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HERMITE INTERPOLATION BY SPACE PH CURVES

Abstract

A new method of construction of the space PH curves matching given C2 border data is presented. The constructed examples show its utility not only for an approximation of space curves but also for a more efficient approximation of planar curves.

Key words

Hermite interpolation, Pythagorean hodograph curves, computer aided geometric design

1 Introduction

Polynomial parametric plane (space) curves $r(t) = [x(t), y(t)]$ ($r(t) = [x(t), y(t), z(t)]$) form the most important mathematical foundation of the various CAGD applications.

Pythagorean Hodograph (PH) curves, introduced in 1990 by R.T. Farouki and T. Sakkalis [1], form their interesting subclass. They are distinguished by the fact that the length of the tangent vector depends on the parameter *t* in a polynomial way. For the planar curves it means that there is a polynomial $\sigma(t)$ satisfying

$$\sigma^2(t) = x'^2(t) + y'^2(t) \quad (1)$$

This condition is similar to the one defining so-called Pythagorean triplets forming the sides of right-angled triangles. The term “hodograph” denotes simply the curve $[x'(t), y'(t)]$ representing the tangent vectors.

Comparing to the ordinary Bézier curves, the PH curves have two important supplementary properties.

First the arc length of any part of a PH curve can be computed by the formula $\int_a^b |\sigma(t)| dt$ and so no numerical integration is necessary. This property is very advantageous in all applications where the speed must be precisely controlled. This is often the case in the constructions of robots and of numerically controlled milling machines.

Secondly the offsets of PH curves can be represented by rational parametric curves. The construction (precise or approximated) of offsets (parallel curves) is extremely important in many applications and for general Bézier curve it is a difficult problem. But for a PH curve we obtain very easily the normal vector in the form $\left[-\frac{y'(t)}{\sigma(t)}, \frac{x'(t)}{\sigma(t)}\right]$ and so an offset in the distance *d* is a rational curve

$$o_d(t) = \left[x(t) - d\frac{y'(t)}{\sigma(t)}, y(t) + d\frac{x'(t)}{\sigma(t)}\right]. \tag{2}$$

In addition it has been observed that PH curves exhibit better curvature behavior, comparing to the “ordinary” polynomial parametric curves.

2 Characterization of PH curves

Similarly to the Pythagorean number triplets, three polynomials satisfy the condition (1) if and only if [2] there are three polynomials $u(t)$, $v(t)$ and $w(t)$ such that

$$x'(t) = [u^2(t) - v^2(t)]w(t),\ y'(t) = 2u(t)v(t)w(t),\ \sigma(t) = [u^2(t) + v^2(t)]w(t) \tag{3}$$

In constructions of PH curves the common factor $w(t)$ is usually supposed to be 1, because it rises the degree of the resulting curve without giving more geometrical flexibility.

In this case the hodograph $r'(t)$ is fully determined by an auxiliary curve $p(t) = [u(t), v(t)]$ which will be called *pre-image*. The final curve $r(t)$ is obtained from its hodograph by integration. If we suppose the integration constants to be zero (in this case $r(0) = [0,0]$) then the pre-image determines fully the curve $r(t)$. So the construction of a PH curve is reduced to the construction of a suitable curve $p(t)$. Clearly if the pre-image is of degree *m*, the resulting PH curve is of degree $n=2m+1$.

The equation (3) can be also interpreted in the complex numbers. If we consider the curve *p* as a complex valued one, i.e. $p(t) = u(t) + iv(t)$, then the holograph is simply its square:

$$p^2(t) = (u^2(t) - v^2(t)) + i(2u(t)v(t))$$

The mapping $p(t) \to r'(t)$ is obviously two to one. This complex representation of the pre-image is very usefully in some constructions (see [3]) because they are reduced to the complex-valued solutions of some system of equations.

The characterization of the space PH curves – satisfying

$$\sigma^2(t) = x'^2(t) + y'^2(t) + z'^2(t) \tag{4}$$

is more complicated. The sufficient-and-necessary condition for the solutions of (4) (see [4]) is the existence of polynomials: $u(t)$, $v(t)$, $r(t)$, $s(t)$ such that

$$\begin{aligned} x'(t) &= u^2(t) + v^2(t) - r^2(t) - s^2(t) \\ y'(t) &= 2u(t)r(t) + 2v(t)s(t) \\ z'(t) &= 2u(t)s(t) - 2v(t)r(t) \\ \sigma(t) &= u^2(t) + v^2(t) + r^2(t) + s^2(t) \end{aligned} \tag{5}$$

We see that in this case the pre-image $p(t) = [u(t), v(t), r(t), s(t)]$ is a four dimensional curve. As a consequence the mapping $p(t) \to r'(t)$ defined by (5) has one-dimensional fibres. This mapping can be also described in a more compact way using quaternions. If we interpret the pre-image as a quaternion $p(t) = u(t) + iv(t) + jr(t) + ks(t)$ then it is not difficult to verify that $p(t)ip(t)^*$ is a pure quaternion of the form $ix'(t) + jy'(t) + kz'(t)$ - see [5]. $p(t)^* = u(t) - iv(t) - jr(t) - ks(t)$ is the conjugate of $p(t)$. It is not difficult to verify that the fibres of the mapping $p(t) \to p(t)ip(t)^*$ are of the form $A(\cos(\Phi) + i\sin(\Phi))$ and so are isomorphic to the circle. The existence of non-trivial fibres complicates the construction of PH curves.

3 Constructions of PH curves

Some construction techniques similar to those existing for ordinary Bézier or B-spline curves (see [6]) were developed for the PH curves. So some local and global constructions matching given C1, C2, G1 and G2 data and the least–squares fitting of PH curves to given two dimensional point data were studied (see the chapter Pythagorean-hodograph curves in [7]).

One of the open problems remains the Hermite interpolation of given C2 space boundary data. We will now describe our attempt to solve this problem.

The task is to construct a space PH curve $r(t)$ satisfying prescribed C2 conditions at the end points. It means that we prescribe the values

$$r(0),\ r'(0),\ r''(0),\ r(1),\ r'(1) \text{ and } r''(1)$$

Without loss of generality we can suppose that $r(0) = [0,0,0]$ and so there remain 15 conditions (5 times 3). A suitable PH curve will be given by a four-dimensional pre-image, so if the pre-image is of degree m, we get $4(m+1)$ degrees of freedom, but one degree of freedom will be lost because of the one-dimensional fibres. So the 15 conditions could be hopefully satisfied in the case $m = 3$ and the resulting PH curve $r(t)$ would be of degree 7 and its hodograph $r'(t)$ will be of degree 6.

We will use the over-mentioned quaternion representation. Let us write the hodograph in its Berstein form

$$r'(t) = \sum_{i=0}^{6} B_i^6(t)\mathbf{h}_i \,,\ t \in \langle 0, 1 \rangle$$

where $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ are the Bernstein and the control points $\mathbf{h}_i$ are pure quaternions. From the elementary theory of Bézier

$$\mathbf{h}_0 = r'(0)\,,\ \mathbf{h}_6 = r'(1)\,,\ \mathbf{h}_1 - \mathbf{h}_0 = \tfrac{1}{6} r''(0) \text{ and } \mathbf{h}_6 - \mathbf{h}_5 = \tfrac{1}{6} r''(1)$$

So the prescribed $r'(0)$, $r''(0)$, $r(1)$, $r'(1)$ uniquely determine the control points $\mathbf{h}_0, \mathbf{h}_1$, $\mathbf{h}_5$ and $\mathbf{h}_6$. The last and the most difficult condition to satisfy is the prescribed value of $r(0)$.

Now let us write the pre-image curve in the Berstein form $p(t) = \sum_{i=0}^{3} B_i^n(t)\mathbf{p}_i$. The identity $r'(t) = p(t)ip(t)^*$ gives us seven equations for the control points, but only following four are important for our purpose:

$$\mathbf{h}_0 = \mathbf{p}_0 i \mathbf{p}_0^*\,,\ \mathbf{h}_6 = \mathbf{p}_3 i \mathbf{p}_3^* \qquad (6)$$

$$\mathbf{h}_1 = \tfrac{1}{2}(\mathbf{p}_0 i \mathbf{p}_1^* + \mathbf{p}_1 i \mathbf{p}_0^*)\,,\ \mathbf{h}_5 = \tfrac{1}{2}(\mathbf{p}_2 i \mathbf{p}_3^* + \mathbf{p}_3 i \mathbf{p}_2^*) \qquad (7)$$

From (6) the control points $\mathbf{p}_0$, $\mathbf{p}_3$ can be computed with one degree of freedom (multiplication by $\cos(\Phi) + i\sin(\Phi)$) but without loss of generality we can suppose $\Phi = 0$ for $\mathbf{p}_0$.

The right-hand-sides of (7) are in fact equal to the vector parts of quaternions $\mathbf{p}_0 i \mathbf{p}_1^*$ and $\mathbf{p}_2 i \mathbf{p}_3^*$ and therefore (7) it can be rewrote as

$$\mathbf{h}_1 + \tau_1 = \mathbf{p}_0 i \mathbf{p}_1^* \text{ and } \mathbf{h}_5 + \tau_2 = \mathbf{p}_2 i \mathbf{p}_3^*,$$

where τ_1 and τ_2 are free real parameters. It means that we can compute the control points $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ with three degrees of freedom τ_1, τ_2 and Φ for the point $\mathbf{p}_3$. These free parameters will be determined by the position of the point $r(1)$.

Now from the pre-image we can compute the hodograph and by integration the final curve $r(t)$ depending on the parameters and the prescribed values for $r(1)$ give us a system of three equations for τ_1, τ_2

and Φ . This system is highly non-linear and up to now we were not able to prove in general the existence and the properties of the solutions. In concrete cases it must be solved by appropriate numerical methods.

4 Examples

One of the main use of the Hermite interpolation is the approximation of a given curve. The first example we give is constructed from the data taken from the space helix of equation (the doted curve on the figure 1), giving the initial conditions

$r(0) = [0, 0, 0]$, $r'(0) = [1, 0, 1]$, $r''(0) = [0, 1, 0]$,

$r(1) = [1, 1, \pi / 2]$, $r'(1) = [0, 1, 1]$, $r''(1) = [-1, 0, 0]$.

The constructed PH curve (the solid line on the figure 1) approximates very well the original helix.

The second example is very interesting. We can approximate by our method plane curves also. Let us consider the ellipse of equation

$$[\sin(\frac{t\pi}{2}), 2 - 2\cos(\frac{t\pi}{2}), 0] .$$

The resulting system of equation for the free parameters is in this case

$$\left(58\tau_1 - 46\tau_2\right)\sqrt{2}\sin(\Phi) + \left(73 + 36\tau_1\tau_2\right)\sqrt{2}\cos(\Phi) + 48\tau_1^2 - 324 = 0$$
$$\left(46\tau_1 - 58\tau_2\right)\sqrt{2}\sin(\Phi) + \left(73 + 36\tau_1\tau_2\right)\sqrt{2}\cos(\Phi) + 48\tau_2^2 - 324 = 0$$
$$-\left(46 + 36\tau_1\tau_2\right)\sqrt{2}\sin(\Phi) + \left(46\tau_1 - 46\tau_2\right)\sqrt{2}\cos(\Phi) + 16(\tau_1 - \tau_2) = 0$$

The solution, which is „closest“ to the ellipse, is presented on the figure 2. It is a space PH curve but its maximal distance from the xy-plane on the interval $t \in \langle 0, 1 \rangle$ is 0.0013 and its projection to the xy-plane is virtually indiscernible from a quarter of the ellipse. For a similar quality of approximation by a planar PH curve it would be necessary to use at least a curve of degree 9.

We were able to construct by our method good approximations for many other curves. So the presented algorithm seems to be a good construction pattern for the space C2 Hermite interpolation. It opens also the possibility to use the higher dimensional curves as auxiliary control objects for the less dimensional ones.

Acknowledgement: This work has been supported by the grant no. 201/03/D113 of the Czech Science Foundation.

Fig. 1

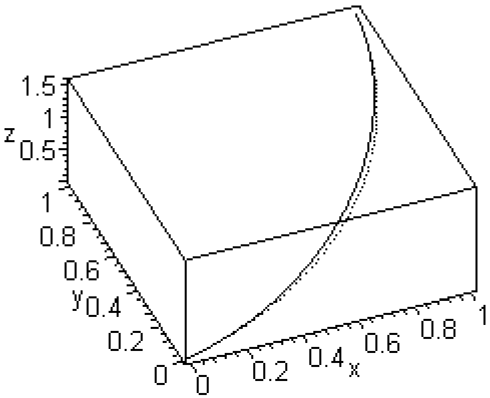
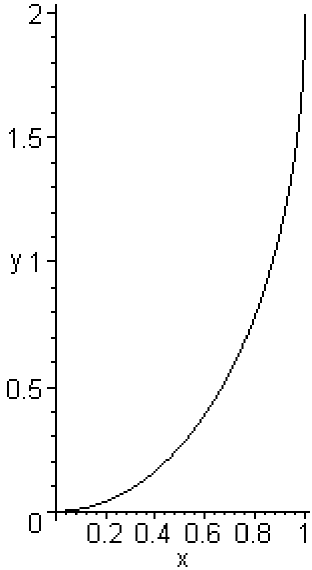


Fig. 2



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