Lifting Cubics into (Minkowski) Space

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Abstract. For a given planar cubic $\mathbf{c}(t) = [X(t), Y(t)]^{\mathrm{T}}$, we identify the system of Pythagorean hodograph (PH) cubics of the form $[X(t), Y(t), Z(t)]^{\mathrm{T}}$ in Euclidean and in Minkowski spaces. We show that there exists one pair of Euclidean and two pairs of Minkowski PH cubics, satisfying $Z(a) = Z(b) = 0$ and therefore interpolating the points $\mathbf{c}(a)$, $\mathbf{c}(b)$. These curves provide upper and lower polynomial bounds on the parametric speed and the arc–length function of $\mathbf{c}(t)$, and can be used for approximating its offset curves. The error and the convergence under subdivision are analyzed.

§1. Introduction

Pythagorean Hodograph (PH) curves, which were introduced in 1990 by Farouki and Sakkalis [4], form a remarkable subclass of polynomial parametric curves. They are distinguished by having a polynomial arc length function and rational offsets. These curves provide an elegant solution of various difficult problems occurring in applications, in particular in the context of CNC (computer-numerical-control) machining. For instance, offsets do not have to be approximated, they can be represented exactly. Also, the arc length of a PH curve can be computed without numerical integration, which speeds up the algorithms for numerically controlled (NC) machining

Various constructions and computational techniques for interpolation and approximation exist. Techniques for Hermite interpolation in the plane were developed e.g. in [11, 15] and in three–dimensional space e.g. in [7, 9, 10]. Global interpolation of point data was studied in [6] and least–squares fitting in [5]. See also the survey [8] and the references cited therein.

PH curves in Minkowski space (MPH curves) were introduced by Moon [13] in the context of the medial axis transform. Recently, several constructions for MPH curves have been described [2, 12]. Clifford algebras provide a unifying approach to PH and MPH curves [3].

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In this paper we explore the application of (M)PH curves to the approximation of the arc length function and of the offsets of a planar cubic curve. The main idea is as follows: The cubic is approximated by (M)PH curves, by lifting it into (Minkowski) space. The parametric speed functions of the (M)PH curves provide upper and lower polynomial bounds on the parametric speed of the cubic, which are used for offset and arc length approximation with certified error bounds.

In [14] we construct Minkowski and Euclidean PH cubics which interpolate the derivative vector at a given point of the planar cubic. In the present paper using different methods, we analyse the whole system of (M)PH curves lying over a planar cubic. The connection to the previous result is summarized in the conclusion.

The remainder of the paper is organized as follows. After recalling some basic concepts, we consider in §3 the system of (Minkowski) Pythagorean curves of the form $[x(t), y(t), z(t)]$ for a given quadratic curve $\mathbf{p}(t) = [x(t), y(t)]^{\mathrm{T}}$. Here, the main tool is the factorization of the polynomial $x^2(t) + y^2(t)$ over $\mathbb{C}$. A similar approach was taken in [1, 12] in order to solve different problems. For a given parametric interval $[a, b]$ we are in particular interested in curves yielding (M)PH space curves interpolating the end points $\mathbf{c}(a)$ and $\mathbf{c}(b)$. In §4 the constructed Pythagorean curves are used in order to bound the distance function $||\mathbf{p}(t)||$, which is in fact the speed function of the cubic $\mathbf{c}(t)$. The bounds on the speed function of $\mathbf{c}(t)$ are then applied in §5 in order to bound the arc-length function of $\mathbf{c}(t)$ and to approximate its offsets. We devote the last section to conclusions.

§2. Preliminaries

Throughout this paper we consider a planar cubic $\mathbf{c}(t)$ and its (quadratic) hodograph $\mathbf{p}(t) = \mathbf{c}'(t) = [x(t), y(t)]^{\mathrm{T}}$. Let Π_2 denote the linear space of all real-valued quadratic polynomials in t, which is identified with $\mathbb{R}^3$. Except for §5 we will always refer to a fixed parameter domain (i.e., an interval) $[a, b] \subset \mathbb{R}$. Finally, we shall use the linear functional $\mathcal{A}$ which represents the average value of a quadratic polynomial on $[a, b]$,

$$\mathcal{A}\,(q) := \frac{1}{b-a}\int_a^b q(t)dt = q_2\frac{a^2+ab+b^2}{3} + q_1\frac{a+b}{2} + q_0, \tag{1}$$

where $q(t) = q_2t^2 + q_1t + q_0 \in \Pi_2$.

We recall the notion of planar Pythagorean curves from [1] and extend it to three–dimensional Euclidean and Minkowski spaces.

Definition 1. *A curve $[x(t), y(t), z(t)]$ is called a Pythagorean curve (PC), or Minkowski Pythagorean curve (MPC), if there exists a polynomial $\sigma(t)$, such that $x^2(t)+y^2(t)+z^2(t) = \sigma^2(t)$, or $x^2(t)+y^2(t)-z^2(t) = \sigma^2(t)$, respectively.*

According to this definition, a curve is PH (or MPH) if its hodograph is a PC (or an MPC).

§3. (Minkowski) Pythagorean Curves Over a Parabola

Let $\mathbf{p}(t) = [x(t), y(t)]$ be a planar non-degenerate quadratic curve (a parabola) not passing through the origin for $t \in \mathbb{R}$ (i.e., $\gcd(x, y) = 1$). Let us consider the factorization over $\mathbb{C}$

$$x(t) + iy(t) = K(t - \xi_1)(t - \xi_2). \tag{2}$$

Neither ξ_1 nor ξ_2 can be real, since $x(t)$ and $y(t)$ would have a common root otherwise, implying that $\mathbf{p}(t)$ passes through the origin.

For future reference, we introduce the real polynomials

$$\begin{array}{rclrcl} \phi_1(t) & = & |K|(t-\xi_1)(t-\bar{\xi}_1), & \phi_2(t) & = & |K|(t-\xi_2)(t-\bar{\xi}_2) \\ \tilde{x}(t) & = & \Re[K(t-\xi_1)(t-\bar{\xi}_2)], & \tilde{y}(t) & = & \Im[K(t-\xi_1)(t-\bar{\xi}_2)], \end{array} \tag{3}$$

the curve

$$\tilde{\mathbf{p}}(t) = [\tilde{x}(t), \tilde{y}(t)]^{\mathrm{T}}, \tag{4}$$

and the distances of the centers of gravity of the segments $\{\mathbf{p}(t), t \in [a, b]\}$ and $\{\tilde{\mathbf{p}}(t), t \in [a, b]\}$ from the origin

$$\delta = \sqrt{\mathcal{A}^2(x) + \mathcal{A}^2(y)}, \quad \tilde{\delta} = \sqrt{\mathcal{A}^2(\tilde{x}) + \mathcal{A}^2(\tilde{y})}. \tag{5}$$

We consider all quadratic curves of the form $[x(t), y(t), Z(t)]$, resp. $[x(t), y(t), z(t)]$, which are Pythagorean in three–dimensional Euclidean, resp. Minkowski space. These curves are said to lie 'over' the given parabola. This is equivalent to characterizing the system of (M)PH curves over a planar cubic, since each (M)PC over $\mathbf{p}$ corresponds to a family of (M)PH curves over $\mathbf{c}$, whose $z-$coordinates differ only by the integration constants.

Theorem 1. *The system of all $Z(t)$ forms a hyperbola in Π_2, while the system of all $z(t)$ forms two ellipses in Π_2.*[1] *All these conics have the common center $0 \in \Pi_2$. If $\mathbf{p}(t)$ is a planar PC, then the hyperbola collapses into a straight line and one of the ellipses into a line segment, otherwise the curves are non-degenerate.*

Proof: The curves $[x(t), y(t), Z(t)]$ resp. $[x(t), y(t), z(t)]$ are (M)PC if and only if there are $\Sigma(t), \sigma(t) \in \Pi_2$, such that

$$\begin{aligned} x^2(t) + y^2(t) &= \Sigma^2(t) - Z^2(t) = [\Sigma(t) + Z(t)]\,[\Sigma(t) - Z(t)] \qquad (6) \\ &= \sigma^2(t) + z^2(t) = [\sigma(t) + iz(t)]\,[\sigma(t) - iz(t)]\,. \qquad (7) \end{aligned}$$

[1]Recall that Π_2 is identified with $\mathbb{R}^3$. Explicit expressions of $Z(t)$, $z(t)$ and corresponding σ and Σ are given in the proof.

From the factorization (2) we get

$$x^2(t) + y^2(t) = |K|^2(t - \xi_1)(t - \bar{\xi}_1)(t - \xi_2)(t - \bar{\xi}_2), \tag{8}$$

and by comparing with (6) we obtain due to the unique factorization of real polynomials

$$\Sigma(t) + Z(t) = \tau\phi_i(t), \quad \Sigma(t) - Z(t) = \frac{\phi_j(t)}{\tau},$$

for $\{i, j\} = \{1, 2\}$ and $\tau \in \mathbb{R} - \{0\}$, where ϕ_1, ϕ_2 are given by (3). Thus, all $Z(t)$ obtained for both choices of i, j are expressed as

$$Z(t) = \frac{\tau\phi_1(t)}{2} - \frac{\phi_2(t)}{2\tau}, \quad \tau \in \mathbb{R} - \{0\}. \tag{9}$$

This equation defines a hyperbola in Π_2, which collapses into a straight line if and only if ϕ_1 is a multiple of ϕ_2, which is equivalent to $\xi_1 = \xi_2$ or $\xi_1 = \bar{\xi}_2$ and therefore to the fact that $\mathbf{p}(t)$ is a Pythagorean curve [1, Prop. 2.3]. The associated polynomials $\Sigma(t)$ are

$$\Sigma(t) = \pm\left(\frac{\tau\phi_1(t)}{2} + \frac{\phi_2(t)}{2\tau}\right). \tag{10}$$

On the other hand, (7) is a factorization of $x^2(t) + y^2(t)$ in two complex conjugate quadratic factors. Each must have precisely one factor from the two sets

$$\{(x - \xi_1), (x - \bar{\xi}_1)\} \text{ and } \{(x - \xi_2), (x - \bar{\xi}_2)\},$$

as proved more generally in [12, Theorem 5]. Considering carefully all the symmetries, the possible polynomials z are expressed as

$$z_1(t) = \Im\left[e^{i\omega}K(t - \xi_1)(t - \xi_2)\right] = \sin(\omega)x(t) + \cos(\omega)y(t), \tag{11}$$

or

$$z_2(t) = \Im\left[e^{i\omega}K(t - \xi_1)(t - \bar{\xi}_2)\right] = \sin(\omega)\tilde{x}(t) + \cos(\omega)\tilde{y}(t), \tag{12}$$

where $\omega \in [0, 2\pi)$ and $\tilde{x}, \tilde{y}$ are defined by (3). These two families of solutions form two ellipses in Π_2. The ellipse $z_1(t)$ (or z_2(t)) collapses to a line segment if and only if $x(t)$ and $y(t)$ (or $\tilde{x}(t)$ and $\tilde{y}(t)$) are linearly dependent, which is equivalent to $\xi_1 = \bar{\xi}_2$ (or $\xi_1 = \xi_2$). In these cases, $\mathbf{p}(t)$ is a Pythagorean curve. The associated polynomials σ are given by

$$\begin{array}{rcl} \sigma_1(t) & = & \pm\Re\left[e^{i\omega}K(t - \xi_1)(t - \xi_2)\right] = \pm\left[\cos(\omega)x(t) - \sin(\omega)y(t)\right] \\ \sigma_2(t) & = & \pm\Re\left[e^{i\omega}K(t - \xi_1)(t - \bar{\xi}_2)\right] = \pm\left[\cos(\omega)\tilde{x}(t) - \sin(\omega)\tilde{y}(t)\right] \end{array} \tag{13}$$

This completes the proof. □

The solutions (11) and (12) will be referred to as the MPC of the first and second kind over $\mathbf{p}$, respectively.

Among the Pythagorean curves described in Theorem 1 we are interested in those satisfying $\mathcal{A}(z) = 0$, see (1). This condition is equivalent to $\int_a^b z(t)dt = 0$ and therefore the corresponding (M)PH cubics interpolate the end points $\mathbf{c}(a), \mathbf{c}(b)$ (have vanishing space coordinate for $t = a$ and $t = b$).

Theorem 2. *Let $\mathbf{p}(t) = [x(t), y(t)]$ be a nondegenerate non–Pythagorean quadratic curve, which does not pass through the origin for $t \in \mathbb{R}$.*

1) *There exists precisely one pair of PC $[x(t), y(t), \pm Z(t)]$, such that $\mathcal{A}(\pm Z) = 0$.*
2) *If $\delta \neq 0$ then there exists exactly one pair of MPC of the first kind $[x(t), y(t), \pm z_1(t)]$, such that $\mathcal{A}(\pm z_1) = 0$.*
3) *If $\tilde{\delta} \neq 0$ then there exists exactly one pair of MPC of the second kind $[x(t), y(t), \pm z_2(t)]$, such that $\mathcal{A}(\pm z_2) = 0$.*

Proof: See (1), (5) for definitions of $\mathcal{A}$ and $\delta, \tilde{\delta}$. Since both $\mathcal{A}(\phi_1)$ and $\mathcal{A}(\phi_2)$ are positive, Eq. (9) implies that

$$\mathcal{A}(Z) = \frac{\tau}{2}\mathcal{A}(\phi_1) - \frac{1}{2\tau}\mathcal{A}(\phi_2) = 0 \tag{14}$$

if and only if

$$\tau = \pm\sqrt{\frac{\mathcal{A}(\phi_2)}{\mathcal{A}(\phi_1)}}. \tag{15}$$

Similarly, if $\delta \neq 0$ then $\mathcal{A}(z_1) = 0$ is satisfied if and only if

$$\sin(\omega) = \frac{\pm\mathcal{A}(y)}{\sqrt{\mathcal{A}^2(x) + \mathcal{A}^2(y)}}, \quad \cos(\omega) = \frac{\mp\mathcal{A}(x)}{\sqrt{\mathcal{A}^2(x) + \mathcal{A}^2(y)}}. \tag{16}$$

The proof of 3) is analogous. $\square$

If $\delta = 0$ then $\mathcal{A}(x) = \mathcal{A}(y) = 0$ and from (11) we obtain that

$$\mathcal{A}(z_1) = \sin(\omega)\mathcal{A}(x) + \cos(\omega)\mathcal{A}(y) = 0 \tag{17}$$

holds for all z_1. Note that $\delta = 0$, (and similarly $\tilde{\delta} = 0$), means that $[0, 0]^{\mathrm{T}}$ is the center of gravity of the segment of $\mathbf{p}$ (resp. $\tilde{\mathbf{p}}$). As confirmed by a short computation, for a given quadratic curve this may happen for at most one interval $[a, b]$.

§4. Bounding the Parametric Speed of the Cubic

The absolute value of the polynomial $\Sigma(t)$ associated to the PC from 1) of Theorem 2 is an upper bound of $||\mathbf{p}(t)||$, which is the parametric speed of the cubic $\mathbf{c}(t)$. Similarly, the absolute values of polynomials $\sigma_1(t)$ and $\sigma_2(t)$ which are associated with the MPC from 2-3) of Theorem 2 are lower bounds on $||\mathbf{p}(t)||$.

Combining (15) with (10) and (16) with (13), one gets

$$\Sigma(t) = \frac{\mathcal{A}(\phi_2)\phi_1(t) + \mathcal{A}(\phi_1)\phi_2(t)}{2\sqrt{\mathcal{A}(\phi_1)\mathcal{A}(\phi_2)}}, \tag{18}$$

$$\sigma_1(t) = \frac{\mathcal{A}(x)x(t) + \mathcal{A}(y)y(t)}{\sqrt{\mathcal{A}^2(x) + \mathcal{A}^2(y)}}, \quad \sigma_2(t) = \frac{\mathcal{A}(\tilde{x})\tilde{x}(t) + \mathcal{A}(\tilde{y})\tilde{y}(t)}{\sqrt{\mathcal{A}^2(\tilde{x}) + \mathcal{A}^2(\tilde{y})}}, \tag{19}$$

where we restrict ourselves to the solutions resulting from choosing the "+" sign. This leads to the polynomial bounds of the parametric speed

$$\Sigma(t) = |\Sigma(t)| \geq ||\mathbf{p}(t)|| \geq |\sigma_j(t)| \geq \sigma_j(t), \quad \text{for } j = 1, 2 \text{ and } t \in \mathbb{R}. \tag{20}$$

Lemma 1. *The differences between the polynomial bounds are bounded by*

$$\Sigma(t) - \sigma_1(t) \leq \frac{|K|^2|\xi_1 - \bar{\xi}_2|^2}{2\delta}\left[\left(t - \frac{a+b}{2}\right)^2 + \frac{1}{3}\left(\frac{b-a}{2}\right)^2\right], \tag{21}$$

$$\Sigma(t) - \sigma_2(t) \leq \frac{|K|^2|\xi_1 - \xi_2|^2}{2\tilde{\delta}}\left[\left(t - \frac{a+b}{2}\right)^2 + \frac{1}{3}\left(\frac{b-a}{2}\right)^2\right]. \tag{22}$$

Proof: From (18), (19) we get

$$\Sigma(t) - \sigma_1(t) = \frac{\mathcal{A}(\phi_2)\phi_1(t) + \mathcal{A}(\phi_1)\phi_2(t)}{2\sqrt{\mathcal{A}(\phi_1)\mathcal{A}(\phi_2)}} - \frac{2\left[\mathcal{A}(x)x(t) + \mathcal{A}(y)y(t)\right]}{2\sqrt{\mathcal{A}^2(x) + \mathcal{A}^2(y)}}. \tag{23}$$

Note that the polynomials x, y, ϕ_1, ϕ_2 are fully determined by the complex numbers K, ξ_1 and ξ_2, see (2), (3). Rewriting these numbers using the real and complex parts, one verifies that

$$\mathcal{A}(\phi_1)\mathcal{A}(\phi_2) \geq \mathcal{A}^2(x) + \mathcal{A}^2(y) = \delta, \tag{24}$$

and therefore

$$\Sigma(t) - \sigma_1(t) \leq \frac{\mathcal{A}(\phi_2)\phi_1(t) + \mathcal{A}(\phi_1)\phi_2(t) - 2\left[\mathcal{A}(x)x(t) + \mathcal{A}(y)y(t)\right]}{2\delta}. \tag{25}$$

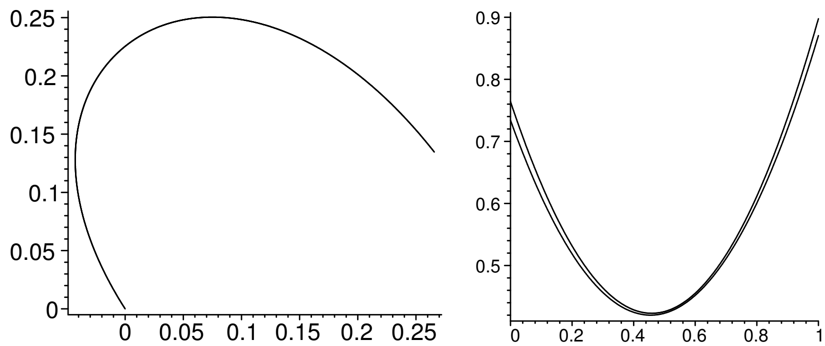

Fig. 1. Planar cubic (left), and the upper and lower bounds on its speed function (right).

Again, considering explicitly the real and complex parts of K, ξ_1 and ξ_2 one can show that the numerator of the last expression equals

$$|K|^2|\xi_1 - \bar{\xi}_2|^2 \left[\left(t - \frac{a+b}{2} \right)^2 + \frac{1}{3} \left(\frac{b-a}{2} \right)^2 \right], \tag{26}$$

which concludes the proof of (21). The result (22) follows similarly. □

Remark. The use of the upper bounds (25), (26) was motivated by considering the limit case $a = b$, where (24) and (25) become equations.

This lemma implies the following estimates of the gaps between the bounding polynomials on the interval $[a, b]$.

Theorem 3. *We have*

$$\begin{aligned} \max_{t\in[a,b]} [\Sigma(t) - \sigma_1(t)] &\leq \frac{|K|^2|\xi_1 - \bar{\xi}_2|^2}{6\delta} (b-a)^2 \quad \textit{and} \\ \max_{t\in[a,b]} [\Sigma(t) - \sigma_2(t)] &\leq \frac{|K|^2|\xi_1 - \xi_2|^2}{6\tilde{\delta}} (b-a)^2 . \end{aligned} \tag{27}$$

Consequently, depending on whether or not $\Im(\xi_1)$ and $\Im(\xi_2)$ have the same signs, $|\xi_1 - \bar{\xi}_2|$ is either greater or less then $|\xi_1 - \xi_2|$, and therefore either the solution of the second or of the first kind (σ_2 or σ_1) yields the better bound.

As a first example, Figure 1 shows a planar cubic and its arc-length function, which is very closely bounded by quadratic functions $\Sigma(t)$ and $\sigma_2(t)$ over the whole interval $[0, 1]$. The maximal gap between the bounds equals 0.027.

§5. Applications

We use the results of the previous sections to approximate the arc-length function of a given cubic $\mathbf{c}(t)$ by a polynomial, and to approximate the offset curves by rational curves.

Algorithm 1. Input: planar cubic $\mathbf{c}(t)$, parameter interval $[a, b]$

1. Calculate the hodograph $\mathbf{p}(t) = \mathbf{c}'(t) = [x(t), y(t)]^{\mathrm{T}}$ and solve the quadratic equation $x(t) + iy(t) = 0$, obtaining complex roots ξ_1, ξ_2, see (2).
2. If $\Im(\xi_1)$ and $\Im(\xi_2)$ have the same sign, use in the sequel the MPC of the second kind, otherwise use the MPC of the first kind. More precisely, depending on the sign distribution, use σ_1 or σ_2 in (19).
3. Split $[a, b]$ into n equal subintervals $[a_j, b_j]$, where $b_j = a_{j+1}$, $a_1 = a$, $b_n = b$. and for each subinterval construct local speed bounds $\Sigma_j(t)$ and $\sigma_j(t)$ using (18) and (19).
4. Collect $\Sigma_j(t)$ and $\sigma_j(t)$ into piecewise quadratic bounds $\Sigma(t)$, $\sigma(t)$ of the parametric speed $||\mathbf{c}'(t)||$ over $[a, b]$.
5. Construct upper and lower polynomial bounds on the *arc-length function* of $\mathbf{c}$:

$$S(t) = \int_a^t \Sigma(u)du, \qquad s(t) = \int_a^t \sigma(u)du. \tag{28}$$

6. Construct rational approximation of the offset of $\mathbf{c}$ at distance d:

$$\mathbf{o}_d(t) = \mathbf{c}(t) + \frac{2d}{\Sigma(u) + \sigma(t)}\mathbf{c}'(t). \tag{29}$$

Remark 1. Note that the bounds $\Sigma(t)$ and $\sigma(t)$ constructed in the step 4 are not continuous. Continuity, or even differentiability, can be obtained by blending neighboring segments $\Sigma_j(t)$, $\sigma_j(t)$. The convergence rate of the modified procedure will be preserved, provided that the blending functions (partition of unity over the interval $[a, b]$) have local support (for example, B-splines).

Remark 2. The offset approximation of the step 6 is obtained by replacing in the exact offset formula the length of the tangent vector $||\mathbf{c}'(t)||$ by the average of its bounds $\Sigma(t)$, $\sigma(t)$. In addition to this result, the offset curve is enclosed by the two rational curves

$$\mathbf{c}(t) + \frac{d}{\Sigma(u)}\mathbf{c}'(t) \quad \text{and} \quad \mathbf{c}(t) + \frac{d}{\sigma(u)}\mathbf{c}'(t). \tag{30}$$

In the second example we approximate the offsets and construct polynomial bounds on the arc-length function of the planar cubic shown in

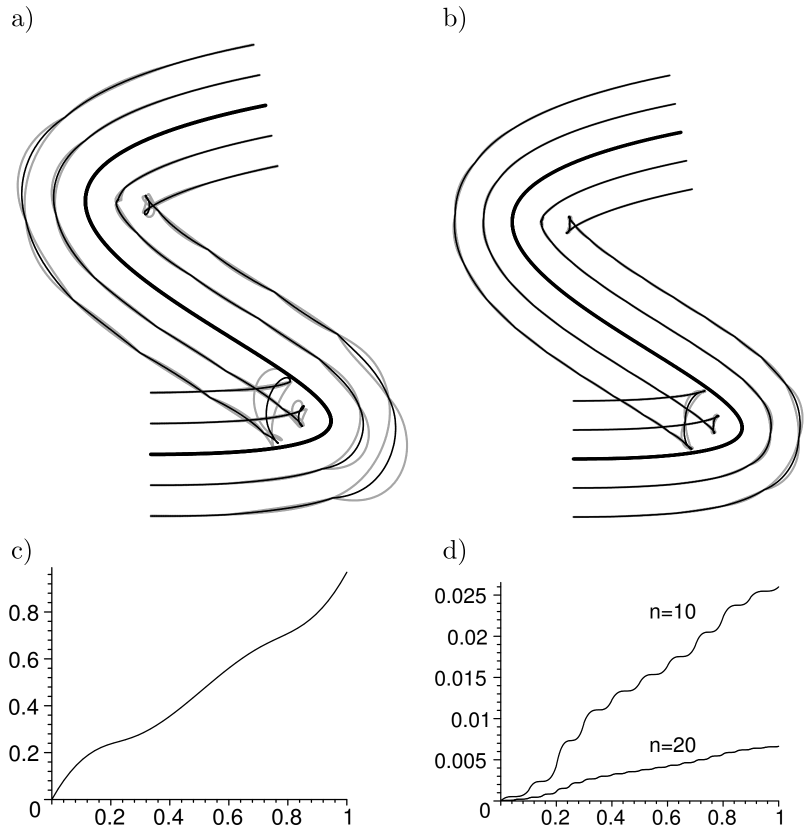

Fig. 2. (a,b): Planar cubic (thick line), with approximate offsets, where the parameter domain was split in 10 (a) and 20 (b) segments. The arc-length function of the cubic (c) and the difference $S(t) - s(t)$ between upper and lower bounds for $n = 10$ and $n = 20$ (d).

Figure 2 (thick curve). The parameter domain is split into 10 and 20 parts.

In order to obtain a continuous approximation of the offset curves, we used linear uniform B-splines as blending functions for the construction of $\Sigma(t)$ and $\sigma(t)$. The curves (30) enclosing the exact offsets are shown in grey. For the arc-length bounds we constructed $\Sigma(t)$ and $\sigma(t)$ by piecing together the bounds over each subinterval. In this situation, the continuity is ensured by the integration. The maximum gap between the arc-length bounds equals 0.019 (2.7% of the curve length) using 10 segments and 0.0048 (0.68%) using 20 segments. Halving the length of the segments reduces the error to a quarter of its previous value.

Theorem 4. *Both the maximal gap between the arc-length bounds* (28) *and the error of the offset approximation* (29) *converge to* 0 *as* $\mathcal{O}(h^2)$, *where* $h = (b-a)/n$ *is the size of the subintervals.*

Proof: Suppose, that $\Im(\xi_1)$ and $\Im(\xi_2)$ have different signs and therefore the MPC of the first kind is used. Then,

$$\max_{t\in[a,b]} [\Sigma(t) - \sigma(t)] = \max_j \max_{t\in[a_j,b_j]} [\Sigma_j(t) - \sigma_j(t)] \leq \max_j \frac{|K|^2|\xi_1 - \bar{\xi}_2|^2}{6\delta_j} h^2,$$

where $|K|$ and $|\xi_1 - \bar{\xi}_2|$ are constants and δ_j is the distance of the center of gravity of the segment $\mathbf{c}(t), t \in [a_j, b_j]$ from the origin – see (5). As the curve $\mathbf{c}$ does not pass through the origin, all δ_j will be greater then some constant $\Delta > 0$ for sufficiently small h. Therefore

$$\max_{t\in[a,b]} [\Sigma(t) - \sigma(t)] \leq \frac{|K|^2|\xi_1 - \bar{\xi}_2|^2}{6\Delta} h^2 = \mathcal{O}(h^2). \tag{31}$$

Now

$$\max_{t\in[a,b]} [S(t) - s(t)] = \int_a^b [\Sigma(t) - \sigma(t)] \mathrm{d}t \leq (b-a) \max_{t\in[a,b]} [\Sigma(t) - \sigma(t)],$$

where $(b-a)$ is the size of the starting interval and does not change under subdivision. Therefore $S(t) - s(t)$ converges to 0 as $\mathcal{O}(h^2)$, too.

The error of the offset approximation (29) evaluates to

$$\max_{t\in[a,b]} |d| \left| \frac{\Sigma(t) + \sigma(t) - 2||\mathbf{c}'(t)||}{\Sigma(t) + \sigma(t)} \right| \leq \max_{t\in[a,b]} |d| \left| \frac{\Sigma(t) - \sigma(t)}{\Sigma(t) + \sigma(t)} \right|. \tag{32}$$

For sufficiently small h the lower bound $\sigma(t)$ will be positive and

$$\Sigma(t) + \sigma(t) \geq \Sigma(t) \geq \min_{t\in[a,b]} ||\mathbf{c}'(t)|| > 0,$$

which together with (31) concludes the proof. The proof for the case when $\Im(\xi_1)$ and $\Im(\xi_2)$ have the same sign is similar. □

Note that the biggest error correspond to the parts of $\mathbf{c}$ where $||\mathbf{c}'||$ is close to 0. This is to be expected from Theorem 3, since the bound on the gap between the two polynomials is of the form $C(b-a)^2/\delta$, where δ (5) is the distance of the center of gravity of the hodograph segment $\{\mathbf{c}'(t) \,|\, t \in [a,b]\}$ from the origin (and similarly for the MPC of the second kind), where $[a, b]$ is the parameter domain. Moreover, the constant C does not depend on the parameter domain.

Based on this observation, a simple adaptive subdivision procedure can be designed. If the parametric speed is slow, then more segments are needed.

§6. Conclusion

We identified the set of all Euclidean and Minkowski space PH curves lying "over" a planar cubic. Among them, we used the curves interpolating two given points of the planar cubic for bounding its parametric speed and arc-length function and for approximating its offsets. The whole procedure is computationally cheap and simple, and it has quadratic convergence with respect to the length of the curve segment.

These results generalize our previous results in [14]. There, using different methods, we constructed Minkowski and Euclidean PH cubics which interpolate the derivative vector at a given point of the planar cubic. This can be seen as the limit case $a = b$ of this paper. In this case, the functional $\mathcal{A}(f) = q(a)$ is still defined by (1) and all the presented results remain valid.

In the limit case, the key results (21), (22) can be rewritten in the simpler form

$$\Sigma(t) - \sigma_1(t) \leq \frac{|K|^2 |\xi_1 - \bar{\xi}_2|^2}{2||\mathbf{p}(a)||}(t-a)^2 \tag{33}$$

$$\Sigma(t) - \sigma_2(t) \leq \frac{|K|^2 |\xi_1 - \xi_2|^2}{2||\mathbf{p}(a)||}(t-a)^2, \tag{34}$$

and even become equations in a certain neighborhood of a.

While the approach of the present paper is more general and the use of complex roots yields more simple expressions, the paper [14] uses directly the control points of the planar cubic and analyses in detail the case $a = b$.

In future we plan to apply similar methods to approximate space cubics and to surfaces of low degree.

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