# Rational Curves of Given Direction

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**Abstract**. We study all the rational curves tangent to a given vector field. We also analyse the degree of these curves and in particular show when occurs a degree reduction.

*Keywords*: Vector Field, Cross Product, Tangent Developabe Surface

## 1 Introduction

This paper is devoted to the construction and analysis of curves of given direction. This concept is rather trivial if we restrict ourselves to general smooth curves and use the apparatus of the classical differential geometry [5]. But the situation becomes much more complicated when the requirement of the rationality of the constructed curves is added [1].

To our knowledge this question was studied only in the special case of the curves with Pythagorean hodograph, [2, 4]. The general case was never considered.

The remainder of the paper is organized as follows. We study the general differential geometry properties of the curves tangent to a given vector field in Section 2. In Section 3 we give a general solution for the rational curves and illustrate this problem on two examples. Section 4 is devoted to the properties of polynomial fields and we show certain simplifications and degree reductions formulas. Eventually we conclude the paper.

## 2 Definition and preliminary observations

We will study curves and vector fields in $\mathbb{R}^3$. By a vector field we mean one parameter family of vectors depending in a smooth way on the parameter $t$. In order to include also the rational field we allow a zero-measure of parameter values for which the field is indefinite.

**Definition 1** *We say, that the curve* $\mathbf{r}(t)$ *is tangent to the field* $\mathbf{v}(t)$ *if and only if*

$$\mathbf{v}(t) \times \mathbf{r}'(t) = \mathbf{0} \tag{1}$$

*for all* $t$.

It is quite obvious from the definition that there is always at least one curve tangent to a given field. Indeed we can directly integrate the input vector field.

**Definition 2** *We will call the curve* $\mathbf{r}(t) = \int \mathbf{v}(t)\,\mathrm{d}t$ *the* primitive tangent curve *to the field* $\mathbf{v}(t)$.

It is also obvious that any other curve tangent to this field can be obtained in the form $\tilde{\mathbf{r}}(t) = \int l(t)\mathbf{v}(t)\,\mathrm{d}t$, where $l(t)$ is a smooth real function.

**Proposition 3** *For a given field let* $\mathbf{r}(t) = \int \mathbf{v}(t)\,\mathrm{d}t$ *be its primitive tangent curve and let it has the curvature function* $\kappa(t)$ *and the torsion function* $\tau(t)$. *Let* $\tilde{\mathbf{r}}(t) = \int l(t)\mathbf{v}(t)\,\mathrm{d}t$ *be another curve tangent to the same field. Then for its curvature and torsion functions the following identities hold*

$$\tilde{\kappa}(t) = \frac{\kappa(t)}{l(t)} \qquad \textit{and} \qquad \tilde{\tau}(t) = \frac{\tau(t)}{l(t)}. \tag{2}$$

**Proof:** By a direct computation we obtain

$$\begin{aligned}
\tilde{\mathbf{r}}'(t) &= l(t)\mathbf{r}'(t) \\
\tilde{\mathbf{r}}''(t) &= l'(t)\mathbf{r}'(t) + l(t)\mathbf{r}''(t) \\
\tilde{\mathbf{r}}'''(t) &= l''(t)\mathbf{r}'(t) + 2l'(t)\mathbf{r}''(t) + l(t)\mathbf{r}'''(t)
\end{aligned}$$

which implies the following identities of the key expressions

$$\begin{aligned}
\tilde{\mathbf{r}}' \times \tilde{\mathbf{r}}'' &= l^2\,\mathbf{r}' \times \mathbf{r}'' \\
\det[\,\tilde{\mathbf{r}}', \tilde{\mathbf{r}}'', \tilde{\mathbf{r}}'''\,] &= l^3\,\det[\,\mathbf{r}', \mathbf{r}'', \mathbf{r}'''\,].
\end{aligned}$$

Consequently we obtain

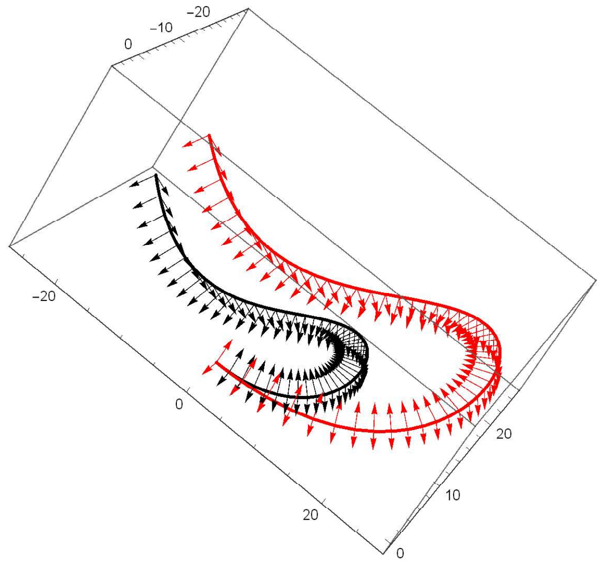
$$\tilde{\kappa} = \frac{||\tilde{\mathbf{r}}' \times \tilde{\mathbf{r}}''||}{||\tilde{\mathbf{r}}'||^3} = \frac{\kappa}{l}, \qquad \tilde{\tau} = \frac{\det[\,\tilde{\mathbf{r}}', \tilde{\mathbf{r}}'', \tilde{\mathbf{r}}'''\,]}{||\tilde{\mathbf{r}}' \times \tilde{\mathbf{r}}''||^2} = \frac{\tau}{l}.$$

Q.E.D.

The identities obtained in the the proof provide also an information about the Frenet frame of the curves tangent to the same vector field.

**Corollary 4** *All the curves tangent to the same vector field have the same binormal vector the same ratio of the curvature and torsion functions. They also have the same tangent and principal normal vectors up to the change of orientation.*

**Example 5** *The following figure displays two curves which are tangent to the same vector field (not shown). They have not only the same Frenet frame but also the same rotation-minimizing frame.*




## 3 Rational curves

Our goal is to construct rational curves tangent to rational vector fields. Let us start this section with a simple example.

**Example 6** Consider the polynomial vector field

$$\mathbf{v}(t) = \begin{pmatrix} 24t^3 - 12t^2 - 12t + 4 \\ 44t^3 - 60t^2 + 24t \\ 12t^2 - 4t^3 \end{pmatrix}$$

Set $l(t) = \frac{1+t}{3+t^2}$ and get $\tilde{\mathbf{r}}(t) = \int l(t)\mathbf{v}(t)\,\mathrm{d}t =$

$$\begin{pmatrix} t^3 + 6t^2 - 22\log\left(t^2+3\right) - 96t + \frac{292\arctan\left(\frac{t}{\sqrt{3}}\right)}{\sqrt{3}} \\ \frac{44t^3}{3} - 8t^2 + 36\log\left(t^2+3\right) - 168t + 168\sqrt{3}\arctan\left(\frac{t}{\sqrt{3}}\right) \\ -\frac{4t^3}{3} + 4t^2 - 12\log\left(t^2+3\right) + 24t - 24\sqrt{3}\arctan\left(\frac{t}{\sqrt{3}}\right) \end{pmatrix}$$

which is of course tangent to the field $\mathbf{v}(t)$.

We see that even from a rational input $\mathbf{v}(t)$ and $l(t)$ we typically obtain a non-rational curve. A different strategy therefore must be used to construct all the rational tangent curves. The geometrical essence of our approach is to construct the curve as the edge of regression of an envelope of (osculating) planes. It can be also expressed purely algebraically as shows the following proposition proved in [3].

**Proposition 7** *Given a rational vector field* $\mathbf{v}(t)$ *all the rational tangent curves* $\mathbf{r}(t)$ *can be expressed in the form*

$$\mathbf{r}(t) = \frac{f(t)\,\mathbf{u}'(t)\times\mathbf{u}''(t) + f'(t)\,\mathbf{u}''(t)\times\mathbf{u}(t) + f''(t)\,\mathbf{u}(t)\times\mathbf{u}'(t)}{\det[\,\mathbf{u}(t),\mathbf{u}'(t),\mathbf{u}''(t)\,]} \qquad (3)$$

*where* $\mathbf{u}(t) = \mathbf{v}(t)\times\mathbf{v}'(t)$ *and* $f(t)$ *is any rational function.*

**Example 8** *Given the field*

$$\mathbf{v} = \begin{pmatrix} 24t^3 - 12t^2 - 12t + 4 \\ 44t^3 - 60t^2 + 24t \\ 12t^2 - 4t^3 \end{pmatrix}$$

*we compute* $\mathbf{u} = \mathbf{v} \times \mathbf{v}'$ *and chose* $f = \frac{1+t}{3+t}$ *and obtain*

$$\begin{aligned} \tilde{\mathbf{r}} &= \frac{f\,(\mathbf{u}' \times \mathbf{u}'') + f'\,(\mathbf{u}'' \times \mathbf{u}) + f''\,(\mathbf{u} \times \mathbf{u}')}{\det[\,\mathbf{u}, \mathbf{u}', \mathbf{u}''\,]} = \\ &= \begin{pmatrix} \frac{144t^7+732t^6+1251t^5+189t^4-1049t^3-51t^2+213t-22}{144(t+3)^3(8t^3-3t^2-t+1)^2} \\ \frac{528t^7+2380t^6+3324t^5-1188t^4-3322t^3+2166t^2+153t-153}{288(t+3)^3(8t^3-3t^2-t+1)^2} \\ -\frac{48t^7+164t^6+24t^5-720t^4-824t^3+180t^2+27t-27}{288(t+3)^3(8t^3-3t^2-t+1)^2} \end{pmatrix}. \end{aligned}$$

## 4 Polynomial vector fields

We will focus on the rational and polynomial vector fields. It is obvious from the definition and equation (1) that a curve is tangent to a vector field if and only if it is tangent to its arbitrary functional multiple. We can use this fact to restrict our attention only to polynomial fields.

**Lemma 9** *If a curve* $\mathbf{r}(t)$ *is tangent to a rational field* $\mathbf{v}(t)$ *then there is up to a constant multiple a unique vector field* $\tilde{\mathbf{v}}(t)$ *with relatively prime components to which the curve is tangent as well.*

**Proof:** Let $k(t)$ be any rational nontrivial function and define $\tilde{\mathbf{v}}(t) = k(t)\mathbf{v}(t)$. Then any curve is tangent to $\tilde{\mathbf{v}}(t)$ if and only if it is tangent to $\mathbf{v}(t)$ because

$$\tilde{\mathbf{v}}(t) \times \mathbf{r}'(t) = k(t)\mathbf{v}(t) \times \mathbf{r}'(t).$$

Now there is up to a scalar multiple precisely one rational function $k(t)$ so that $\tilde{\mathbf{v}}(t) = k(t)\mathbf{v}(t)$ is polynomial with relatively prime components. Indeed, if $k_1(t)$ denotes the polynomial least common multiple of the denominators of the components of $\mathbf{v}(t)$ then $k_1(t)\mathbf{v}(t)$ is a polynomial field. Let $k_2(t)$ be a polynomial greatest common divisor of its components. Eventually set $k(t) = k_1(t)/k_2(t)$. Q.E.D.

The previous lemma implies that we can restrict our input to the polynomial fields with relatively prime components. Let as analyze the degrees of expressions occurring in (3).

**Lemma 10** *Let* $\mathbf{v}(t)$ *a polynomial vector field of degree* $n$ *with relatively prime components. Then* $\mathbf{u}(t) = \mathbf{v}(t) \times \mathbf{v}'(t)$ *is a polynomial field of degree* $2n-2$ *and* $\det[\mathbf{v}(t), \mathbf{v}'(t), \mathbf{v}''(t)]$ *is a polynomial of degree* $3n-6$.

**Proof:** Writing the components of the vector field $\mathbf{v}(t)$ explicitly shows that the leading terms of the components $\mathbf{v}(t) \times \mathbf{v}'(t)$ cancel to 0 and the three leading terms of $\det[\mathbf{v}(t), \mathbf{v}'(t), \mathbf{v}''(t)]$ cancel to 0 as well. Q.E.D.

**Example 11** Consider the polynomial vector field

$$\mathbf{v}(t) = \begin{pmatrix} 24t^3 - 12t^2 - 12t + 4 \\ 44t^3 - 60t^2 + 24t \\ 12t^2 - 4t^3 \end{pmatrix}$$

which is of degree $n = 3$. By a direct computation we get

$$\mathbf{u}(t) = \mathbf{v}(t) \times \mathbf{v}'(t) = \begin{pmatrix} -288t^4 - 192t^3 + 288t^2 \\ 240t^4 - 96t^3 + 192t^2 - 96t \\ 912t^4 - 2208t^3 + 1536t^2 - 480t + 96 \end{pmatrix}$$

which is of degree 4 and

$$\det[\mathbf{v}(t), \mathbf{v}'(t), \mathbf{v}''(t)] = 1152\left(16t^3 - 6t^2 - 2t + 2\right)$$

which is of degree 3.

We can obtain explicit expressions for the formulae appearing in (3) as follows.

**Proposition 12** *Let $\mathbf{v}(t)$ be a vector field and $\mathbf{u}(t) = \mathbf{v}(t) \times \mathbf{v}'(t)$. Then*

$$\begin{aligned} \mathbf{u} \times \mathbf{u}' &= \det[\,\mathbf{v}, \mathbf{v}', \mathbf{v}''\,]\,\mathbf{v} \\ \mathbf{u}' \times \mathbf{u}'' &= \det[\,\mathbf{v}, \mathbf{v}'', \mathbf{v}'''\,]\,\mathbf{v} + \det[\,\mathbf{v}, \mathbf{v}', \mathbf{v}''\,]\,\mathbf{v}'' \\ \mathbf{u} \times \mathbf{u}'' &= \det[\,\mathbf{v}, \mathbf{v}', \mathbf{v}'''\,]\,\mathbf{v} + \det[\,\mathbf{v}, \mathbf{v}', \mathbf{v}''\,]\,\mathbf{v}' \\ \det[\,\mathbf{u}, \mathbf{u}', \mathbf{u}''\,] &= (\det[\,\mathbf{v}, \mathbf{v}', \mathbf{v}''\,])^2. \end{aligned}$$

**Proof:** By a direct differentiation we obtain

$$\mathbf{u} = \mathbf{v} \times \mathbf{v}', \quad \mathbf{u}' = \mathbf{v} \times \mathbf{v}'', \quad \mathbf{u}'' = \mathbf{v}' \times \mathbf{v}'' + \mathbf{v} \times \mathbf{v}'''$$

and the proof can be concluded using the standard vector identity

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\,\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d})\,]\,\mathbf{c} - [\,\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\,]\,\mathbf{d}.$$

Q.E.D.

This Proposition explains the special form of the denominator of the tangent curve in Example 5. It might also help to understand possible simplifications of the formula (3) for special choices of $f(t)$.

## 5 Conclusion

We presented several results for curves tangent to polynomial vector fields. We have seen that the Frenet frame and the ratio between the curvature and torsion are essentially determined by the field. We have also presented the general formula for rational tangent curves and proved several results about the degrees of expressions occurring in this formula. We hope that these results will lead to the full understanding of possible cancellation of the denominator in this formula.

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