Hermite interpolation with HE-splines

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Abstract. Hypo/epicycloids (shortly HE-cycloids) are well-known curves, in detail studied in classical geometry. Hence, one may wonder what new can be said about these traditional geometric objects. It has been proved recently, cf. [12], that all rational HE-cycloids are curves with rational offsets, i.e. they belong to the class of rational Pythagorean hodograph curves. In this paper, we present an algorithm for G^1 Hermite interpolation with hypo/epicycloidal arcs which results from their support function representation.

Keywords: Hypocycloids; epicycloids; Pythagorean hodograph curves; rational offsets; support function; Hermite interpolation

1 Introduction

Hypocycloids and epicycloids are a subclass of the so called roulettes, i.e., planar curves traced by a fixed point on a closed convex curve as that curve rolls without slipping along a second curve. In this case, both curves are circles, cf. [13, 1, 9]. Various sizes of circles generate different hypo/epicycloids, which are closed when the ratio of the radius of the rolling circle and the radius of the other circle is rational.

It has been proved recently, cf. [12], that all rational hypo/epicycloids are curves with Pythagorean normals and therefore they provide rational offsets. This new result was obtained with the help of support function (SF) representation of hypo/epicycloids. In this paper, we present the main idea of the algorithm for G^1 Hermite interpolation with hypo/epicycloidal arcs. Due to the their simple support function representation, the G^1 interpolation with hypo/epicycloidal arcs becomes a linear problem.

The remainder of the paper is organized as follows. Section 2 recalls some basic facts concerning elementary theory of hypo/epicycloids and their support functions. In, Section 3 we formulate an algorithm for G^1 Hermite interpolation with hypo/epicycloidal arcs and discuss the main idea. Then, the algorithm is demonstrated on examples. Then, we conclude the paper.

2 Support function representation of hypo/epicycloids

A *hypocycloid* is a plane curve generated by the trace of a fixed point $\mathbf{c}$ on a circle with radius r that rolls without sliding within a fixed circle with radius $R > r$. If the fixed circle is centered at the origin, the hypocycloid is parameterized by

$$\mathbf{c}(\varphi) = (R - r) \cdot \mathbf{n}(\varphi) - r \cdot \mathbf{n}(k\varphi), \quad k = 1 - \frac{R}{r}, \tag{1}$$

where $\mathbf{n}(\varphi) = (\sin\varphi, \cos\varphi)^\top$. In case the moving circle roles outside the fixed circle, we generate an *epicycloid*, whose equation is obtained by using a negative value r in (1). W.l.o.g. we may assume $R > 2r$ in the rest of this paper (for more details see the so-called *Double Generating Theorem* (1785) by D. Bernoulli). We introduce the name *HE-cycloid* which is a unified term for both hypo/epicycloids.

Recently, the support function representation of curves and surfaces has been applied to some problems in CAGD – see [10, 5]. It was shown that this representation is, among others, very suitable for describing offsets and convolutions as these operations correspond to simple algebraic operations of the associated support functions.

We consider algebraic curves in the two-dimensional Euclidean plane, which is identified with $\mathbb{R}^2$. Recall that an algebraic curve $\mathcal{C}$ in $\mathbb{R}^2$ which is not a line has the *dual representation* of the form

$$D(\mathbf{n}, h) = 0, \tag{2}$$

where D is a homogeneous polynomial in $\mathbf{n} = (n_1, n_2)^\top$ and h. The set of all lines

$$T_{\mathbf{n},h} := \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{n} \cdot \mathbf{x} = h\}, \text{ for } D(\mathbf{n}, h) = 0 \tag{3}$$

forms the system of *tangents* of the curve $\mathcal{C}$. The vector $\mathbf{n}$ is the normal vector. If $|\mathbf{n}| = 1$ then the value of h is the oriented distance of the tangent to the origin.

If the partial derivative $\partial D/\partial h$ does not vanish at $(\mathbf{n}_0, h_0) \in \mathbb{R}^3$ and $D(\mathbf{n}_0, h_0) = 0$ holds, then (2) implicitly defines a function

$$\mathbf{n} \mapsto h(\mathbf{n}) \tag{4}$$

in a certain neighborhood of $(\mathbf{n}_0, h_0) \in \mathbb{R}^2$. The restriction of this function to the unit circle $S_1 = \{\mathbf{n} \in \mathbb{R}^2 : |\mathbf{n}| = 1\}$ is then called the *support function* and analogously $D(\mathbf{n}, h) = 0$ is the *implicit support function* (or shortly *ISF*) *representation* of the curve $\mathcal{C}$.

Inversely, from any smooth real function on S_1 we can reconstruct the corresponding curve by the mapping $x_h : S_1 \to \mathbb{R}^2$

$$x_h(\mathbf{n}) = h(\mathbf{n})\mathbf{n} + h'(\mathbf{n})\mathbf{n}^\perp, \tag{5}$$

where $\mathbf{n}^\perp \in S_1$, $\mathbf{n} \cdot \mathbf{n}^\perp = 0$, cf. [11]. The vector-valued function x_h gives a parameterization of the envelope of the set of tangents (3). Hence, all curves with the associated rational support function are rational — it is enough to substitute to (5) a rational parameterization of S_1, e.g. $\mathbf{n} = (2t/(1+t^2), (1-t^2)/(1+t^2))^\top$.

The following list summarizes how the support function $h(\mathbf{n})$ is related to translation, rotation and scaling of the curve (for more details see [11]):

1. *translation*: $h(\mathbf{n}) \mapsto h(\mathbf{n}) + \mathbf{v} \cdot \mathbf{n}$, where $\mathbf{v}$ is the translation vector;
2. *rotation*: $h(\mathbf{n}) \mapsto h(\mathbf{A}\mathbf{n})$, where $\mathbf{A}$ is a matrix from $SO(2)$;
3. *scaling*: $h(\mathbf{n}) \mapsto \lambda h(\mathbf{n})$, where $\lambda \in \mathbb{R}$ is the scaling factor.

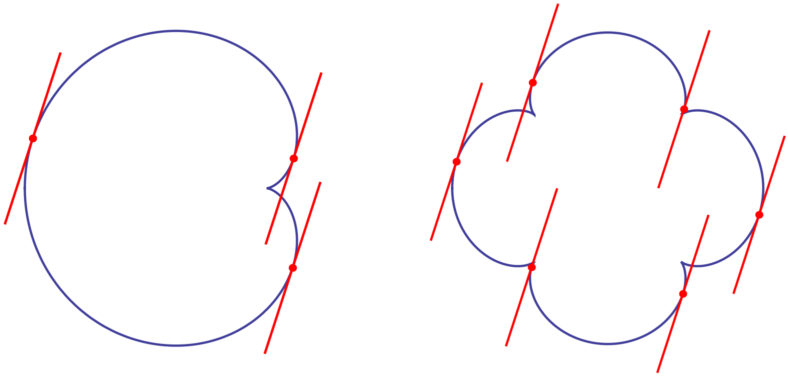

Figure 1: Left: Cardioid (HE-cycloid $\mathcal{C}_3^1$); Right: HE-cycloid $\mathcal{C}_3^2$ – with tangents of the given direction.

Lemma 1. *Any HE-cycloid given by the parameterization* (1) *possesses the support function in the form*

$$h(\theta) = (R - 2r)\cos\left(\frac{R}{R-2r}\theta\right) \tag{6}$$

with respect to the parameterization $\mathbf{n}(\theta) = (\sin\theta, \cos\theta)^\top$.

Proof. The main idea is analogous to the proof in [10] where so called quasi-convex hypocycloids were partially studied. □

In what follows we assume $R : r$ is rational and we set

$$\varrho = \gcd(R, r),\ a = R/\varrho,\ b = (R-2r)/\varrho. \tag{7}$$

Hence, (6) can be rewritten into the form

$$h(\theta) = (b\varrho)\cos\left(\frac{a}{b}\theta\right). \tag{8}$$

The multiplication by a constant factor represents scaling and thus we can omit the factor $(b\varrho)$ from (8).

Definition 2. *The curve given by the support function*

$$h(\theta) = \cos\left(\frac{a}{b}\theta\right) \tag{9}$$

with relatively prime $a, b \in \mathbb{N}$ will be called canonical HE-cycloid with parameters a, b *and denoted $\mathcal{C}^a_b$.*

Let us emphasize that the support function (9) represents an epicycloid for $a < b$ and a hypocycloid for $a > b$. Furthermore, the angular distance between cusps in the parameter domain is always $\frac{b}{a}\pi$.

3 Hermite interpolation algorithm

In this section we will present a method for G^1 Hermite interpolation with HE-cycloids. Reader who is more interested in the related interpolation problems is referred e.g. to [7].

As proved in [12], all rational HE-cycloids are rational PH curves. For more details about PH curves see [2, 8, 3, 4] and references therein. Thus, the so called *HE-splines* brings an important extra-feature, the rationality of offsets, which is a property very useful for applications, cf. [6].

Lemma 3. *The curve given by the support function*

$$h(\theta) = v_x \sin\theta + v_y \cos\theta + s \sin\frac{a}{b}\theta + c\cos\frac{a}{b}\theta. \tag{10}$$

is a scaled, rotated and translated $\mathcal{C}^a_b$ and all such transformations can be obtained by a suitable choice of coefficients $v_x, v_y, s, c \in \mathbb{R}$.

Proof. $\mathcal{C}^a_b$ scaled through a factor λ and rotated through angle α has the support function

$$\lambda\cos\left[\frac{a}{b}(\theta-\alpha)\right] = \underbrace{\lambda\sin\left(\frac{a}{b}\alpha\right)}_{s}\sin\left(\frac{a}{b}\theta\right) + \underbrace{\lambda\cos\left(\frac{a}{b}\alpha\right)}_{c}\cos\left(\frac{a}{b}\theta\right)$$

and the translation by vector $(v_x, v_y)^\top$ of any curve is realized by adding the term

$$v_x \sin\theta + v_y\cos\theta$$

to its support, as stated in Section 2. □

HE-cycloid with the support function (10) has the parameterization

$$\mathbf{x}(\theta) = h(\theta)(\sin\theta, \cos\theta)^\top + h'(\theta)(\cos\theta, -\sin\theta)^\top, \tag{11}$$

where coefficients v_x, v_y, s, c appears also linearly, cf. (5). This means, that given G^1 Hermite boundary data, we will be able to find v_x, v_y, s, c solving linear system of equation, so that (a segment of) $\mathcal{C}_b^a$ interpolates the data.

Now we suppose that they are given the following G^1 Hermite data

$$\begin{aligned} P_0 &= (x_0, y_0)^\top, & \mathbf{n}_0 &= (\sin\theta_0, \cos\theta_0)^\top, \\ P_1 &= (x_1, y_1)^\top, & \mathbf{n}_1 &= (\sin\theta_1, \cos\theta_1)^\top. \end{aligned} \tag{12}$$

Let us denote $\omega = \theta_1 - \theta_0$. We say, that the data are *regular* with respect to $\mathcal{C}_b^a$, if $P_0 \neq P_1$ and $0 < |\omega| < \min\left(\pi, \frac{b}{a}\pi\right)$. We say that an Hermite interpolant interpolating these data is *simple* if its normals vary only within the interval (θ_0, θ_1) (or within (θ_1, θ_0) if $\theta_1 < \theta_0$).

The restriction on ω is motivated by the fact that the angular distance between two consecutive cusps of $\mathcal{C}_b^a$ is $\frac{b}{a}\pi$. Taking the interpolation conditions for the given data, we obtain the following system of linear equations in the matrix form

$$\mathbf{M} \cdot \begin{bmatrix} v_x \\ v_y \\ c \\ s \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ x_1 \\ y_1 \end{bmatrix}, \tag{13}$$

where $\mathbf{M} =$

$$\begin{bmatrix} 1\ 0 & \left(\sin\theta_0 \cos\frac{a}{b}\theta_0 - \frac{a}{b}\cos\theta_0 \sin\frac{a}{b}\theta_0\right) & \left(\sin\theta_0 \sin\frac{a}{b}\theta_0 + \frac{a}{b}\cos\theta_0 \cos\frac{a}{b}\theta_0\right) \\ 0\ 1 & \left(\cos\theta_0 \cos\frac{a}{b}\theta_0 + \frac{a}{b}\sin\theta_0 \sin\frac{a}{b}\theta_0\right) & \left(\cos\theta_0 \sin\frac{a}{b}\theta_0 - \frac{a}{b}\sin\theta_0 \cos\frac{a}{b}\theta_0\right) \\ 1\ 0 & \left(\sin\theta_1 \cos\frac{a}{b}\theta_1 - \frac{a}{b}\cos\theta_1 \sin\frac{a}{b}\theta_1\right) & \left(\sin\theta_1 \sin\frac{a}{b}\theta_1 + \frac{a}{b}\cos\theta_1 \cos\frac{a}{b}\theta_1\right) \\ 0\ 1 & \left(\cos\theta_1 \cos\frac{a}{b}\theta_1 + \frac{a}{b}\sin\theta_1 \sin\frac{a}{b}\theta_1\right) & \left(\cos\theta_1 \sin\frac{a}{b}\theta_1 - \frac{a}{b}\sin\theta_1 \cos\frac{a}{b}\theta_1\right) \end{bmatrix}. \tag{14}$$

This represents the main computational part of our interpolation algorithm.

Proposition 4. *For any regular G^1 Hermite data, there is precisely one interpolant similar to a segment of $\mathcal{C}_b^a$ and it can be found by solving a system of linear equations.*

Proof. See [12] for the detailed proof of this statement. □

Now, we demonstrate the presented approach on two examples.

Example 5. For the given data (Figure 2, left), we show the unique interpolant and the three positions of a similar segment on $\mathcal{C}_5^3$ in canonical position (see Fig. 2, right – green color). Note that the number of various intervals $[\theta_0 + 2k\pi, \theta_1 + 2k\pi]$, on which the interpolant can be parameterized, is $b = 5$. The number of geometrical occurrences of the similar

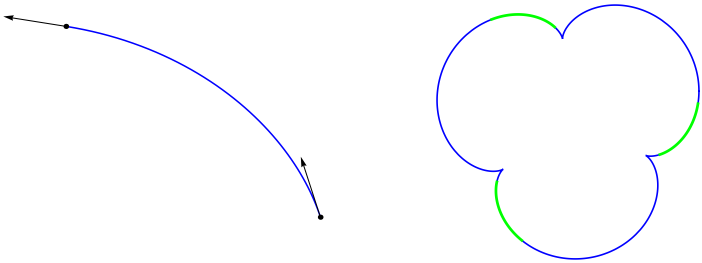

Figure 2: Left: Hermite interpolation of the given data (black) by an arc of $\mathcal{C}_5^3$ (blue); Right: $\mathcal{C}_5^3$ (blue) together with three segments (green) similar to the used interpolant.

segment is given by the symmetry group of the particular HE-cycloid and is equal to $a = 3$ in this case.

Example 6. Let us consider a Bézier quartic curve on the interval $t \in [0, 1]$ with the control points $[0, 0]$, $[0, 1]$, $[1, 2]$, $[2, 1]$ and $[1, 0]$ (see Fig. 3). The curve is replaced by 2,4 and 8 interpolating arcs of $\mathcal{C}_3^1$ (cardioid).

It can be shown by measuring the approximation error and its improvement (ratio of two consecutive errors), that when no inflections are present then the approximation order is 4 and at the inflections it drops to 3 (which is due to the absence of inflection points on HE-cycloids).

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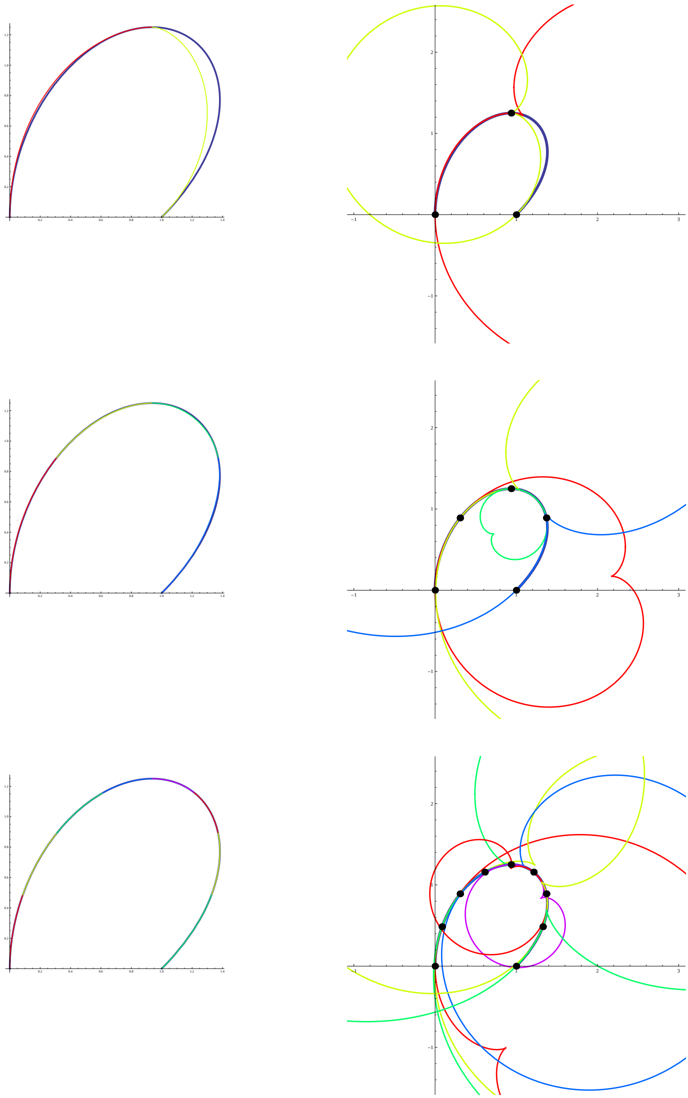

Figure 3: Bézier curve and its conversion in HE-spline curve composed of 2, 4 and 8 arcs of $\mathcal{C}_3^1$ (cardioid).

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