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Smooth Cubic Pythagorean Hodograph Splines

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Abstract

In this paper we describe how to control the transformations, reparameterizations and continuity of planar cubic Pythagorean Hodograph splines. More precisely we show how these features can be ensured by conditions on the curve preimage.

Key words: PH curve, spline, signed curvature, rotation

1 Introduction and preliminaries

Pythagorean hodograph (PH) curves (see [1, 3, 2, 4, 5] and the references cited therein), form a remarkable subclass of polynomial parametric curves. They have a piecewise polynomial arc length function and, in the planar case, rational offset curves. These curves provide an elegant solution of various difficult problems occurring in applications, in particular in the context of CNC (computer-numerical-control) machining. Our paper is devoted to the modification of the planar PH curves via their preimage and on the connection of two PH cubics in a smooth way.

A Bézier curve is called *Pythagorean Hodograph (PH)* if the length of its tangent vector, taken in the appropriate metric, depends in a polynomial way on the parameter. In particular in the planar case $\mathbf{p}(t) = [x(t), y(t)]$ is called *planar PH curve* if there exists a polynomial $\sigma(t)$ such that

$$x'(t)^2 + y'(t)^2 = \sigma^2(t). \tag{1}$$

The degree of $\sigma(t)$ equals $n-1$, where n is the degree of the PH curve. The curve $\mathbf{h}(t) = [x'(t), y'(t)]$ is called the *hodograph* of $\mathbf{p}(t)$.

The planar polynomial curve $\mathbf{p}(t)$ can be identified with complex valued polynomial $\mathbf{p}(t) = x(t) + iy(t)$. The hodograph $\mathbf{h}(t) = x'(t) + iy'(t)$ then satisfy the equation (1) if and only if it is of the form $\mathbf{h}(t) = \lambda(t)\mathbf{w}(t)^2$, where $\mathbf{w}(t) = u(t) + iv(t)$ is a complex valued polynomial called *preimage*, [1, 6].

In order to study the $\mathcal{G}^2$ continuity we will need the notion of the signed curvature.

Definition 1.1 Let $\mathbf{c} : I \to \mathbb{R}^2$ be a regular parameterized curve. We define its signed curvature at the point $t \in I$ by the formula

$$\kappa_z(t) = \frac{\det(c'(t),\, c''(t))}{\||c'(t)\||^3}. \tag{2}$$

The signed curvature is not changed by the orientation-preserving reparameterizations.

2 Controling PH curves via their preimage

In this section we will show how a PH curve can be transformed and reparameterized via its preimage. We will also provide the formula for the signed curvature based on the preimage.

Lemma 2.1 Let $\mathbf{p} : I \to \mathbb{R}^2$ be a PH curve nad $\mathbf{q} : I \to \mathbb{R}^2$ its preimage. Then $\mathbf{p}$ is regular if and only if $\forall t \in I : \mathbf{q}(t) \neq \mathbf{0}$.

Proof: Let the complex preimage is of the form $\mathbf{q}(t) = a(t) + ib(t)$. Suppose

$$\forall t \in I : \mathbf{q}(t) \neq \mathbf{0} \Leftrightarrow \forall t \in I : a(t) \neq 0 \vee b(t) \neq 0.$$

Thus for

$$\mathbf{h}(t) = q^2(t) = a^2(t) - b^2(t) + i2a(t)b(t)$$

we get $\forall t \in I : \mathbf{h}(t) \neq (0, 0)$, because if for some $t_0 \in I : a(t_0) = b(t_0) \neq 0$, then $a(t_0)b(t_0) \neq 0$. If on the other hand for some $t_1 \in I : a(t_1)b(t_1) = 0$, then $a^2(t_1) - b^2(t_1) \neq 0$. As $\mathbf{p}'(t) = \mathbf{h}(t) \neq \mathbf{0} t \in I$, we obtain the regularity.
For the other implication consider the curve $\mathbf{p}$ with the hodograph

$$\mathbf{h}(t) = a^2(t) - b^2(t) + i2a(t)b(t),$$

which due to the regularity satisfies $\mathbf{h}(t) \neq \mathbf{0}, t \in I$. Then

$$\forall t \in I : a^2(t) - b^2(t) \neq 0 \vee 2a(t)b(t) \neq 0 \Leftrightarrow \forall t \in I : a(t) \neq 0 \vee b(t) \neq 0.$$

Thus we get

$$\forall t \in I : \mathbf{q}(t) = a(t) + ib(t) \neq \mathbf{0}.$$

□

Lemma 2.2 If the peimage is rotated clockwise by an angle α then the resulting PH curve is rotated by the angle 2α.

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Proof: Rotation by the angle α can be realized by the multiplication with the complex unit $\cos\alpha + i\sin\alpha$. Suppose that the starting preimage is $\mathbf{q} = a + ib$ (we omit the parameter t on which the functions a, b depend) and obtain the rotated preimage

$$\begin{aligned} \mathbf{q}_r &= (a+ib)(\cos\alpha + i\sin\alpha) = \\ &= (a\cos\alpha - b\sin\alpha) + i(a\sin\alpha + b\cos\alpha) = c + id. \end{aligned}$$

Taking its square we obtain the rotated hodograph

$$\begin{aligned} \mathbf{h}_r &= (c^2 - d^2) + i(2cd) = \\ &= a^2\cos^2\alpha - 2ab\cos\alpha\sin\alpha + b^2\sin^2\alpha - a^2\sin^2\alpha - 2ab\sin\alpha\cos\alpha - \\ &- b^2\cos^2\alpha + i2(a^2\sin\alpha\cos\alpha + ab\cos^2\alpha - ab\sin^2\alpha - b^2\sin\alpha\cos\alpha) = \\ &= a^2\cos 2\alpha - 2ab\sin 2\alpha - b^2\cos 2\alpha + i(a^2\sin 2\alpha - b^2\sin 2\alpha + 2ab\cos(2\alpha) \\ &= ((a^2 - b^2) + i(2ab))(\cos 2\alpha + i\sin 2\alpha) = \mathbf{h}(\cos 2\alpha + i\sin 2\alpha). \end{aligned}$$

We thus obtain the starting hodograph rotated by the angle 2α. The same holds for the PH curve, as the integration commutes with the rotation. □

Translation can be realized by the integration constant. We have seen, that the rotation is obtained via rotating the preimage. The following lemma shows that the linear reparameterization of the preimage provides a scaled PH curve.

Lemma 2.3 Let the PH curve $\mathbf{p}$ has the r $\mathbf{q}$ and let $k \neq 0$, $l \in \mathbb{R}$. Then the linearly reparameterized preimage $\mathbf{q}(kt + l)$ provides the PH curve $\frac{1}{k}\mathbf{p}(kt + l)$.

Proof: We obtain this result by a direct use of the substitution in the integral. □

The following result is straightforward, too.

Lemma 2.4 Let $\mathbf{q}$ is the preimage of the PH curve $\mathbf{p}$. The $k\mathbf{q}, k \in \mathbb{R}$, is the preimage of the curve $k^2\mathbf{p}$.

Combining the two previous observation we get the following method for obtaining the pure reparameterization of the curve.

Proposition 2.5 Let $\mathbf{q}(t)$ be the preimage of the PH curve $\mathbf{p}(t)$ and k, $l \in \mathbb{R}, k > 0$. Then $\sqrt{k} \cdot \mathbf{q}(kt + l)$ is the preimage of the PH curve $\mathbf{p}(kt + l)$.

The signed curvature of the PH curve can be obtained from its preimage in the following way.

Proposition 2.6 Let a PH kivka
$\mathbf{p} : I \to \mathbb{R}^2$ has the preimage $\mathbf{q}$. Then the signed curvature of $\mathbf{p}$ can be expressed as

$$\kappa_z(t) = 2\frac{\mathrm{Im}(\bar{\mathbf{q}}(t)\mathbf{q}'(t))}{|\mathbf{q}(t)|^4}. \tag{3}$$

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Proof: This formula can be found at [1] without the proof. Let us denote $\mathbf{h} = \mathbf{p}' = (x', y')$, $\mathbf{h} = \mathbf{q}^2$, thus $\mathbf{h}' = 2\mathbf{q}\mathbf{q}' = (x'', y'')$. using the formula (2) we get

$$\kappa_z = \frac{\det(\mathbf{p}',\, \mathbf{p}'')}{\|\mathbf{p}\|^3} = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}}.$$

Moreover

$$\begin{aligned} 2\frac{\mathrm{Im}(\bar{\mathbf{q}}\mathbf{q}')}{|\mathbf{q}|^4} &= \frac{\mathrm{Im}(2\mathbf{q}\mathbf{q}'\bar{\mathbf{q}}^2)}{|\mathbf{q}|^6} = \frac{\mathrm{Im}(\mathbf{h}'\bar{\mathbf{h}})}{|\mathbf{h}|^3} = \\ &= \frac{\mathrm{Im}((x'' + iy'')(x' - iy'))}{(x'^2 + y'^2)^{3/2}} = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}} = \kappa_z. \end{aligned}$$

□

3 Smooth joints of PH cubics

In the reminder of the paper we will restrict our investigations to the cubic PH curves. In particular we will consider the joint between two cubic PH curves and the sufficient end necessary conditions for the joint to be smooth. We will suppose that the curves are connected in the $\mathcal{C}^0$ way, which can be achieved by setting suitably the integration constants. We will use the following notation: Consider two PH cubics $\mathbf{p}_1$ a $\mathbf{p}_2$ with the linear preimages $\mathbf{q}_1$ and $\mathbf{q}_2$ of the form

$$\begin{gathered} \mathbf{q}_1(t) = \mathbf{w}_0(1-t) + \mathbf{w}_1 t, \mathbf{q}_2(t) = z_0(1-t) + z_1 t, \\ \mathbf{w}_0,\, \mathbf{w}_1,\, \mathbf{z}_0,\, \mathbf{z}_1 \in \mathbb{C}, \mathbf{w}_0 = w_{01} + iw_{02},\, \mathbf{w}_1 = w_{11} + iw_{12}. \end{gathered} \tag{4}$$

The following proposition gives the conditions for the $\mathcal{G}^1$ and $\mathcal{C}^1$ continuity.

Proposition 3.1 Let $\mathbf{p}_1$, $\mathbf{p}_2$ are PH cubics with the preimages of the form (4). The curves $\mathbf{p}_1$, $\mathbf{p}_2$ are connected in the $\mathcal{G}^1$ way if and only if $\mathbf{z}_0 = c\mathbf{w}_1, c \in \mathbb{R}, c \neq 0$.
In particular the curves are connected in the $\mathcal{C}^1$ way if and only if $c = \pm 1$.

Proof: Let $\mathbf{h}_1 = \mathbf{p}_1'$, $\mathbf{h}_2 = \mathbf{p}_2'$.
First suppose $\mathbf{z}_0 = c\mathbf{w}_1, c \in \mathbb{R}, c \neq 0$. Then

$$\begin{gathered} \mathbf{h}_1(t) = \mathbf{w}_0^2(1-t)^2 + \mathbf{w}_0\mathbf{w}_1 2t(1-t) + \mathbf{w}_1^2(1-t)^2, \\ \mathbf{h}_2(t) = c^2\mathbf{w}_1^2(1-t)^2 + c\mathbf{w}_1\mathbf{z}_1 2t(1-t) + \mathbf{z}_1^2(1-t)^2. \end{gathered}$$

We thus have $\mathbf{p}_1'(1) = \mathbf{h}_1(1) = \mathbf{w}_1^2$ a $\mathbf{p}_2'(0) = \mathbf{h}_2(0) = c^2\mathbf{w}_1^2$. The tangent vectors thus have the same direction and orientation and we have the $\mathcal{G}^1$ continuity.

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Moreover the reparameterization of $\mathbf{p}_2$ by the function $\phi(t) = t/c^2$ provides the $\mathcal{C}^1$ continuity. Denote

$$\mathbf{P}_0 = \frac{\mathbf{w}_0^2 + \mathbf{w}_0\mathbf{w}_1 + \mathbf{w}_1^2}{3}, \mathbf{P}_1 = \mathbf{P}_0 + \frac{c^2\mathbf{w}_1^2}{3},$$
$$\mathbf{P}_2 = \mathbf{P}_1 + \frac{c\mathbf{w}_1\mathbf{z}_1}{3}, \mathbf{P}_3 = \mathbf{P}_2 + \frac{\mathbf{z}_1^2}{3}.$$

It holds $\mathbf{p}_2(t) = \mathbf{P}_0(1-t)^3 + \mathbf{P}_1 3t(1-t)^2 + \mathbf{P}_2 3t^2(1-t) + \mathbf{P}_3 t^3$, tedy

$$(\mathbf{p}_2 \circ \phi)(t) = \mathbf{P}_0\left(1-\frac{t}{c^2}\right)^3 + \mathbf{P}_1\frac{3t}{c^2}\left(1-\frac{t}{c^2}\right)^2 + \mathbf{P}_2\frac{3t^2}{c^4}\left(1-\frac{t}{c^2}\right) + \mathbf{P}_3\frac{t^3}{c^6}.$$

Differentiating we get

$$(\mathbf{p}_2 \circ \phi)'(t) = \mathbf{P}_0\left(-\frac{3}{c^2} + \frac{6t}{c^4} - \frac{3t^2}{c^6}\right) + 3\mathbf{P}_1\left(\frac{1}{c^2} - \frac{4t}{c^4} + \frac{3t^2}{c^6}\right) +$$
$$+ 3\mathbf{P}_2\left(\frac{2t}{c^4} - \frac{3t^2}{c^6}\right) + \mathbf{P}_3\frac{3t^2}{c^6},$$

and thus $(\mathbf{p}_2 \circ \phi)'(0) = -\frac{3\mathbf{P}_0}{c^2} + \frac{3\mathbf{P}_1}{c^2} = \mathbf{w}_1^2 = \mathbf{p}_1'(1)$.
For the inverse implication if the connection is $\mathcal{G}^1$ we have $\mathbf{p}_1'(1) = k\mathbf{p}_2'(0), k > 0$, and thus $\mathbf{h}_1(1) = \mathbf{w}_1^2 = k\mathbf{h}_2(0) = k\mathbf{z}_0^2$. Setting $c = \pm\sqrt{k} \neq 0$ we get the statement.
For $c = \pm 1$ there holds $\phi(t) = t/c^2 = t$, the reparameterization is thus the identity and we have the $\mathcal{C}^1$connection. □

The following proposition gives the conditions for the $\mathcal{G}^2$ and $\mathcal{C}^2$ continuity.

Proposition 3.2 Let $\mathbf{p}_1$, $\mathbf{p}_2$ are PH cubics with the preimages (4) and let $\mathbf{z}_0 = c\mathbf{w}_1, c \neq 0$. The curves $\mathbf{p}_1, \mathbf{p}_2$ are connected in the $\mathcal{G}^2$ way if and only if the point $\mathbf{z}_1$ is on the straight line

$$w_{12}x - w_{11}y + c^3(w_{01}w_{12} - w_{11}w_{02}) = 0.$$

The curves are connected in the $\mathcal{C}^2$ way if and only if $c = \pm 1$ and
$c(\mathbf{z}_1 - \mathbf{z}_0) = c(\mathbf{z}_1 - c\mathbf{w}_1) = \mathbf{w}_1 - \mathbf{w}_0$, tedy $\mathbf{z}_1 = c(2\mathbf{w}_1 - \mathbf{w}_0)$.

Proof: The $\mathcal{G}^1$ connection is ensured by the previous proposition. We must obtain the same signed curvature on both segments. Let us denote κ_1 (resp. κ_2) the signed curvature of the curve $\mathbf{p}_1$ (resp. $\mathbf{p}_2$). For $\mathbf{q}_1 = \mathbf{w}_0(1-t) + \mathbf{w}_1 t$ we get using 2.6

$$\kappa_1(t) = 2\frac{\operatorname{Im}(\bar{\mathbf{q}}_1(t)\mathbf{q}_1'(t))}{|\mathbf{q}_1(t)|^4} =$$
$$= \frac{2[(w_{01} + t(w_{11} - w_{01}))(w_{12} - w_{02}) - (w_{11} - w_{01})(w_{02} + t(w_{12} - w_{02}))]}{[(w_{01} + t(w_{11} - w_{01}))^2 + (w_{02} + t(w_{12} - w_{02}))^2]^2}$$

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and for $\mathbf{q}_2 = c\mathbf{w}_1(1-t) + \mathbf{z}_1 t$, $\mathbf{z}_1 = [z_{11}, z_{12}]$ we get

$$\kappa_2(t) = 2\frac{\mathrm{Im}(\bar{\mathbf{q}}_2(t)\mathbf{q}_2'(t))}{|\mathbf{q}_2(t)|^4} =$$
$$= \frac{2[(cw_{11} + t(z_{11} - cw_{11}))(z_{12} - cw_{12}) - (z_{11} - cw_{11})(cw_{12} + t(z_{12} - cw_{12}))]}{[(cw_{11} + t(z_{11} - cw_{11}))^2 + (cw_{12} + t(z_{12} - cw_{12}))^2]^2}.$$

We require $\kappa_1(1) = \kappa_2(0)$, which leads to

$$\frac{2(w_{01}w_{12} - w_{11}w_{02})}{(w_{11}^2 + w_{12}^2)^2} = \frac{2c(w_{11}z_{12} - w_{12}z_{11})}{((cw_{11})^2 + (cw_{12})^2)^2},$$

which gives the condition for $\mathbf{z}_1$, namely

$$c^3(w_{01}w_{12} - w_{11}w_{02}) = w_{11}z_{12} - w_{12}z_{11}.$$

For the $\mathcal{C}^2$ continuity we must have $\mathbf{p}_1''(1) = \mathbf{p}_2''(0)$. There holds

$$\mathbf{p}_1''(1) = \mathbf{p}_2''(0) \Leftrightarrow \mathbf{h}_1'(1) = \mathbf{h}_2'(0) \Leftrightarrow 2\mathbf{w}_1(\mathbf{w}_1 - \mathbf{w}_0) = 2c\mathbf{w}_1(\mathbf{z}_1 - c\mathbf{w}_1) \Leftrightarrow$$
$$\Leftrightarrow (\mathbf{w}_1 - \mathbf{w}_0) = c(\mathbf{z}_1 - c\mathbf{w}_1), c = \pm 1.$$

□

4 Examples

Let us demonstrate the continuity results on several examples.

Example 4.1 *Let*

$$\mathbf{w}_0 = [1,\ 1],\ \mathbf{w}_1 = [0,\ -1],\ \mathbf{z}_0 = c\mathbf{w}_1 = [0,\ -3/2],\ \mathbf{z}_1 = [-1,\ 0], c = 3/2.$$

By Proposition 3.1 the resulting curves should be connected in the $\mathcal{G}^1$ way. Indeed

$$\mathbf{p}_2'(0) = (-9/4,\ 0) = c^2\mathbf{p}_1'(1) = \frac{9}{4}(-1, 0).$$

On Figure 1 there is the linear preimage $\mathbf{q}_1$, $\mathbf{q}_2$, and on Figure 2 the resulting PH curves. $\mathcal{C}^1$ connection is a special case of the $\mathcal{G}^1$ connection and is obtained by setting $c = \pm 1$. For $c = 1$ we get $\tilde{\mathbf{z}}_0 = [0,\ -1]$ and indeed $\tilde{\mathbf{p}}_2'(0) = \mathbf{p}_1'(1) = (-1, 0)$, see Figures 3 and 4. The preimage $\tilde{\mathbf{q}}_2$ has the control points $\tilde{\mathbf{z}}_0$, $\mathbf{z}_1$ and $\tilde{\mathbf{p}}_2$ is the resulting PH curve.

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Example 4.2 *Let*

$$\mathbf{w}_0 = [1,\, 0],\, \mathbf{w}_1 = [2,\, 2],\, \mathbf{z}_0 = c\mathbf{w}_1 = [4,\, 4],\, \mathbf{z}_1 = [0,\, 8], c = 2.$$

For $\mathbf{z}_1$ *on the line* $x-y+8=0$ *we get by 3.2 the* $\mathcal{G}^2$ *connectivity. On the figure 5 we see the preimage of the resulting PH along the the condition line (red). At the Figure 6 we see the resulting PH curves having at the joint points identical signed curvatures* $\kappa_{z1}(1) = \kappa_{z2}(0) = 1/16$.
In order to have the $\mathcal{C}^1$ *continuity let us set* $c = -1$*, and thus* $\tilde{\mathbf{z}}_0 = [-2,\, -2]$*. To obtain the* $\mathcal{G}^2$ *continuity the control point* $\mathbf{z}_1$ *must be on the line* $x - y - 1 = 0$*. If we set* $\mathbf{z}_1 = -2\mathbf{w}_1 + \mathbf{w}_0 = [-3,\, -4]$ *we obtain even the* $\mathcal{C}^2$ *continuity. We can see the results on Figures 7 and 8.*

5 Conclusion

We have solved the problem of the connection of order one and two for planar PH cubics. In the future we plan to investigate higher degree PH curves and express the connection results in the formalism of B-splines.

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