Pythagorean Hodograph Curves of Degree Four

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Abstract. All Euclidean similitudes and reparameterizations of planar curves with Pyhtagorean Hodograph (PH curves) can be induced by Euclidean similitudes and reparameterizations of their preimage. We exploit this fact in order to describe the explicit and implicit form of all planar PH curves of degree 4. We also classify these curves with respect to the type and distribution of their singularities. Our results are illustrated by examples.

Keywords: Euclidean similitudes, curves with Pythagoream Hodograph, curves of degree 4, algebraic curves, singularity, genus

1 Introduction

Pythagorean Hodograph (PH) curves (see the survey [3], the book [4] and the references cited therein) form an interesting subclass of polynomial parametric curves. They possess a piecewise polynomial arc length function and rational offset curves in the planar case.

Since their introduction in [2] planar PH curves have been studied in many publications. The greatest attention was devoted to various construction techniques. Structural results seem to be much less frequent, the original publication [2] and later [1] studying PH curves from the point of view of complex numbers seem to be the most important. In [7] the conditions for the monotonicity of curvature were studied for PH quintics.

In [5] using a new approach the classification of PH cubics was obtained in a different way and the classification of PH quintics was outlined and solved for the case of symmetrical curves. This paper uses similar techniques for the case of PH quartics. After reviewing some basic facts about PH curves we provide in Proposition 2 a new and improved formulation of the classification principle for PH curves of any degree. In Section 3 we apply this result to the case of PH quartics and we fully classify these curves.

2 Planar Pythagorean Hodograph curves

A planar parametric polynomial curve $\mathbf{c}(t) = [c_x(t), c_y(t)]$ is called Pythagorean Hodograph (PH) [2] if there exists a polynomial $\sigma(t)$ so that

$$c'_x(t)^2 + c'_y(t)^2 = \sigma(t)^2 \tag{1}$$

It follows from a more general theory valid in any unique factorization domain that all the solutions of (1) have the form

$$c'_x(t) = \lambda(t)(u(t)^2 - v(t)^2) \tag{2}$$
$$c'_y(t) = \lambda(t)(2u(t)v(t)) \tag{3}$$
$$\sigma(t) = \lambda(t)(u(t)^2 + v(t)^2). \tag{4}$$

where $u(t)$, $v(t)$, $\lambda(t)$ are any real polynomials. The proof is given in [6] and is quite straightforward. It is obvious that (2)–(4) yield a solution of (1). Conversely, given $c'_x(t)$, $c'_y(t)$ and $\sigma(t)^2$ one can see that $\sigma(t) + c'_x(t)$ and $\sigma(t) + c'_x(t)$ must have the same (up to a scalar multiple) maximal square-free factor which will be denoted $\lambda(t)$. Eventually define $u(t)$ and $v(t)$ via the relations

$$u(t)^2 = \frac{\sigma(t) + c'_x(t)}{2\lambda(t)}, \qquad v(t)^2 = \frac{\sigma(t) - c'_x(t)}{2\lambda(t)}.$$

Let us remark that $\lambda(t)$ is a common factor of the components of $\mathbf{c}(t)$ a thus its roots will typically correspond to the cusps of $\mathbf{c}(t)$. The solution (2)–(3) can be expressed using the complex numbers. Defining $\mathbf{z}(t) = u(t) + \mathbf{i}v(t)$, we see that $c'_x(t) + \mathbf{i}c'_y(t) = \lambda(t)\mathbf{z}(t)^2$. Consequently using the convention of convex–valued polynomials we can state the following

Proposition 1 *A planar curve $\mathbf{c}(t)$ is a PH curve if and only if there exist polynomials $\mathbf{z}(t)$, $\lambda(t)$ which are complex and real valued respectively, and*

$$\mathbf{c}(t) = \int \lambda(t)\mathbf{z}(t)^2 \, dt. \tag{5}$$

We say, that $\mathbf{c}(t)$ is generated by $[\mathbf{z}(t), \lambda(t)]$.

Note that the integration is done for each component separately and provides free integration constants which geometrically correspond to a translation. Several (mutually translated) curves are thus generated by the same pair of functions. From the point of view of the classification of PH curves with respect to suitable geometric transformations it means that the translations are taken into account automatically.

The PH property is naturally preserved by the Euclidean similitudes. It can be shown that there is a four dimensional system of transformations and reparameterizations of the generating polynomials $\mathbf{z}(t)$ and $\lambda(t)$ which change curve $\mathbf{c}(t)$ only by a linear reparameterization and Euclidean similitudes. Consequently the following algebraic reformulation of the theorems proven in [5] holds.

Proposition 2 *Any PH curve in the plane can be obtained via a linear reparameterization and Euclidean similitude from a PH curve generated by* $[\mathbf{z}(t), \lambda(t)]$ *where the leading coefficients of both* $\mathbf{z}(t)$ *and* $\lambda(t)$ *is equal to* 1*, and the coefficient of the second highest power of* $\mathbf{z}(t)$ *is equal to* 0 *or* $\mathbf{i}$.

3 Pythagrean hodograph curves of degree 4

Let us study in which ways a PH curves of degree 4 can be obtained in the form (5). Using the Proposition 2 the following classification is obtained.

Proposition 3 *Any planar PH curve of degree* 4 *is either a reparameterized straight line or a curve obtained by a linear reparameterization and Euclidean similitude from the curve*

$$\mathbf{c}_A(t) = \left(\frac{1}{4}t^4 + \frac{A}{3}t^3 - \frac{1}{2}t^2 - At, \frac{2}{3}t^3 + At^2\right) \tag{6}$$

for some value of the real parameter A.

Proof. The only two possibilities for the degree of $\mathbf{z}(t)$ are 0 or 1, leaving us with degree of $\lambda(t)$ being 3 or 1 respectively. Together with Proposition (2) this means, that any PH curve of degree 4 is a reparameterization and similitude transformation of a curve generated by one of the following pairs of functions

1. $[1, \lambda_3(t)]$ where $\lambda_3(t)$ is a real polynomial of degree 3 with the leading coefficient equal to 1
2. $[t, t + B]$ where $B \in \mathbb{R}$ is a free parameter.
3. $[t + \mathbf{i}, t + A]$ where $A \in \mathbb{R}$ is a free parameter.

The first two cases involve only the real functions and thus lead do a degree 4 parameterization of the x axis. The last case provides the the curve (6). □

We thus see that there is only one parametric system (6) of non–trivial PH curves of degree 4. Let us investigate this system thoroughly. First of all, the formula (6) exhibits following symmetry. $\mathbf{c}_A(t)$ and $\mathbf{c}_{-A}(-t)$ have the same x coordinate and opposite y coordinate. In other words, the curves $\mathbf{c}_A(t)$ and $\mathbf{c}_{-A}(t)$ are related by a reflection and reparameterization. For this reason we will only study the parameter values $A \geq 0$.

Implicitizing the parametric curve (6) we obtain the following implicit equation.

$$\begin{aligned} 0 \quad = \quad & 384A^2x^2 + 144A^4x^2 - 1024x^3 - 384A^3y - 144A^5y \\ & +1152Axy + 48A^3xy - 768Ax^2y - 288y^2 + 1044A^2y^2 \\ & +96A^4y^2 - 864xy^2 + 24A^2xy^2 - 540Ay^3 - 16A^3y^3 + 81y^4. \end{aligned}$$

Studying the first partial derivatives of (7) we conclude that there is one singularity at the point

$$\left[\frac{A^2(6-A^2)}{12}, \frac{A^3}{3}\right]. \tag{7}$$

and two singularities (for two different choices of the sign) at the points

$$\left[\frac{A^4-12A^2-96 \pm A\left(\sqrt{A^2-8}\right)^3}{96}, \frac{24A+A^3 \pm \left(\sqrt{A^2-8}\right)^3}{12}\right]. \tag{8}$$

Looking at these formulas the following facts transpire.

- The singularity (7) is always real, it corresponds to the parametric value $t = A$ and it is always a cusp.
- For $0 \le A < \sqrt{8}$ the singularities (8) are complex non–real and mutually distinct. Each of them is an (imaginary) ordinary double point.
- For $A > \sqrt{8}$ the singularities (8) are real and mutually distinct. It can also be easily checked that they are both distinct from the singularity (7). Each of them is a real ordinary double point.
- For $A = \pm\sqrt{8}$ the two singularities (8) coincide and as a matter of fact they also coincide with the singularity (7). Indeed, in this particular case there is one triple point $\left[-\frac{4}{3}, \frac{16\sqrt{2}}{3}\right]$. It contains two branches, one with a simple tangent and the other one with a double tangent.

In order to visualize the three possible configurations of the singularities of a planar PH curve of degree 4 we display the relevant part of their locus for the parameter values $A = 2.3$, $A = \sqrt{8} \approx 2.83$ and $A = 3.4$, see Figure 1.

4 Conclusion

We have classified all the PH curves of degree 4 up to the Euclidean similitudes. We have also shown which classes of algebraic curves can be obtained within this classification (from the point of view of the distribution of the singularities). In future we intend to study the PH curves on degree 5 in a similar way.

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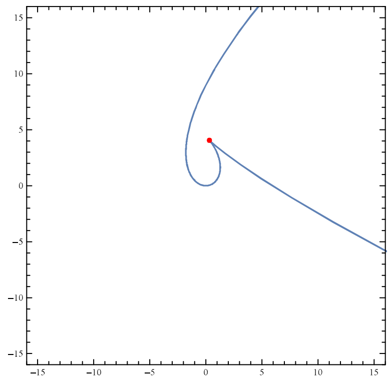
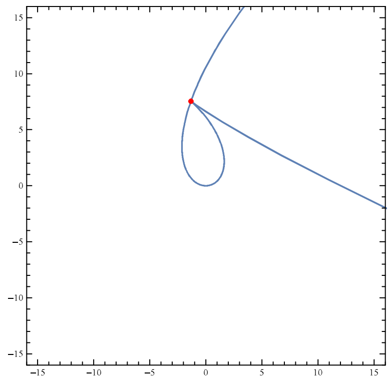
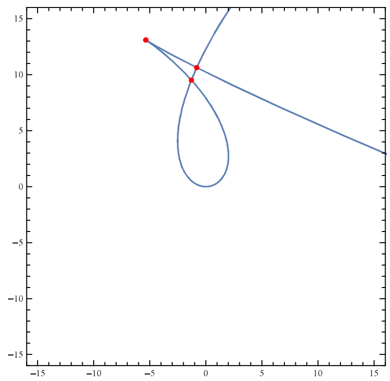


$A = 2$



$A = \sqrt{8} \approx 2.83$



$A = 3.4$

Figure 1: The three types of planar PH curves of degree 4 obtained setting the indicated values of A at (6).

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