Local Properties of Algebraic Curves Using Rational Puiseux Series

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Abstract. In this paper we apply the rational Puiseux series to study the local properties of algebraic curves at their singular points. In particular we exploit the existence of a bijection between the curve real branches and set of rational Puiseux series at a given point of the curve. We determine the quadrant which contains any curve half-branch and find the mutual position of all the branches. All this information is extracted from a certain tree representation without the necessity of computing the Puiseux series explicitly. This study is meant as an element for our new method for a topologically accurate approximation of algebraic curves.

Keywords: rational Puiseux series, local topology, algebraic curve, singularity, branch

1 Introduction

In [2, 3] we proposed a novel approach to study the topology of algebraic curves. It is based on three fundamental steps. In the first we identify the singular (and some other critical) points and all the branches at these points. Then we study the connectivity of these branches. Eventually we approximate all the connecting segments.

For the seek of completeness let us recall that their exist also a completely different approach which is based on subdivision. The only certified algorithm (i.e. one which gives the correct output for every input) based on subdivision is [1]. This algorithm subdivides the studied region into *regular regions* (the curve is smooth inside) and *regions with singular points*, which can be made sufficiently small. The topology inside the regions containing a singular point is recovered from the information on the boundary using the topological degree.

In this paper we present some results which improves the first step of our algorithm, more precisely the identification of all the branches at a singular point. First we describe the system of rational Puiseux series introduced in [5]. These are certain generalizations of the standard Puiseux series [6, chapter IV] which are in bijection with real branches (trough the origin) of a given algebraic curve. We also shortly recall the algorithm to find rational Puiseux series. In the second section we improve the ideas given in [4]. We deduce the local position of the branches. The singular part of Puiseux series determines in which quadrant(s) the given branch lies. Also, the singular part implies the clockwise order of the curve branches.

2 Rational Puiseux Series

Through this paper we suppose that the polynomial $f(x, y) \in \mathbb{Q}[x, y]$ is a monic and irreducible in $\mathbb{C}[x, y]$. We denote by $\mathcal{C}_f$ the set of corresponding affine points

$$\mathcal{C}_f = \{[x, y] \in \mathbb{R}^2 \mid f(x, y) = 0\}.$$

Definition 1. Let C_f be a curve, t be a variable and $[\tilde{x}, \tilde{y}] \in (\mathbb{C}[t] \backslash \mathbb{Q})^2$. $[\tilde{x}, \tilde{y}]$ is called a *parametrization* of C_f if $f(\tilde{x}, \tilde{y}) = 0$.

Parametrization $[\tilde{x}_1, \tilde{y}_1]$ and $[\tilde{x}_2, \tilde{y}_2]$ are *equivalent* if there exists $z \in \mathbb{R}[[t]]$ linear in t such that $\tilde{x}_1(z) = \tilde{x}_2$ and $\tilde{y}_1(z) = \tilde{y}_2$.

The equivalence classes of irreducible parametrizations of $\mathcal{C}_f$ are called *branches*.

Definition 2. The field of *Puiseux series* is

$$\mathbb{Q}\langle\langle t \rangle\rangle = \bigcup_{k=1}^{\infty} \mathbb{Q}((t^{1/k})),$$

where $\mathbb{Q}((t^{1/k}))$ denotes formal Laurent series in $t^{1/k}$.

In [6, IV.3] it is shown that $\mathbb{Q}\langle\langle x \rangle\rangle$ is algebraically closed. It means that the roots of f as a polynomial in y are Puiseux series in x. We will call these roots *Puiseux series of f*.

Definition 3. To each Puiseux series $\sum_{i \in \mathbb{Z}} a_i t^{i/n}$ corresponds the *parametrization* $[\tilde{x}, \tilde{y}]$ defined as follows:

$$\tilde{x}(t) = t^n \qquad \tilde{y}(t) = \sum_{i \in \mathbb{Z}} a_i t^i. \tag{1}$$

For applications, the disadvantage of Puiseux series is that more parametrizations corresponding to Puiseux series can be equivalent, i.e. that more Puiseux series can describe the same branch of the curve. Duval in [5] introduced the system of rational Puiseux series, where each parametrization corresponds to precisely one branch.

Definition 4. Let $y_1, y_2, \ldots, y_s$ be Puiseux series of f and $\{[\tilde{x}_j, \tilde{y}_j]\}_{j=1}^{s}$ be corresponding parametrization. Let r denote the maximal number of non-equivalent parametrization (branches) between $\{[\tilde{x}_j, \tilde{y}_j]\}_{j=1}^{s}$. *System of rational Puiseux series* of f over $\mathbb{Q}$ is the set

$$\{[\tilde{x}'_j, \tilde{y}'_j]\}_{j=1}^{r}$$

of pairwise non-equivalent irreducible parametrization of $\mathcal{C}_f$, which is invariant under the action of the Galois group $\mathcal{G}(\mathbb{C}/\mathbb{Q})$ and for each j, $\tilde{x}'_j = \lambda_j t^{n_j}$, where $n_j > 0$ and $\lambda_j \neq 0$.

In the following text we will use *rational Puiseux series* to refer to a parametrization from the system of rational parametrizations.

The key property of rational Puiseux series, proved in [5], is summarized here:

Theorem 1. *Let $f(x,y) \in \mathbb{R}[x,y]$ and $\{[\tilde{x}_k, \tilde{y}_k]\}_k$ be a rational Puiseux series of f over $\mathbb{R}$. Then the branch $[\tilde{x}_k, \tilde{y}_k]$ is real if and only if the coefficient of $\tilde{x}_k$ and every coefficient of $\tilde{y}_k$ are real numbers.*

Algorithm for finding a system of rational Puiseux series

The algorithm describes how to find the local parametrization of all branches above the origin. If we are interested in the branches above an another point, we can translate the coordinate system. The structure above the point $[a,b]$ is obtained by examination $\tilde{f} = f(x+a, y+b)$ at the origin.

The algorithm is recursive and very similar to the algorithm to find the standard Puiseux series (see [6, IV.3]). It is usually described using the recursion, but we find more transparent use the terminology of trees. We can say that the algorithm is based on the tree traversal.

Let describe how to compute the tree of the given polynomial f. The tree is generally infinite and has nodes of three types

- the root of the tree – a given polynomial,
- nodes of type N (shortly N-nodes) – "Newton polygon edge" and
- nodes of type C (shortly C-nodes) – "coefficients of Puiseux series."

N-nodes and C-nodes periodically alternate.

The number of tree branches is same as the number of curve branches. The Puiseux series of a given curve branch is fully determined by the information attached to the nodes on the tree branch.

At the root and in C-nodes we compute Newton polygon of f. It consists of several edges (children of type N). Every edge is fully described by its equation $pi + qj = l$ and certain characteristic equation $h(z)$. To the node we attach the quadruple (q, p, l, h).

In N-nodes we search for coefficients of Puiseux series (γ and δ). They depends on p, q and root ϱ of $h(z)$, so for every ϱ we have one child of type C. More precisely $\gamma = \varrho^{-v}$ and $\delta = \varrho^{u}$, where $uq + vp = 1$. We also compute new polynomial g. To the node we attach the quadruple $(\varrho, \gamma, \delta, g)$.

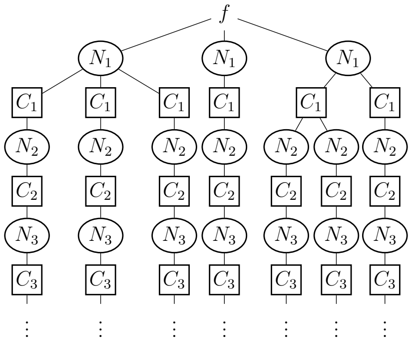

Figure 1: Example of the tree corresponding to some polynomial f.

Usually we are interested in finite number of terms of rational Puiseux series, i.e. in finite sub-tree. The natural choice is singular sub-tree defined in the following definition. The reason why it is so important is in Proposition 1.

Definition 5 (with proposition)**.** Let the tree branch be the sequence of nodes $B = (f, N_1,\ C_1,\ N_2, C_2, \dots)$. There exists j_0 such that for every $k > k_0$ it holds

$$\gamma_k = q_k = 1 \quad \text{ and } \quad \delta_k \in \mathbb{Q}(\gamma_1, \delta_1, \gamma_2, \delta_2, \dots, \gamma_{k_0}, \delta_{k_0}).$$

The part of the tree branch $(f, N_1, C_1, N_2, C_2, \dots, N_{k_0}, C_{k_0})$ is called *singular*. The rest of the branch $(N_{k_0+1}, C_{k_0+1}, N_{k_0+2}, C_{k_0+2}, \dots)$ is called *regular* part of the branch. The sub-tree consisting of singular part of each branch is called *singular sub-tree* of T.

Proof. The existence can be proven using observation in [6, page 102] and direct computations. □

Proposition 1. *Let $B_1, B_2 \dots B_r$ be the branches of the tree of f. The local topology at the origin is influenced only by singular parts of B_k, where $k = 1, 2, \dots, r$.*

Proof. See [5]. □

Directly from the recursive equations, we can deduce the parametrizations of Puiseux series:

Theorem 2. *Let $B = (f, N_1, C_1, N_2, C_2, \dots, N_w, C_w)$ be a part of a branch of the tree of f. Let the length of singular part of B be k_0. The*

Puiseux series of the curve branch is

$$P(B) = [x, y] = \left[\lambda t^n, \sum_{j=1}^{k_0} \xi_j t^{c_j} + \sum_{j=k_0+1}^{w} \chi_j t^{d_j} + \cdots \right] \tag{2}$$

where

$$\lambda = \prod_{k=1}^{k_0} \gamma_k^{\prod_{i=1}^{k-1} q_i}, \qquad n = \prod_{k=1}^{k_0} q_k,$$

$$\xi_k = \delta_k \prod_{l=1}^{k} \left(\prod_{i=l+1}^{k_0} \gamma_i^{\prod_{k=l+1}^{i-1} q_k} \right)^{p_l}, \quad c_k = \sum_{i=1}^{k} \left(p_i \prod_{k=i+1}^{k_0} q_k \right),$$

$$\chi_k = \delta_k \prod_{l=1}^{k_0} \left(\prod_{i=l+1}^{k_0} \gamma_i^{\prod_{k=l+1}^{i-1} q_k} \right)^{p_l}, \quad d_k = c_{k_0} + \sum_{i=k_0+1}^{k} p_i. \tag{3}$$

3 Local geometry over given point

In this part we describe the local topology over a given point. In the first part we choose two quadrants, where the given branch can lie. In the second part we determine in which of possible quadrants the branch really lies. And in the last section we determine the mutual position of the branches. Any such information easily follows from the first terms of Puiseux series of the curve branch. Here, we describe how to extract these characteristics from the singular part of the tree branch.

3.1 Branch position - possible quadrants

In this subsection we describe in which quadrants a given branch can lie. We can obtain this information very fast from the first two tree nodes. The disadvantage is that the found quadrants are only possible ones, i.e. the branch can lie in only one or in both of them.

Assume that the quadrants are numbered as usual (the quadrant number 1 is $x > 0$ & $y > 0$ and then counterclockwise.

Definition 6. The point $[x_0, y_0]$ is called *regular* point of f if at least one derivative $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ is nonzero. Otherwise it is called *singular*.

The topology in regular point is always clear. We are interested in local topology in singular points, where should be more branches.

Proposition 2. *Let B be a real branch through a singular point determined by the tree branch. The first N-node (namely p, q) and first C-node (namely the sign of ϱ) restricts the local position of B as stated in Table 1.*

q	p	sign(ϱ)	possible quadrants
odd	odd	+	1 & 3
odd	odd	-	2 & 4
even	odd	+	1 & 4
even	odd	-	2 & 3
odd	even	+	1 & 2
odd	even	-	3 & 4

Table 1: Local position of a given curve branch using only first two inner nodes of the tree branch.

Proof. Let $(f, N_1, C_1, N_2, C_2, \dots)$ be the tree branch corresponding to B. The Puiseux series using only the first N and C-node is

$$[\gamma_1 t^{q_1}, \delta_1 t^{p_1}].$$

The local parametrization of B using the whole singular part of the tree branch has the form

$$[\gamma_1 x_1^{q_1}, \delta_1 x_1^{p_1} + x_1^{p_1} y_1],$$

with $x_1 = \lambda t^n$ and $y_1 = \sum_{j=2}^{k_0} \xi_j t^{c_j}$ where $\lambda = \prod_{k=2}^{k_0} \gamma_k^{\prod_{i=1}^{k-1} q_i}$, $n = \prod_{k=2}^{k_0} q_k$, $\xi_k = \delta_k \prod_{l=2}^{k} \left(\prod_{i=l+1}^{k_0} \gamma_i^{\prod_{k=l+1}^{i-1} q_k} \right)^{p_l}$ and $c_k = \sum_{i=2}^{k} \left(p_i \prod_{k=i+1}^{k_0} q_k \right)$. Note that the regular part of the tree branch has no effect on topology (see Prop. 1). The position in quadrants is given by the sign of x-coordinate $(\gamma_1 x_1^{q_1})$ and the sign of y-coordinate, which is near zero influenced only by the first term $\delta_1 x_1^{p_1}$.

From the previous paragraphs it is clear that the Puiseux series corresponding to first two nodes really approximate the position of the branch. When the signs of x_1 for $t = \pm\epsilon$ (suppose ϵ positive infinitesimal) are different the branch lies in both quadrants, when the signs are equal, the branch B lies only in one of given quadrants.

Denote $B = [B_x, B_y] = [\gamma t^q, \delta t^p]$. Recall that $\gamma = \varrho^{-v}$ and $\delta = \varrho^u$. We can distinguish 3 cases according to parities of q and p. In every case we treat only the case $\varrho < 0$, the case $\varrho > 0$ is similar. Denote $s = \mathrm{sign}(\varrho) = -1$.

Case p and q odd

We need to find the sign of γ and δ.

$$\operatorname{sign}(\gamma) = \operatorname{sign}(\varrho^{-v}) = \operatorname{sign}(\varrho^{-vp})$$
$$\operatorname{sign}(\delta) = \operatorname{sign}(\varrho^{u}) = \operatorname{sign}(\varrho^{uq})$$

Using the fact that $uq + vp = 1$ we have

$$\operatorname{sign}(\gamma)\operatorname{sign}(\delta) = \operatorname{sign}(\varrho^{-vp})\operatorname{sign}(\varrho^{uq}) = \operatorname{sign}(\varrho^{-vp})\operatorname{sign}(\varrho^{1-vp}) = -1$$

The last equation is because the difference between $-vp$ and $1 - vp$ is 1 and therefore one term is odd and one even.

So we have to distinguish two cases:
If $\operatorname{sign}(\varrho^{-v}) = -1$ and $\operatorname{sign}(\varrho^{u}) = 1$ then for $t = \epsilon$ we have $B_x < 0, B_y > 0$ and B lies in second quadrant and for $t = -\epsilon$ we have $B_x > 0, B_y < 0$ and B lies in quadrant 4.

If $\operatorname{sign}(\varrho^{-v}) = 1$ and $\operatorname{sign}(\varrho^{u}) = -1$ then for $t = \epsilon$ we have $B_x > 0, B_y < 0$ and B lies in quadrant 4 and for $t = -\epsilon$ we have $B_x < 0, B_y > 0$ and B lies in quadrant 2.

Case q even and p odd
As in the previous case $\operatorname{sign}(\gamma) = \operatorname{sign}(\varrho^{-v}) = \operatorname{sign}(\varrho^{-vp})$ and because q is even, we have $\operatorname{sign}(\varrho^{uq}) = 1$ therefore

$$\operatorname{sign}(\gamma) = \operatorname{sign}(\varrho^{-vp})\operatorname{sign}(\varrho^{uq}) = \operatorname{sign}(\varrho^{-vp})\operatorname{sign}(\varrho^{1-vp}) = -1$$

So, we have two possibilities:
If u is even, then $\operatorname{sign}(\varrho^{u}) = 1$ and for $t = \epsilon$: $B_x < 0, B_y > 0$ and B lies in second quadrant and for $t = -\epsilon$ we have $B_x < 0, B_y < 0$ and B lies in quadrant 3.

If u is odd, then $\operatorname{sign}(\varrho^{u}) = 1$ and for $t = \epsilon$: $B_x < 0, B_y < 0$ and B lies in third quadrant and for $t = -\epsilon$ we have $B_x < 0, B_y > 0$ and B lies in quadrant 2.

Case q odd and p even is analogous to the previous one.

Case q and p even can not arise, because it will be in contradiction with the irreducibility of Puiseux series. □

Example 1. To demonstrate the proposition we use the curve defined by equation $f = 2y^5 - xy^3 + 3x^2y^3 + 2x^2y^2 - x^5y^2 - x^3y + 2x^5$. We know that $\mathcal{C}_f$ has three real branches through the origin, see Fig. 2. First branch is solid, second is dashed and third is dotted.

First branch has first two nodes $N_1 = (1, 2, 5, -z+2)$, $C_1 = (2, 1, 2, f_1)$, more explicitly $q = 1$ is odd, $p = 2$ is even, $\varrho = 2$ is positive and therefore the branch can lie in quadrants 1 and 2.

Similarly second branch has nodes $(1, 1, 4, -(z-1)^2)$, $(1, 1, 1, f_1)$, i.e. $q = 1$ is odd, $p = 1$ is odd, $\varrho = 1$ is positive and the branch has position 1 & 3.

And third branch has nodes $N_1 = (2, 1, 5, 2z-2)$, $C_1 = (1/2, 2, 1, f_1)$, i.e. $q = 2$ is even, $p = 1$ is odd, $\varrho = 1/2$ is positive and the branch has position 1 & 4.

3.2 Branch position - exact quadrants

The branch can lie in both possible quadrants but does not have to. In this section we describe how to decide in which of possible quadrants the branch really lies. Again, this information is contained in the first terms of Puiseux series. We extract the information from the singular part of the tree branch, so we do not need to compute the Puiseux series explicitly.

The following observation is corollary of proof of Proposition 2.

Corollary 1. *Let B be a real curve branch through the singular point determined by the tree branch. Let the singular part of the branch has height 2. Then the branch B lies in both quadrants given in Prop. 2.*

Example 1 (continuous). The singular part of first and third tree branch has height 2. Due to Corollary 2 the first branch pass through both quadrants 1 and 2 and the third branch pass through quadrants 1 and 4.

If the singular part of the tree branch is higher than two, it is possible that the curve branch lies only in one of quadrants given in Prop. 2.

Proposition 3. *Let B be a real curve branch going through a singular point and the corresponding singular part of the tree branch have nodes $(f, N_1, C_1, N_2, C_2, \ldots, N_{k_0}, C_{k_0})$. Let i & j are possible quadrants of position of B given by Prop. 2.*

If q_1 is odd and $q_2 q_3 \cdots q_{k_0}$

- *is odd then B lies in both quadrants i and j.*
- *is even then B lies in only one quadrant i or j. Which one can be recognized by determining the sign of $B_x = \lambda t^n$ (see (2)).*

If q_1 is even and $p_1 q_2 q_3 \cdots q_{k_0}$,

- *is odd then B lies in both quadrants i and j.*
- *is even then B lies in only one quadrant i or j. The quadrant is given by the sign of first term of B_y, i.e. $\xi_1 t^{c_1}$.*

Proof. In the first case (q_1 odd) the possible quadrants are given by $\gamma_1 x_1^{q_1}$. Because $x_1 = \lambda t^n$, the signs of x_1 for $t = \pm\epsilon$ are different when $n = \prod_{i=2}^{k_0} q_i$ is odd. If n is even, the sign of x_1 is the same for $t = \pm\epsilon$. The sign of B_x is dependent also on λ, i.e. the parities of q_i and the signs of γ_i $(i = 2, 3, \ldots, k_0)$. The sign of B_x determines the quadrant, where B lies.

In the second case (q_1 even) the sign of B_x is same for possible quadrants and the question is whether B_y has same sign for $t = \pm\epsilon$. The y-coordinate is near zero influenced only by the first term $\xi_1 t^{c_1}$. If $c_1 = p_1 q_2 q_3 \cdots q_{k_0}$ is odd, the signs of B_y are different for $t = \pm\epsilon$ and B lies in both quadrants. If c_1 is even, the signs of B_y are same and B lies only in one quadrant which is fully determined by the signs of $\delta_1, \gamma_2, \gamma_3, \ldots, \gamma_{k_0}$ and the parities of $q_2, q_3, \ldots, q_{k_0}$. $\square$

Example 2 (continuation). The singular part of the second branch B_2 is $(f,\ N_1 = (1,1,4,-(z-1)^2),\ C_1 = (1,1,1,f_1),\ N_2 = (2,1,2,-z+7),\ C_2 = (7,1/7,1,f_2))$ $q_1 = 1$ is odd and $q_2 = 2$ is even therefore B_2 lies only in one quadrant. $\lambda > 0$ because $\gamma_1 = 1 > 0$ and $\gamma_2 = 1/7 > 0$. n is odd, because $q_2 = 2$ is odd. We conclude that $B_x > 0$ for $t = \pm\epsilon$ and B_2 lies in second quadrant.

3.3 Mutual position of branches

Every curve branch has two natural half-branches, one for $t > 0$ and the second for $t < 0$. Using singular parts of tree branches it is possible to resolve the order of half-branches of the curve through a given singular point. We are interested in the order of half-branches on the right side (quadrants 1, 4) and left side (quadrants 2, 3) of the point separately. In this section (without loss of generality) we assume that all the half-branches are in quadrants 1 and 4. The case of quadrants 2, 3 is analogous.

To compare the half-branches we need to number them. One natural numbering follows from previous section - let the branch i (numbered in the tree from the left) consists of half-branches π_{2i-1} and π_{2i}. It will be useful to define for every branch π_j associated functions q_j which gives the number of quadrant of branch π_j.

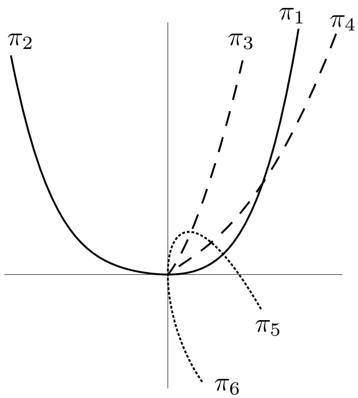

Figure 2: Numbering of branches

Definition 7. Assume that branches $q_i = q_{i'}$. Due to the implicit function theorem, we can consider the branches as the functions of x coordinate, i.e. $y = \pi_i(x)$ resp. $y = \pi_{i'}(x)$. We say that $\pi_i < \pi_{i'}$ if there exists a neighborhood of the origin, where for every x is $\pi_i(x) < \pi_{i'}(x)$.

Suppose that $q_i = 4$ and $q_{i'} = 1$. It is clear that $\pi_i < \pi_{i'}$.

Otherwise both compared curve branches are in the same quadrant. Assume that $B = [B_x, B_y] = [\lambda t^n, \sum_{k=1}^{k_0} \xi_k t^{c_k}]$ and $B' = [B'_x, B'_y] = [\lambda' t'^{n'}, \sum_{k=1}^{k'_0} \xi'_k t'^{c'_i}]$ are the parametrizations of the branches using theirs singular parts of tree branches. We consider same (infinitesimal) values of B_x and B'_x and we ask whether B_y is greater than B'_y or vise versa.

Without loss of generality assume that $c_1 \leq c'_1$. To simplify the notation denote $w = (\lambda/\lambda')^{1/n'}$. From $B_x = B'_x$, we can deduce that $t' = (\lambda/\lambda')^{1/n'} t^{n/n'} = wt^{n/n'}$. We denote $\tilde{c}_1 := (n/n')c'_1$.

The reciprocal position of branches follows from the following scheme

- $c_1 = \tilde{c}_1$
  - $\xi_1 = w^{c'_1}\xi'_1$. Repeat this decision procedure with c_2, c'_2 (resp. c_{i+1}, c'_{i+1}). As the branches are different, they have different singular parts of tree branches and the process terminates.
  - $\xi_1 < w^{c'_1}\xi'_1$ then $B < B'$.
  - $\xi_1 > w^{c'_1}\xi'_1$ then $B > B'$.
- $c_1 < \tilde{c}_1$
  - $\xi_1 > 0$ then $B > B'$
  - $\xi_1 < 0$ then $B < B'$
- $c_1 > \tilde{c}_1$
  - $\xi'_1 w^{c'_1} > 0$ then $B < B'$
  - $\xi'_1 w^{c'_1} < 0$ then $B > B'$

If the half-branch corresponds to $t < 0$ we substitute $\tilde{t} = -t$ and compare the modified local parametrization.

Proof. $B < B'$ if and only if $\lim\limits_{t \to 0^+} \operatorname{sign}(B_y - B'_y) = 1$. We have

$$\lim_{t \to 0^+} \operatorname{sign}(B_y - B'_y) = \lim_{t \to 0^+} \operatorname{sign}\left(\sum_{k=1}^{k_0} \xi_k t^{c_k} - \sum_{k=1}^{k'_0} \xi'_k t^{c'_k}\right) =$$

$$= \lim_{y \to \infty} \operatorname{sign}\left(\sum_{k=1}^{k_0} \xi_k \frac{1}{y^{c_k}} - \sum_{k=1}^{k'_0} \xi'_k \frac{1}{y^{c_k}}\right) = \lim_{y \to \infty} \operatorname{sign}\left(\sum_{i=m}^{m'} \frac{1}{y^i}(\xi_{(i)} - \xi'_{(i)})\right),$$

where $m = \min_{i'=1,\dots,k'_0}^{i=1,\dots,k_0} (c_i, c'_{i'})$, $m' = \max_{i'=1,\dots,k'_0}^{i=1,\dots,k_0} (c_i, c'_{i'})$ and $\xi_{(i)}, \xi'_{(i)}$ are the coefficients of t^i in B_y resp. B'_y.

Let o be the smallest number for which $\xi_{(o)} \neq \xi'_{(o)}$. Then

$$\lim_{t\to 0^+} \operatorname{sign}(B_y - B'_y) = \lim_{y\to\infty} \operatorname{sign}\left(\frac{1}{y^o}(\xi_{(o)} - \xi'_{(o)})\right),$$

which implies the proposition. □

Example 1 (continuation). We can number the half-branches of first branch π_1, π_2 with $q_1 = 1$, $q_2 = 2$. Denote the half-branches of second branch π_2, π_3 with $q_2 = 1$, $q_3 = 1$. Third branch half-branches are π_5, π_6 with $q_5 = 1$, $q_6 = 4$. Everything is marked in Figure 2.

On the left side of the origin is only one branch π_2. On the right side, we have five half-branches. In the quadrant 4 is only one branch π_6. The rest of branches is in the first quadrant. Their position is following:

- $\pi_5 > \pi_3$ because $n = n' = 2$ and $c_1 = 1 < 2 = c'_1$ and $\xi_1 = 1 > 0$.
- $\pi_3 > \pi_4$ because $n = n' = 2$, $c_1 = c'_1 = 2$, $w = 1$ and $\xi_1 = \xi'_1$, but $c_2 = c'_2 = 3$ and $\xi_2 = 1/7 > -1/7 = \xi'_2$.
- $\pi_4 > \pi_1$ because $n = 2 \neq 1 = n'$, $t' = t^2/7$ and $c_1 = 2 < 4 = \tilde{c}_1$ and $\xi_1 = 1/7 > 0$.

4 Conclusion

We have presented an improvement on the existing algorithm for finding the rational Puiseux series and the corresponding curve branches. We used the formalism of trees instead of the terminology of recursion. The output we use to find the position of branches in quadrants and the mutual position of branches without the necessity of computing whole Puiseux series.

These information are very important to deduce the correct topology of the given algebraic curve. We plan to exploit it to improve the global topology results in [2].

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