-\!\underset{-l}{\times}\!-\!\overset{l+m}{\bullet}) \to C_0^B(B_{r_1}, \overset{l}{\bullet}\!-\!\underset{-k-l}{\times}\!-\!\overset{k+l+m}{\bullet})$$

where $C_k^B(B_{r_1}, \overset{k+l}{\bullet}\!-\!\underset{-l}{\times}\!-\!\overset{l+m}{\bullet})$ denotes the space of all sections over B_{r_1} bounded on B_{r_1} together their derivatives to order k.

There is a standard norm $\| \, . \, \|_k$ on $C_k^B(B_{r_1}, \overset{k+l}{\bullet}\!\!-\!\!\underset{-l}{\times}\!\!-\!\!\overset{l+m}{\bullet})$ such that we get the Banach space. By the Banach theorem about open mapping we see, that $\mathrm{Im}\ D_2$ is a closed subspace of $C_0^B(B_{r_1}, \overset{l}{\bullet}\!\!-\!\!\underset{-k-l}{\times}\!\!-\!\!\overset{k+l+m}{\bullet})$. From the global surjectivity of D_2 follows, that all the polynomes in coordinates 5.2.2. are in $\mathrm{Im}\ D_2$. If we consider the Taylor serie for f with centre in 0, then this serie converges uniformly to f on B_{r_1} and so

$$f \in \ \mathrm{Im}\ D_2$$

and $D_2^{k,l,m}$ is surjective.

Similarly the operator $D_3^{k,l,m}$ is surjective, too. Because $D_4^{k,l,m} \circ D_2^{k,l,m} = D_5^{k,l,m} \circ D_3^{k,l,m}$ we have:

$$Im\ D_4^{k,l,m} = Im\ D_5^{k,l,m} = Im\ [D_4^{k,l,m} \oplus D_5^{k,l,m}]$$

and the exactness of (1) and (2) follows from the exactness of (DIG). $\square$

Chapter 6 - Results

6.1. Theorem. *Consider the diagram from 5.2.*

$$\begin{tikzcd} & \times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ \times\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet & & \bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet \end{tikzcd}$$

Let C' be a convex subset of $\mathbb{M}^I$, $B' = \tau^{-1}(C')$ and $A' = \eta(B')$. The Penrose transform gives the following isomorphisms and exact sequences of $\mathfrak{g}$-modules. Operators are considered on section of bundles over C'.

For the zero cohomology on $\times\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet$:

$\boxed{a \geq 1}$

$$H^0(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) \simeq \text{Ker } D_1^{a,b,c}$$

This gives kernels of all operators of type [1] from (DIG)

$\boxed{a < 1}$

$$H^0(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) = 0$$

For the first cohomology on $\times\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet$:

$\boxed{a \geq 1}$

$$H^1(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) \simeq \text{Ker } D_4^{a,b,c}$$

This gives, together with the case $a = 0$, kernels of all operators of type [4] from (DIG)

$\boxed{a = 0}$

$$H^1(A', \overset{0}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) \simeq \text{Ker } D_3^{0,b,c} = D_4^{0,b,c}$$

$\boxed{-b < a < 0}$

$$0 \to \text{Ker } D_4^{-a,a+b,c} \to H^1(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) \to \text{Ker } D_8^{-a,a+b,c} \to 0$$

This gives, together with the case $a = -b$, kernels of all operators of type [8] from (DIG)

$\boxed{a = -b}$

$$H^1(A', \overset{-b}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) \simeq \text{Ker } D_8^{b,0,c} = D_7^{b,0,c}$$

$\boxed{-b-c<a<-b}$

$$0 \to H^1(A',\ \overset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!-\!\!\overset{c}{\bullet}\,) \to \mathrm{Ker}\ D_5^{b,c,-a-b-c} \xrightarrow{d_2^{0,1}} \overset{-a-b-c}{\bullet}\!\!-\!\!-\!\!\underset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet} \to 0$$

$\boxed{a=-b-c}$

$$H^1(A',\ \overset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!-\!\!\overset{c}{\bullet}\,) \simeq \mathrm{Ker}\ D_2^{b,c,0} = D_5^{b,c,0}$$

$\boxed{a<-b-c}$

$$H^1(A',\ \overset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!-\!\!\overset{c}{\bullet}\,) \simeq \mathrm{Ker}\ D_5^{b,c,-a-b-c}$$

This gives, together with the case $a=-b-c$, kernels of all operators of type [5] from (DIG)

Proof. We just apply the principle from chapter 4. By 5.2.10. our C' satisfies conditions from 4.1. and 4.4. and so we have the spectral sequence

$$E_1^{pq} = \Gamma(C', \tau_*^q \Delta^p(\,\overset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!-\!\!\overset{c}{\bullet}\,)) \tag{SEQ}$$

and

$$E^{pq} \Rightarrow H^r(B', \eta^{-1}\mathcal{O}(\,\overset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!-\!\!\overset{c}{\bullet}\,)) = H^r(A', \mathcal{O}(\,\overset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!-\!\!\overset{c}{\bullet}\,)). \tag{SUM}$$

The bundles $\Delta^p(\,\overset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!-\!\!\overset{c}{\bullet}\,))$ are described in 5.4.2. and recipe for direct images τ_* in 5.4.4.

Now we will compute (SEQ) for different cases. The operators between sections of sheaves in spectral sequences will be identified up to a multiple by a non zero complex number with some of operators from (DIG). For this we need proof that these operators are differential and G-equivariant. This fact is precised and proved in 6.3.

<u>case $a>1$</u> We can using 5.4.2. and 5.4.4. rewrite the first term of the spectral sequence (SEQ) as follows (We omit to write section over C' and write just the sheaves):

$$E_1^{p,q} = \left|\begin{array}{ccccc} \cdots & & \cdots & & \cdots \\ 0 & & 0 & & 0 \\ 0 & & 0 & & 0 \\ \overset{a}{\bullet}\!\!-\!\!-\!\!\overset{b}{\times}\!\!-\!\!-\!\!\overset{c}{\bullet} & \xrightarrow{d_1^{0,0}} & \overset{a+b}{\bullet}\!\!-\!\!-\!\!\overset{-b}{\times}\!\!-\!\!-\!\!\overset{b+c}{\bullet} & \xrightarrow{d_1^{1,0}} & \overset{a+b+c}{\bullet}\!\!-\!\!-\!\!\underset{-b-c}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet} \\ \hline \end{array}\right.$$

The maps $d_1^{0,0}$and $d_1^{0,1}$ are both $\mathfrak{g}$-equivariant local operators by lemma 6.3. They are non-trivial, because they coincide with operators in BGG. We can identify them using theorem 5.5.3 with operators from (DIG). Up to multiple by a non zero complex number it holds:

$$d_1^{0,0} = D_1^{a,b,c}$$
$$d_1^{0,1} = D_3^{a,b,c}$$

Now the second member of the spectral sequence is :

$$E_2^{p,q} = E_\infty^{p,q} = \begin{array}{|ccc}
\dots & \dots & \dots \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\text{Ker } D_1^{a,b,c} & \text{Ker } D_3^{a,b,c} / \text{ Im } D_1^{a,b,c} & \text{Coker } D_3^{a,b,c} \\
\hline
\end{array}$$

$$d_2 = 0$$

So we have by (SUM) the following identifications of $\mathfrak{g}$ modules:

$$\text{Ker } D_1^{a,b,c} \simeq H^0(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) \tag{1}$$

$$\text{Ker } D_3^{a,b,c} / \text{ Im } D_1^{a,b,c} \simeq H^1(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) \tag{2}$$

$$\text{Coker } D_3^{a,b,c} \simeq H^2(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) \tag{3}$$

The (2) is equal to Ker $D_4^{a,b,c}$ by 5.5.7. and the (3) is equal to 0 for the case $A = \mathbb{M}^I$ by 5.2.11. So $D_3^{a,b,c}$ is surjective on global sections and by 5.5.8. for all trivial A', too. (1) and (2) gives us results of theorem for this case.

We can see, that we obtain the kernels of all the operators of type [1] and [4] from (DIG):

The operator $D_1^{a,b,c}$ is well defined for $a, c > 0, b \geq 0$ but for $b = 0$ has trivial kernel. Our case gives all the solutions of $D_1^{a,b,c}$ for $a, b, c > 0$.

The operator $D_4^{a,b,c}$ is well defined for $a, c \geq 0, b > 0$ but for $c = 0$ has trivial kernel and so no solution. Our case exhausts all $a, b, c > 0$ and the operator with $a = 0$ will be obtained in the following case.

In all the remaining cases $E_1^{0,0} = 0$ and so $H^0(A, \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}) = 0$ for $a < 0$. Computation will be very similar and so we will be brief.

<u>case $a = 0$</u> The first term of the spectral sequence (SEQ) is:

$$E_1^{p,q} = \begin{array}{|cccc}
\dots & \dots & & \dots \\
0 & 0 & & 0 \\
0 & 0 & & 0 \\
0 & \overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{-b}{\times}\!\!-\!\!\!-\!\!\overset{b+c}{\bullet} & \xrightarrow{d_1^{1,0}} & \overset{b+c}{\bullet}\!\!-\!\!\!-\!\!\underset{-b-c}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet} \\
\hline
\end{array}$$

$$d_1^{1,0} = D_3^{0,b,c} = D_4^{0,b,c}$$

The second member of the spectral sequence is :

$$E_2^{p,q} = E_\infty^{p,q} = \left| \begin{array}{ccc} \dots & \dots & \dots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \text{Ker } D_3^{0,b,c} & \text{Coker } D_3^{0,b,c} \\ \hline \end{array} \right.$$

$$d_2 = 0$$

And we obtain the results for this case. The $D_3^{0,b,c}$ is surjective and it follows, that operators from the position [3] are all surjective.

<u>case $-b < a < 0$</u>
The first term of the spectral sequence (SEQ) is:

$$E_1^{p,q} = \left| \begin{array}{cccc} \dots & \dots & & \dots \\ 0 & 0 & & 0 \\ \overset{-a}{\bullet}\!-\!\overset{a+b}{\times}\!-\!\overset{c}{\bullet} & 0 & & 0 \\ 0 & \overset{a+b}{\bullet}\!-\!\overset{-b}{\times}\!-\!\overset{b+c}{\bullet} & \xrightarrow{d_1^{1,0}} & \overset{a+b+c}{\bullet}\!-\!\underset{-b-c}{\times}\!-\!\overset{b}{\bullet} \\ \hline \end{array} \right.$$

$$d_1^{0,1} = D_4^{-a,a+b,c}$$

From the lemma 5.5.7. it follows that Coker $D_4^{-a,a+b,c} \simeq \overset{c}{\bullet}\!-\!\underset{-b-c}{\times}\!-\!\overset{-a}{\bullet}$. The second member of the spectral sequence is so:

$$E_2^{p,q} = \left| \begin{array}{ccc} \dots & \dots & \dots \\ 0 & 0 & 0 \\ \overset{-a}{\bullet}\!-\!\overset{a+b}{\times}\!-\!\overset{c}{\bullet} & 0 & 0 \\ 0 & \text{Ker } D_4^{-a,a+b,c} & \overset{c}{\bullet}\!-\!\underset{-b-c}{\times}\!-\!\overset{-a}{\bullet} \\ \hline \end{array} \right.$$

$$d_2^{0,1} : \overset{-a}{\bullet}\!-\!\overset{a+b}{\times}\!-\!\overset{c}{\bullet} \to \overset{c}{\bullet}\!-\!\underset{-b-c}{\times}\!-\!\overset{-a}{\bullet}$$

Because $H^2(\mathbb{P}^I, \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}) = 0$, $d_2^{0,1}$ must be surjective on sections over $\mathbb{M}^I$ and so $d_2^{0,1} \neq 0$. It follows that

$$d_2^{0,1} = D_8^{-a,b+a,c}.$$

The third member of spectral sequence is:

$$E_3^{p,q} = E_\infty^{p,q} = \left|\begin{array}{ccc} \dots & \dots & \dots \\ 0 & 0 & 0 \\ \text{Ker } D_8^{-a,b+a,c} & 0 & 0 \\ 0 & \text{Ker } D_4^{-a,a+b,c} & 0 \\ \hline \end{array}\right.$$

Considered as vector spaces it holds:

$$H^1(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}\,) \simeq \text{Ker } D_8^{-a,a+b,c} \oplus \text{Ker } D_4^{-a,a+b,c}$$

From the construction of spectral sequence follows that considered as $\mathfrak{g}-$modules we have only the sequence [B-T,14.13.]:

$$0 \to E_\infty^{1,0} \to E_\infty^{1,0} \oplus E_\infty^{0,1} \to E_\infty^{0,1} \to 0$$

and this gives exactly the exact sequence from 6.1.

We ask which operators of type [8] we obtain. It is, for which $k, m > 0, l \geq 0$ the system of equations:

$$\begin{aligned} k &= -a \\ l &= a + b \\ m &= c \end{aligned}$$

has solution with $c > 0, b > 0, 0 > a > -b$ It has such solution except the case $l = 0$ which will be obtained in the following case.

<u>case $a = -b$</u>

$$E_1^{p,q} = E_2^{p,q} = \left|\begin{array}{ccc} \dots & \dots & \dots \\ 0 & 0 & 0 \\ \overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\times}\!\!-\!\!\!-\!\!\overset{c}{\bullet} & 0 & 0 \\ 0 & 0 & \overset{b+c}{\bullet}\!\!-\!\!\!-\!\!\underset{-b-c}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet} \\ \hline \end{array}\right.$$

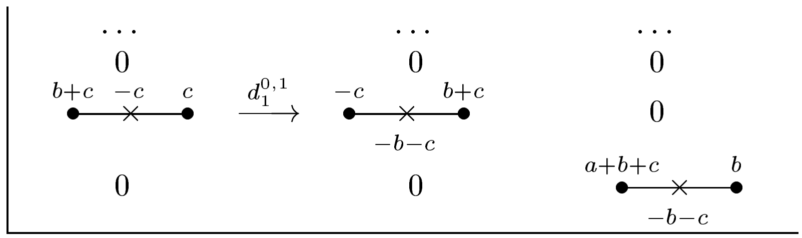
$d_2^{0,1}$ must be surjective. We identify:

$$d_2^{0,1} = D_7^{b,0,c} = D_8^{b,0,c}$$

$$E_3^{p,q} = E_\infty^{p,q} = \left|\begin{array}{ccc} \dots & \dots & \dots \\ 0 & 0 & 0 \\ \text{Ker } D_8^{b,0,c} & 0 & 0 \\ 0 & 0 & 0 \\ \hline \end{array}\right.$$

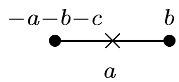
and all is clear.

<u>case $-b-c<a<-b$</u>

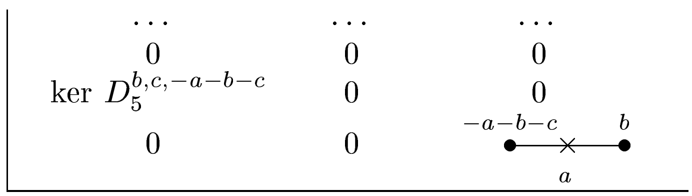
$E_1^{p,q} =$ 

The map $d_1^{0,1}$ must be surjective and so non zero, hence

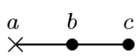
$$d_1^{0,1} = D_5^{b,c,-a-b-c}$$

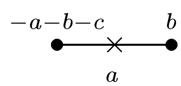
by 5.5.8. Coker $D_5^{b,c,-a-b-c} =$ 

and the second term is:

$E_2^{p,q} =$ 

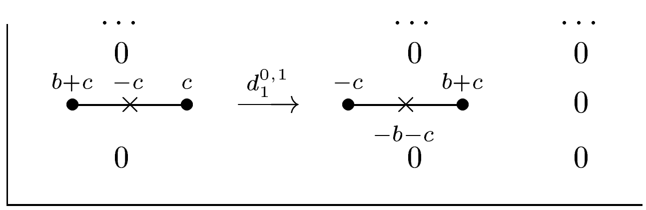
$d_2^{0,1}$ must be surjective and we obtain exact sequence:

$0 \to H^1(A',$ 

$) \to ker D_5^{b,c,-a-b-c} \xrightarrow{d_2^{0,1}}$ 

$\to 0$

<u>case $a=-b-c$</u>

$E_1^{p,q} =$ 

$d_1^{0,1}$ must be surjective and so non zero.

$$d_1^{0,1} = D_5^{b,c,0} = D_2^{b,c,0}$$

And this case is done.

case $a < -b-c$

$$E_1^{p,q} = \left|\begin{array}{ccccc} \cdots & & \cdots & & \cdots \\ 0 & & 0 & & 0 \\ \overset{-a}{\bullet}\!\!-\!\!\overset{a+b}{\times}\!\!-\!\!\overset{c}{\bullet} & \xrightarrow{d_1^{0,1}} & \overset{-a-b}{\bullet}\!\!-\!\!\overset{a}{\times}\!\!-\!\!\overset{b+c}{\bullet} & \xrightarrow{d_1^{1,1}} & \overset{-a-b-c}{\bullet}\!\!-\!\!\underset{a}{\times}\!\!-\!\!\overset{b}{\bullet} \\ 0 & & 0 & & 0 \\ \hline \end{array}\right.$$

$d_1^{1,1}$ must be surjective. We identify:

$$d_1^{1,1} = D_6^{b,c,-a-b-c}$$

$$E_2^{p,q} = E_\infty^{p,q} = \left|\begin{array}{ccc} \cdots & \cdots & \cdots \\ 0 & 0 & 0 \\ \text{Ker } d_1^{0,1} & \text{Ker } D_6^{b,c,-a-b-c} / \text{ Im } d_1^{0,1} & 0 \\ 0 & 0 & 0 \\ \hline \end{array}\right.$$

The operators from position [6] are not injectif for $m > 0$. Hence Ker $D_6^{b,c,-a-b-c} \neq 0$ and Im $d_1^{0,1} = \text{Ker } D_6^{b,c,-a-b-c}$, because $H^2(A', \overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\bullet}) = 0$. So $d_1^{0,1}$ is non-trivial and

$$d_1^{0,1} = D_5^{b,c,-a-b-c}$$

which gives the isomorphism of this case.

The operator $D_5^{k,l,m}$ has the non-trivial kernel for $k, l > 0$ and $m \geq 0$. In this case we obtain all the possibilities of coefficients k,l,m except $m = 0$, which was obtained in the previous case.

The reason for $\mathfrak{g}$-equivariance is the equivariance of BGG, of computation of direct images, of interpretations 5.5.7. and 5.5.8. and lemma 2.10. saying, that equivariance is preserved by derivation in spectral sequence. □

Theorem 6.2. *Consider the diagram*

$$\begin{tikzcd} & \times\!\!-\!\!\times\!\!-\!\!\times \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ \times\!\!-\!\!\bullet\!\!-\!\!\times & & \bullet\!\!-\!\!\times\!\!-\!\!\bullet \end{tikzcd}$$

Let C' be a convex subset of $\mathbb{M}^I$, $B' = \tau^{-1}(C')$ and $A' = \eta(B')$.
The Penrose transform gives the following isomorphisms of $\mathfrak{g}$-modules.

For the zero cohomology on $\times\!\!-\!\!\bullet\!\!-\!\!\times$ *:*

$\boxed{a \geq 1 \ \text{and}\ c \geq 1}$

$$H^0(A', \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\times}) \simeq \text{Ker } D_1^{a,b,c} : \Gamma(C', \overset{a}{\bullet}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet}) \to \Gamma(C', \overset{a+b}{\bullet}\!-\!\overset{-b}{\times}\!-\!\overset{b+c}{\bullet})$$

This gives all the operators of type [1] from [DIG]

$\boxed{a < 1 \ \text{or}\ c < 1}$

$$H^0(A', \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\times}) = 0$$

For the first cohomology on $\times\!-\!\bullet\!-\!\times$

$\boxed{a \geq 1 \ \text{and}\ c \geq 1}$

$$H^1(A', \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\times}) \simeq \ Coker D_1^{a,b,c} : \Gamma(C', \overset{a}{\bullet}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet}) \to \Gamma(C', \overset{a+b}{\bullet}\!-\!\overset{-b}{\times}\!-\!\overset{b+c}{\bullet})$$

This gives all the operators of type [1] from [DIG]

$\boxed{a = 0 \ \text{and}\ c > -b}$ or $\boxed{c = 0 \ \text{and}\ a > -b}$ or $\boxed{-b < a, c < 0}$

$$H^1(A', \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\times}) = \overset{a+b}{\bullet}\!-\!\overset{-b}{\times}\!-\!\overset{b+c}{\bullet}$$

This gives all the homogeneous bundles of the type [2].

$\boxed{a \geq 1 \ \text{and}\ -b < c < 0}$

$$0 \to \overset{a+b}{\bullet}\!-\!\overset{-b}{\times}\!-\!\overset{b+c}{\bullet} \to H^1(A', \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\times}) \to \overset{a}{\bullet}\!-\!\overset{b+c}{\times}\!-\!\overset{-c}{\bullet} \to 0$$

$\boxed{c \geq 1 \ \text{and}\ -b < a < 0}$

$$0 \to \overset{a+b}{\bullet}\!-\!\overset{-b}{\times}\!-\!\overset{b+c}{\bullet} \to H^1(A', \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\times}) \to \overset{-a}{\bullet}\!-\!\overset{a+b}{\times}\!-\!\overset{c}{\bullet} \to 0$$

$\boxed{c \geq 1 \ \text{and}\ a = -b}$

$$H^1(A', \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\times}) \simeq \overset{-a}{\bullet}\!-\!\overset{a+b}{\times}\!-\!\overset{c}{\bullet}$$

$\boxed{a \geq 1 \ \text{and}\ c = -b}$

$$H^1(A', \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\times}) \simeq \overset{a}{\bullet}\!-\!\overset{b+c}{\times}\!-\!\overset{-c}{\bullet}$$

$\boxed{a \geq 1 \ \textit{and} \ c < -b \ \textit{and} \ a+b+c \geq 0}$

$$H^1(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\times}) \simeq \operatorname{Ker} D_3^{a+b+c,-c-b,b}$$

We obtain kernels of all operators of type [3].

$\boxed{a \geq 1 \ \textit{and} \ c < -b \ \textit{and} \ a+b+c < 0}$

$$H^1(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\times}) \simeq \operatorname{Ker} D_4^{-a-b-c,a,b}$$

We obtain kernels of all operators of type [4].

$\boxed{c \geq 1 \ \textit{and} \ a < -b \ \textit{and} \ a+b+c \geq 0}$

$$H^1(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\times}) \simeq \operatorname{Ker} D_2^{b,-a-b,a+b+c}$$

We obtain kernels of all operators of type [2].

$\boxed{c \geq 1 \ \textit{and} \ a < -b \ \textit{and} \ a+b+c < 0}$

$$H^1(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\times}) \simeq \operatorname{Ker} D_5^{b,c,-a-b-c}$$

We obtain kernels of all operators of type [5].

$\boxed{\textit{otherwise}}$

$$H^1(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\times}) = 0$$

For the second cohomology on $\times\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\times$

$\boxed{a < -b \ \textit{and} \ c < -b}$

$$H^2(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\times}) \simeq \operatorname{Ker} D_6^{-b-c,b,-b-a}$$

This gives all the operators of type [6] from [DIG]

$\boxed{a = -b \ \textit{and} \ c < 0}$ *or* $\boxed{c = -b \ \textit{and} \ a < 0}$ *or* $\boxed{-b < a, c < 0}$

$$H^2(A', \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\times}) = \overset{-a}{\bullet}\!\!-\!\!\!-\!\!\underset{a+b+c}{\times}\!\!-\!\!\!-\!\!\overset{-c}{\bullet}$$

This gives all the homogeneous bundles of the type [5].

$\boxed{a < -b \ \textit{and} \ -b < c < 0}$

$$0 \to \overset{-a-b}{\bullet}\!\!-\!\!\overset{a}{\times}\!\!-\!\!\overset{b+c}{\bullet} \to H^2(A', \overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\times}) \to \overset{-a}{\bullet}\!\!-\!\!\underset{a+b+c}{\times}\!\!-\!\!\overset{-c}{\bullet} \to 0$$

$\boxed{c < -b \ \textit{and} \ -b < a < 0}$

$$0 \to \overset{a+b}{\bullet}\!\!-\!\!\overset{c}{\times}\!\!-\!\!\overset{-b-c}{\bullet} \to H^2(A', \overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\times}) \to \overset{-a}{\bullet}\!\!-\!\!\underset{a+b+c}{\times}\!\!-\!\!\overset{-c}{\bullet} \to 0$$

$\boxed{c < -b \ \textit{and} \ a = 0}$

$$H^2(A', \overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\times}) \simeq \overset{a+b}{\bullet}\!\!-\!\!\overset{c}{\times}\!\!-\!\!\overset{-b-c}{\bullet}$$

$\boxed{a < -b \ \textit{and} \ c = 0}$

$$H^2(A', \overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\times}) \simeq \overset{-a-b}{\bullet}\!\!-\!\!\overset{a}{\times}\!\!-\!\!\overset{b+c}{\bullet}$$

$\boxed{a \geq 1 \ \textit{and} \ c < -b \ \textit{and} \ a+b+c \geq 0}$

$$H^2(A', \overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\times}) \simeq \textit{Coker} \ D_3^{a+b+c,-c-b,b}$$

We obtain kernels of all operators of type [3].

$\boxed{a \geq 1 \ \textit{and} \ c < -b \ \textit{and} \ a+b+c < 0}$

$$H^2(A', \overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\times}) \simeq \textit{Coker} \ D_4^{-a-b-c,a,b}$$

We obtain kernels of all operators of type [4].

$\boxed{c \geq 1 \ \textit{and} \ a < -b \ \textit{and} \ a+b+c \geq 0}$

$$H^2(A', \overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\times}) \simeq \textit{Coker} \ D_2^{b,-a-b,a+b+c}$$

We obtain kernels of all operators of type [2].

$\boxed{c \geq 1 \ \textit{and} \ a < -b \ \textit{and} \ a+b+c < 0}$

$$H^2(A', \overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\times}) \simeq \textit{Coker} \ D_5^{b,c,-a-b-c}$$

We obtain kernels of all operators of type [5].

$\boxed{otherwise}$

$$H^2(A',\ \overset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!-\!\!\overset{c}{\times}\ ) = 0$$

Proof. The proof is essentially the same as in 6.1. By 5.3.6. our C' is good in sense of chapter 4 and so we have the spectral sequence

$$E_1^{pq} = \Gamma(C', \tau_*^q \Delta^p(\ \overset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!-\!\!\overset{c}{\times}\ )) \tag{SEQ}$$

and

$$E^{pq} \Rightarrow = H^r(B', \eta^{-1}\mathcal{O}(\ \overset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!-\!\!\overset{c}{\times}\ )) = H^r(A', \mathcal{O}(\ \overset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!-\!\!\overset{c}{\times}\ )). \tag{SUM}$$

The spaces $\Delta^p(\ \overset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!-\!\!\overset{c}{\times}\ ))$ are given in 5.4.3. and recipe for direct images τ_* in 5.4.5.

Now we will compute (SEQ) for different cases. It will be very simple, because $\Delta^p(\ \overset{a}{\times}\!\!-\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!-\!\!\overset{c}{\times}\ ))$ is 0 for $p \geq 2$. It follows that $E_2 = E_\infty$ for (SEQ) and we must only to identify E_1 and d_1. It is completely same as in 6.1. The results are given bellow: <u>case $a > 1$ and $c > 1$</u>

$$E_1^{p,q} = \left|\begin{array}{ccccc} \cdots & & \cdots & \cdots \\ 0 & & 0 & 0 \\ 0 & & 0 & 0 \\ \overset{a}{\bullet}\!-\!-\!\overset{b}{\times}\!-\!-\!\overset{c}{\bullet} & \xrightarrow{d_1^{0,0}} & \overset{a+b}{\bullet}\!-\!-\!\overset{-b}{\times}\!-\!-\!\overset{b+c}{\bullet} & 0 \\ \hline \end{array}\right.$$

$$d_1^{0,0} = D_1^{a,b,c}$$

<u>case $a > 1$ and $1 > c > -b$</u>

$$E_1^{p,q} = \left|\begin{array}{ccc} \cdots & \cdots & \cdots \\ 0 & 0 & 0 \\ \overset{a}{\bullet}\!-\!-\!\overset{b+c}{\times}\!-\!-\!\overset{-c}{\bullet} & 0 & 0 \\ 0 & \overset{a+b}{\bullet}\!-\!-\!\overset{-b}{\times}\!-\!-\!\overset{b+c}{\bullet} & 0 \\ \hline \end{array}\right.$$

$$d_1^{0,1} = 0$$

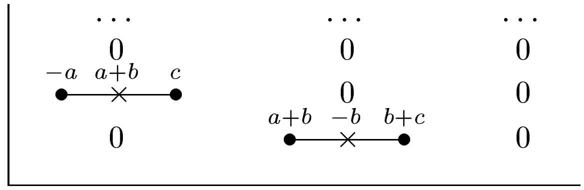
<u>case $a > 1$ and $c < -b$</u>

$$E_1^{p,q} = \left|\begin{array}{ccccc} \cdots & & \cdots & \cdots \\ 0 & & 0 & 0 \\ \overset{a}{\bullet}\!-\!-\!\overset{b+c}{\times}\!-\!-\!\overset{-c}{\bullet} & \xrightarrow{d_1^{0,1}} & \overset{a+b}{\bullet}\!-\!-\!\overset{c}{\times}\!-\!-\!\overset{-b-c}{\bullet} & 0 \\ 0 & & 0 & 0 \\ \hline \end{array}\right.$$

$$d_1^{0,1} = \begin{cases} D_3^{a+b+c,-c-b,b} & \text{if } a+b+c \geq 0 \\ D_4^{-a-b-c,a,b} & \text{if } a+b+c < 0 \end{cases}$$

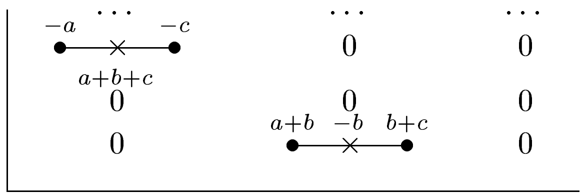
This give kernels of all operators of type [3] and [4] from (DIG).

case $-b < a < 1$ and $c > 1$

$E_1^{p,q} =$ 

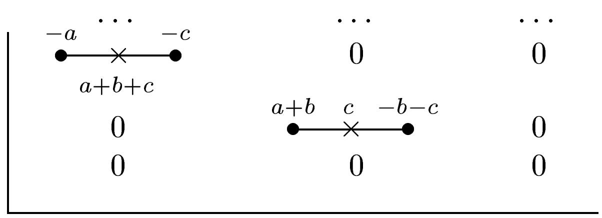
$$d_1 = 0$$

case $-b < c, a < 1$

$E_1^{p,q} =$ 

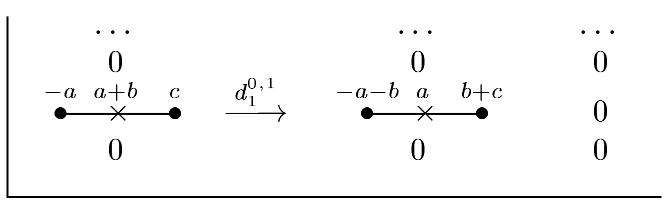
$$d_1 = 0$$

case $-b < a < 1$ and $c < -b$

$E_1^{p,q} =$ 

$$d_1 = 0$$

case $a < -b$ and $c > 1$

$E_1^{p,q} =$ 

$$d_1^{0,1} = \begin{cases} D_2^{b,-a-b,a+b+c} & \text{if } a+b+c \geq 0 \\ D_5^{b,c,-a-b-c} & \text{if } a+b+c < 0 \end{cases}$$

We get kernels of all operators of type [2] and [5].

case $a < -b$ and $-b < c < 1$

$$E_1^{p,q} = \left| \begin{array}{ccc} \cdots & \cdots & \cdots \\ \overset{-a}{\bullet}\!\!-\!\!\underset{a+b+c}{\times}\!\!-\!\!\overset{-c}{\bullet} & 0 & 0 \\ 0 & \overset{-a-b}{\bullet}\!\!-\!\!\overset{a}{\times}\!\!-\!\!\overset{b+c}{\bullet} & 0 \\ 0 & 0 & 0 \\ \hline \end{array} \right.$$

$$d_1 = 0$$

case $a, c < -b$

$$E_1^{p,q} = \left| \begin{array}{cccc} \cdots & & \cdots & \cdots \\ \overset{-a}{\bullet}\!\!-\!\!\underset{a+b+c}{\times}\!\!-\!\!\overset{-c}{\bullet} & \xrightarrow{d_1^{0,2}} & \overset{-a-b}{\bullet}\!\!-\!\!\underset{a+b+c}{\times}\!\!-\!\!\overset{-b-c}{\bullet} & 0 \\ 0 & & 0 & 0 \\ 0 & & 0 & 0 \\ \hline \end{array} \right.$$

$$d_1^{0,2} = D_6^{-b-c,b,-b-a}$$

All the results follows by (SUM) from these diagrams. There is only one problem - how to prove, that operators $d_1^{0,1}$ are non-trivial. (The nontriviality of $d_1^{0,2}$ follows from fact that $H^3(\mathbb{A}^I, \overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\times}) = 0$ and the nontriviality of $d_1^{0,0}$ from the fact, that it is exactly an operator from relative BGG). This nontriviality is supposed for example in [Ea2], but we do not know the proof.

The $\mathfrak{g}$-equivariance has the same reasons like in 6.1. $\square$

6.3. Explanation. In proofs of 6.1. and 6.2. we got mappings $L_{C'}$ between sections over C' of two vector bundles. By 4.1. the maps $L_{C'}$ are defined for such C' which form a basis of topology of C'. We want to prove, that L is a local G-equivariant operator.

For this we must prove three conditions:

(1) L commutes with sheaf restrictions.
(2) L is G-equivariant.
(3) $L_{C'}$ is continuous.

Let us see how the $L_{C'}$ is defined: In fact it is the mapping d in spectral sequence associated with double complex:

$$K_{B'}^{p,q} = \Gamma(B', \mathcal{E}^{0,q}(E_\lambda))$$

There is action of G on the sheaves $\mathcal{E}^{0,q}(E_\lambda)$ given in 3.9. and the both horizontal and vertical operators of K are G-equivariant by 3.5. and 3.9. It means, that if $B_1 \subset B$ and $B_2 = gB_1$ for some $g \in G$ then we have

$$f_g^{p,q} : K_{B_1}^{p,g} \to K_{B_2}^{p,q}.$$

These f_g commute with both operators of K. It follows, that there are induced mappings

$$f_{g,r}^{p,q} : E_{B_1,r}^{p,q} \to E_{B_2,r}^{p,q},$$

commuting with operators d_r. The computation of direct images is G-equivariant (3.14.) and so if we identify $E_{B_i}^{p,q}$ with $\Gamma(C_i, E_\lambda)$, the $f_{g,r}$ will be identical with action of g on $\mathcal{O}(E_\lambda)$, which sends the sections over C_1 to sections over C_2. So L is G-equivariant. Absolutely same argument works if we replace f_g by restriction for $B_1 \subset B_2$. We obtain following commutative diagram:

$$\begin{tikzcd}
\Gamma(C_1, E_\lambda) \arrow[r, "L_{C_1}"] \arrow[d, "r_{C_1}^{C_2}"'] & \Gamma(C_1, E_\rho) \arrow[d, "r_{C_1}^{C_2}"] \\
\Gamma(C_2, E_\lambda) \arrow[r, "L_{C_2}"'] & \Gamma(C_2, E_\rho)
\end{tikzcd}$$

and L is well defined homomorphism of sheaves.

For the proof of continuity (see 2.12. for definition) we will use the Dolbeault resolution of sheaves on $\times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet$ see 3.7. We will use the description of identification

$$\Gamma(C_2, \tau_*^i \mathcal{O}(E)) \simeq \Gamma(C_2, \mathcal{H}^i(\mathbb{P}^1, E)) \simeq H^i(B_2, \mathcal{O}(E))$$

given in lemma 5.4.6. and the definition of mappings in spectral sequence given in 2.10.

There were only the operators d_1 and d_2 which were identified with some differential operators. First let us see the situation for d_1. Let $\{h_j\}$ be a basis for $\mathcal{H}^i(\mathbb{P}_1, E)$ and

$$s \in \Gamma(C_2, \mathcal{H}^i(\mathbb{P}^1, E)); s = \sum_1^m s^j.h_j$$

where s^j are holomorphic functions on C_2. We define a (0,i)-form on B_2.

$$\alpha(z, v) := \sum_1^m s^j(z).h_j(v).$$

Evidently $P(\alpha) = s$ and

$$L_{C_2}(s) = P(D(\alpha)),$$

where D is operator in BGG-resolution. Now we will prove the continuity of L. Let

$$s_n \in \mathit{\Gamma}(C_2, \mathcal{H}^i(\mathbb{P}^1, E)); s_n \xrightarrow{d} 0 \text{ on } C_2,$$

where $\xrightarrow{d}$ means uniform convergence for all derivatives. By definition

$$s_n^j \xrightarrow{d} 0 \text{ on } C_2 \text{ for all } j.$$

It is easy to see that $\alpha_n \xrightarrow{d} 0$ on B_2, because

$$d_I(\alpha) = \sum_0^m d_{I_z} s^j . d_{I_v} h_j$$

$d_{I_z} s^j \xrightarrow{d} 0$ and $d_{I_v} h_j$ are bounded on compact $\mathbb{P}^1$. The operator D is a differential operator and so

$$D(\alpha_n) \xrightarrow{d} 0 \text{ on } B_2,$$

P is given by projection, which is continuous in topology of uniform convergence and so

$$P(D(\alpha_n)) \rightrightarrows 0$$

and L_{C_2} is continuous.

The situation for d_2 is a little more complicated, namely we have :

$$\begin{tikzcd}
\mathcal{E}^{0,1}(B_2, E_0) \arrow[r, "D_1"] & \mathcal{E}^{0,1}(B_2, E_1) & \\
& \mathcal{E}^{0,0}(B_2, E_1) \arrow[u, "\bar{\partial}"'] \arrow[r, "D_2"'] & \mathcal{E}^{0,0}(B_2, E_d)
\end{tikzcd}$$

and

$$L_{C_2}(s) = P(D_2(\beta))$$

where β is some element of $\mathcal{E}^{0,0}(B_2, E_1)$ for which

$$D_1(\alpha) = \bar{\partial}(\beta).$$

Such an element must exist from the construction of the spectral sequence. Now let us prove the continuity of L_{C_2}. Let $\bar{C_1} \subset C_2$ and:

$$s_n \in \mathit{\Gamma}(C_2, \mathcal{H}^i(\mathbb{P}^1, E_n)); s_n \xrightarrow{d} 0 \text{ on } C_2$$

then define $\alpha_n \in \mathcal{E}^{0,1}(B_2, E_0)$ like above and $\alpha_n \xrightarrow{d} 0$ on B_2.

$$\gamma_n := D_1(\alpha_n) \xrightarrow{d} 0 \text{ on } B_2$$

Lemma. *There are forms $\beta_n \in \mathcal{E}^{0,0}(B_j, E_1)$ such that $\bar\partial(\beta_n) = \gamma_n$ and $\gamma_n \xrightarrow{d_k} 0$ on B_1, where $\xrightarrow{d_k}$ means uniform convergence for all the derivatives up to order k.*

Proof. Let $\mathcal{C}_k^{0,i}(B_1, E_1)$ denotes the space of all k-smooth (0,i)-forms on B_1 bounded with their derivations on B_1. This a Banach space with respect to the norm $\| \, . \, \|_k$. There is operator

$$\bar\partial : \mathcal{C}_{k+1}^{0,0}(B_1, E_1) \to \mathcal{C}_k^{0,1}(B_1, E_1)$$

continuous with respect to the norms. Evidently $\gamma_n \in \mathcal{C}_k^{0,1}(B_1, E_1)$ and even in Im $\bar\partial$ because there are some β'_n even on B_2. By Banach theorem on open mappings there are some $\beta_n \in \mathcal{C}_{k+1}^{0,0}(B_1, E_1)$ such that $\bar\partial(\beta_n) = \gamma_n$ and $\beta_n \xrightarrow{\|\cdot\|_{k+1}} 0$ This β_n must be C^∞, because β'_n is smooth and $\beta_n - \beta'_n$ is a holomorphic function on B_1. $\square$

If we choose k to be big enough, we have

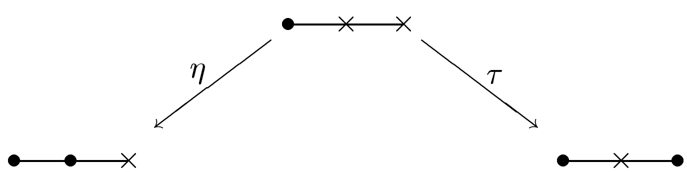
$$D_2(\beta_n) \rightrightarrows 0 \text{ on } B_1$$

and from the continuity of P we have

$$L_{C_1}(s_n) = P(D_2(\beta_n)) \rightrightarrows 0$$

and this give us the continuity of L.

6.4. Remark. Let us discuss which solutions were obtained by Penrose transform: In 6.1. was obtained all the solutions of all operators of types [1],[4],[5],[8] from diagram DIG (see 5.5.6.). If we consider the double fibration



we obtain by the symmetry solutions of all the operators of type [1],[3],[4],[8]. In 6.2. we obtain solutions of all the operators of type [1],[2],[3],[4],[5],[6]. Thus we see, that we have obtained solutions of all G-equivariant operators on $\bullet\!\!-\!\!\times\!\!-\!\!\bullet$ with exception of operators of type [7]. But in fact $D_7 = D_2 \circ D_4$ and from surjectivity of D_2 follows the following exact sequence of $\mathfrak{g}$-modules:

$$0 \to \text{Ker } D_2 \to \text{Ker } D_7 \to \text{Ker } D_4 \to 0,$$

which gives us information about Ker D_7.

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