Mathematics I

November 4, 2020

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- Preparation for other courses — Statistics, Microeconomics, ...

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- Training of logical thinking and mathematical exactness

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- understand the analysis for **sequences** and **functions**.
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- understand definitions (give positive and negative examples) and theorems (explain their meaning, neccessity of the assumptions, apply them in particular situations)
- understand mathematical proofs, give mathematically exact arguments

Mathematics I

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- Introduction

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- Sequences

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- Sequences
- Functions of one real variable

Textbooks

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- **Hájková et al: Mathematics 1**

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- Trench: Introduction to real analysis

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- Zorich: Mathematical analysis I
- Rudin: Principles of mathematical analysis

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- $A_1 \times \cdots \times A_m = \{[a_1, \ldots, a_m];\ a_1 \in A_1, \ldots, a_m \in A_m\}$ ... the Cartesian product

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Infinitely many sets: $A_1 \cup A_2 \cup A_3 \cup \dots$ is equivalent to $\bigcup\limits_{i=1}^{\infty} A_i$, and also to $\bigcup\limits_{i\in\mathbb{N}} A_i$.

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- $\Leftrightarrow$ . . . equivalence; “if and only if”

Theorem 1 (de Morgan rules)

Let S, A_α, $\alpha \in I$, where $I \neq \emptyset$, be sets. Then

$$S \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (S \setminus A_\alpha) \qquad \textit{and} \qquad S \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (S \setminus A_\alpha).$$

Example (irrationality of $\sqrt{2}$)

If a real number x solves the equation $x^2 = 2$, then x is not rational.

Rational numbers

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- The set of rational numbers

$$\mathbb{Q} = \left\{\frac{p}{q};\ p \in \mathbb{Z}, q \in \mathbb{N}\right\},$$

where $\frac{p_1}{q_1} = \frac{p_2}{q_2}$ if and only if $p_1 \cdot q_2 = p_2 \cdot q_1$.

Real numbers $\mathbb{R}$

The real numbers are sometimes also called the real line. It is the continuum of numbers where there are no gaps. We will explain later, what ”no gap” means.

Definition.

By the set of real numbers $\mathbb{R}$ we will understand a set on which there are operations of addition and multiplication (denoted by $+$ and $\cdot$), and a relation of ordering (denoted by $\leq$), such that it has the following three groups of properties.

I. The properties of addition and multiplication and their relationships.

II. The property of being ordered.

III. The infimum axiom (completion).

Definition

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The infimum axiom:
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The number g is denoted by $\inf M$ and is called the infimum of the set M.

Remark

- The infimum axiom says that every non-empty set bounded from below has infimum.

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- The infimum of the set M is its greatest lower bound.

Let $a, b \in \mathbb{R}$, $a \leq b$. We denote:

- An open interval $(a, b) = \{x \in \mathbb{R};\ a < x < b\}$,
- A closed interval $[a, b] = \{x \in \mathbb{R};\ a \leq x \leq b\}$,
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Unbounded intervals:

$$(a, +\infty) = \{x \in \mathbb{R};\ a < x\}, \quad (-\infty, a) = \{x \in \mathbb{R};\ x < a\},$$

analogically $(-\infty, a]$, $[a, +\infty)$ and $(-\infty, +\infty)$.

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A real number that is not rational is called irrational. The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of irrational numbers.

Suprema and Maxima

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Let $M \subset \mathbb{R}$. A number $G \in \mathbb{R}$ satisfying

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Theorem 2 (Supremum theorem)

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The following holds: $\sup M = -\inf(-M)$.

Definition

Let $M \subset \mathbb{R}$. We say that a is a maximum of the set M (denoted by $\max M$) if a is an upper bound of M and $a \in M$. Analogously we define a minimum of M, denoted by $\min M$.

Theorem 3 (Archimedean property)

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Theorem 4 (existence of an integer part)

For every $r \in \mathbb{R}$ *there exists an integer part of* r*, i.e. a number* $k \in \mathbb{Z}$ *satisfying* $k \leq r < k + 1$*. The integer part of* r *is determined uniquely and it is denoted by* $[r]$.

Theorem 5 (*n*th root)

For every $x \in [0, +\infty)$ *and every* $n \in \mathbb{N}$ *there exists a unique* $y \in [0, +\infty)$ *satisfying* $y^n = x$.

Theorem 5 (nth root)

For every $x \in [0, +\infty)$ and every $n \in \mathbb{N}$ there exists a unique $y \in [0, +\infty)$ satisfying $y^n = x$.

Theorem 6 (density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$)

Let $a, b \in \mathbb{R}$, $a < b$. Then there exist $r \in \mathbb{Q}$ satisfying $a < r < b$ and $s \in \mathbb{R} \setminus \mathbb{Q}$ satisfying $a < s < b$.

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A sequence $\{a_n\}_{n=1}^{\infty}$ is equal to a sequence $\{b_n\}_{n=1}^{\infty}$ if $a_n = b_n$ holds for every $n \in \mathbb{N}$.

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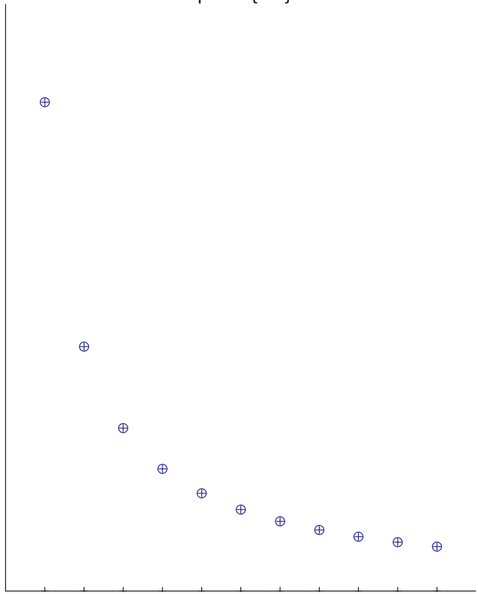
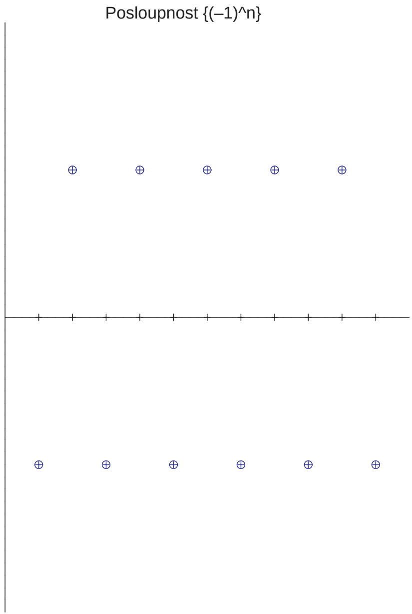
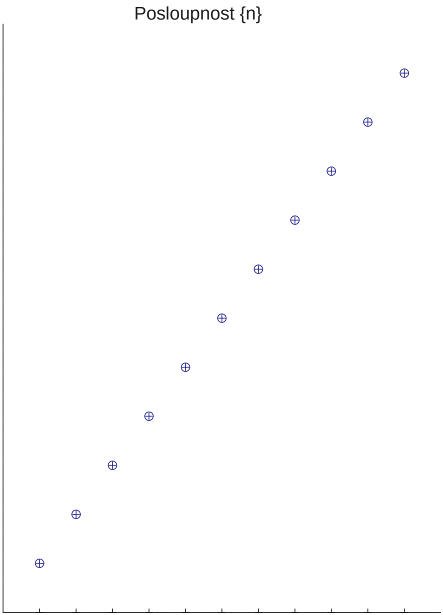
By the set of all members of the sequence $\{a_n\}_{n=1}^{\infty}$ we understand the set

$$\{x \in \mathbb{R};\ \exists n \in \mathbb{N}\colon a_n = x\}.$$

Posloupnost {1/n}




Posloupnost {(–1)^n}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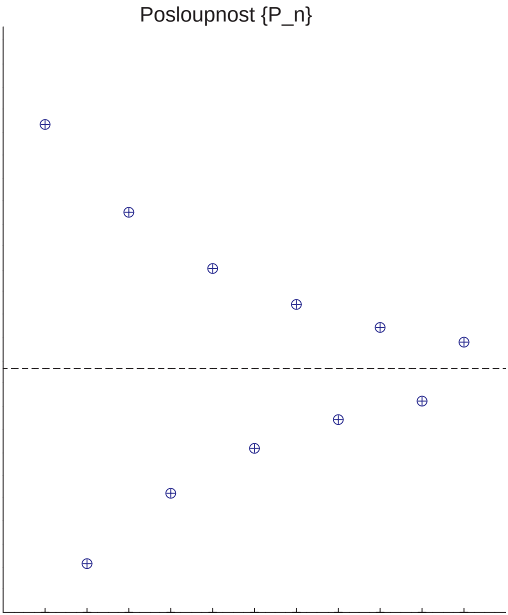
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A sequence $\{a_n\}$ is monotone if it satisfies one of the conditions above. A sequence $\{a_n\}$ is strictly monotone if it is increasing or decreasing.

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- Suppose all the members of the sequence $\{b_n\}$ are non-zero. Then by the quotient of sequences $\{a_n\}$ and $\{b_n\}$ we understand a sequence $\{\frac{a_n}{b_n}\}$.

Definition

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

- By the sum of sequences $\{a_n\}$ and $\{b_n\}$ we understand a sequence $\{a_n + b_n\}$.
- Analogously we define a difference and a product of sequences.
- Suppose all the members of the sequence $\{b_n\}$ are non-zero. Then by the quotient of sequences $\{a_n\}$ and $\{b_n\}$ we understand a sequence $\{\frac{a_n}{b_n}\}$.
- If $\lambda \in \mathbb{R}$, then by the λ-multiple of the sequence $\{a_n\}$ we understand a sequence $\{\lambda a_n\}$.

Definition

We say that a sequence $\{a_n\}$ has a limit which equals to a number $A \in \mathbb{R}$ if to every positive real number ε there exists a natural number n_0 such that for every index $n \geq n_0$ we have $|a_n - A| < \varepsilon$, i.e.

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon\ |a_n - A| < \varepsilon.$$

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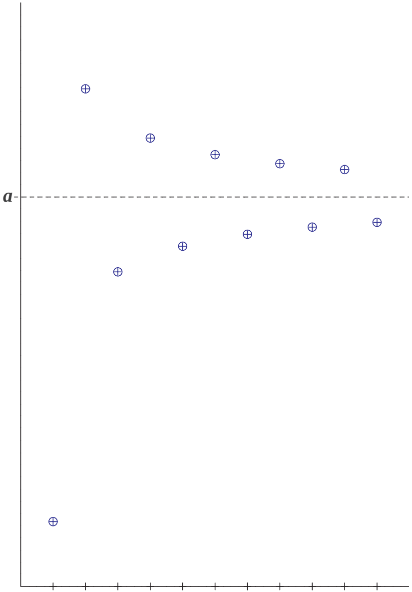
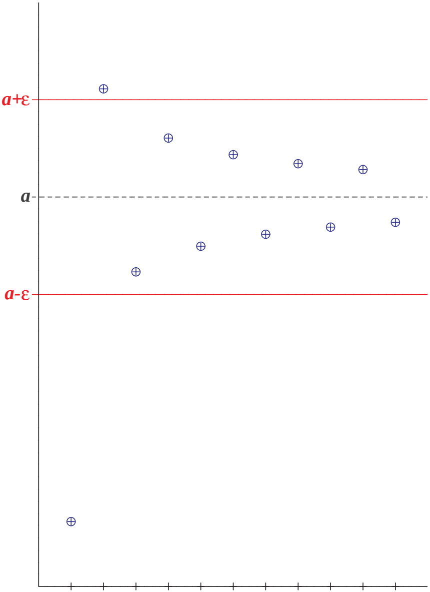
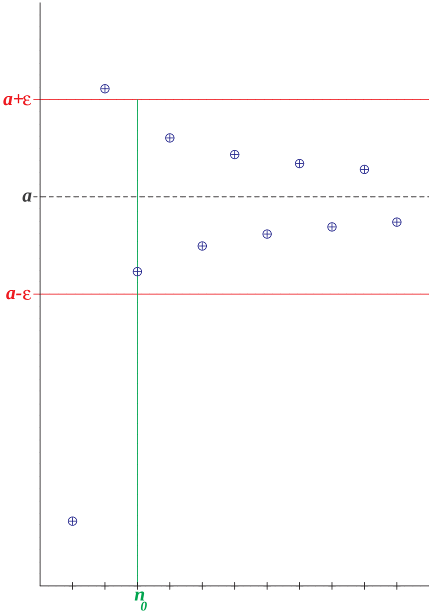
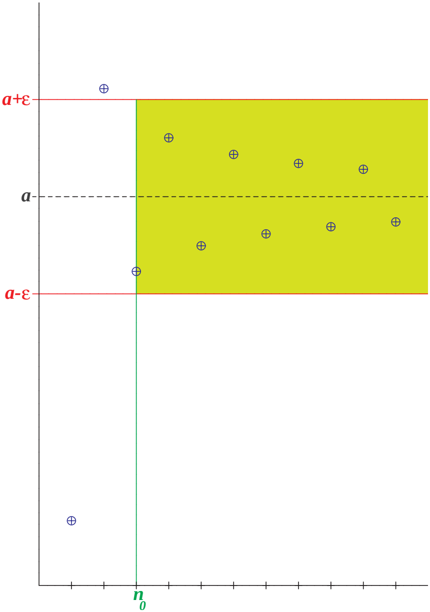
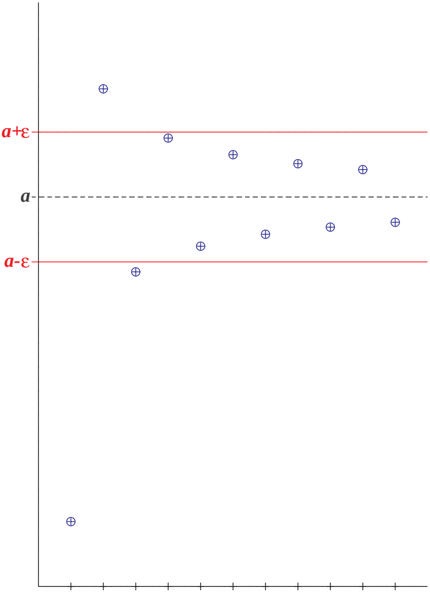
We say that a sequence $\{a_n\}$ is convergent if there exists $A \in \mathbb{R}$ which is a limit of $\{a_n\}$.


a

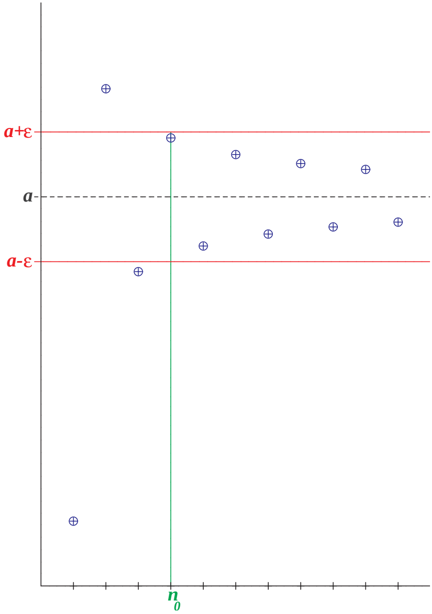

a+ε
a
a-ε

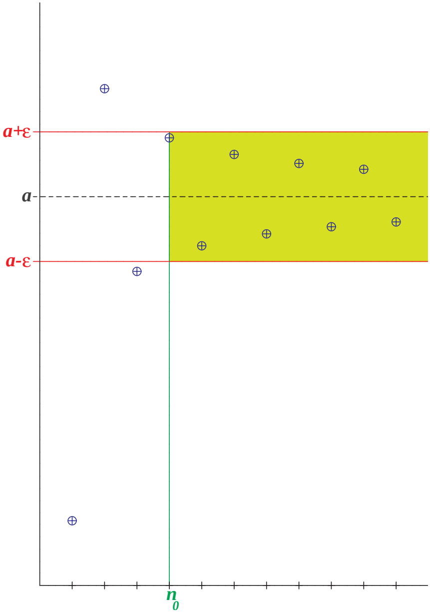

a+ε
a
a-ε
n
0

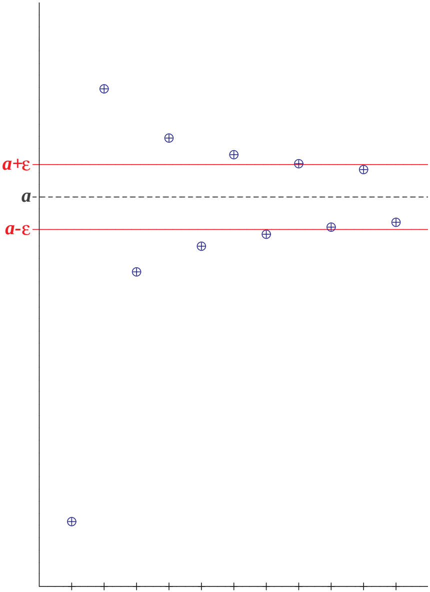

a+ε
a
a-ε
n
0

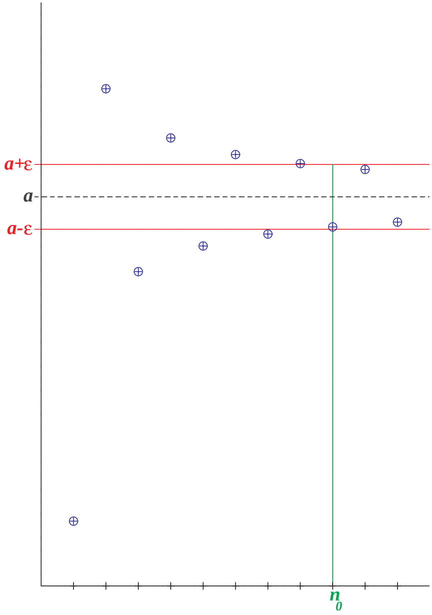


a+ε
a
a-ε

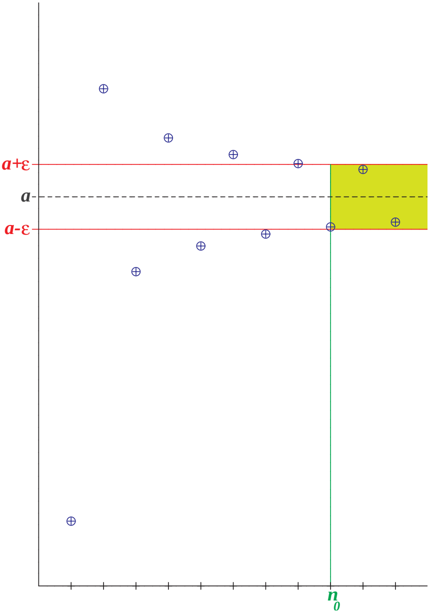

a+ε
a
a-ε
n0


a+ε
a
a-ε
n0


a+ε
a
a-ε



a+ε
a
a-ε
n0


a+ε
a
a-ε
n0

Theorem 7 (uniqueness of a limit)

Every sequence has at most one limit.

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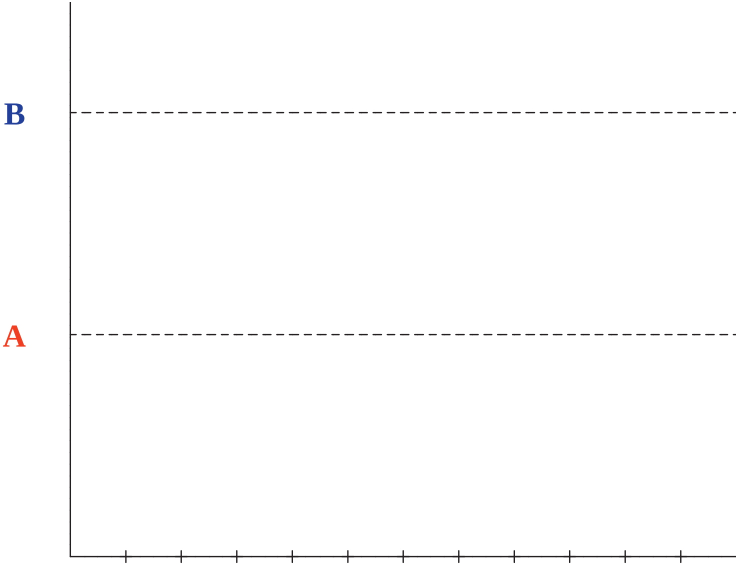
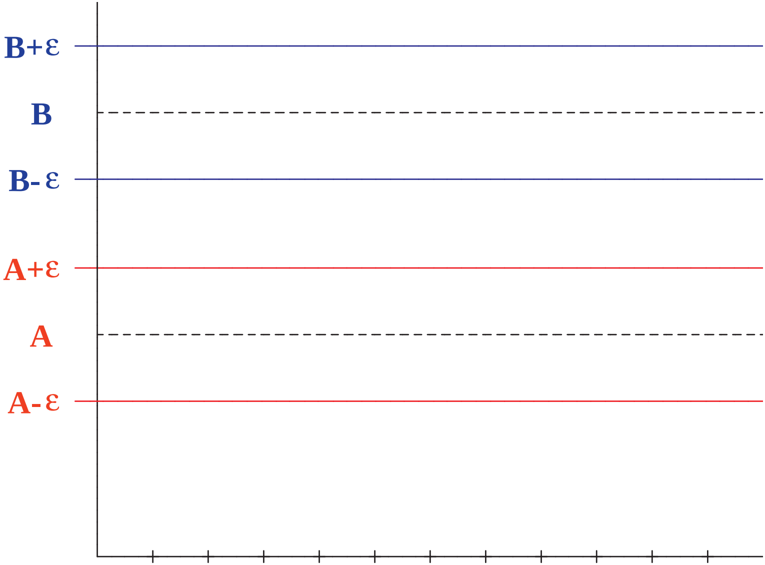
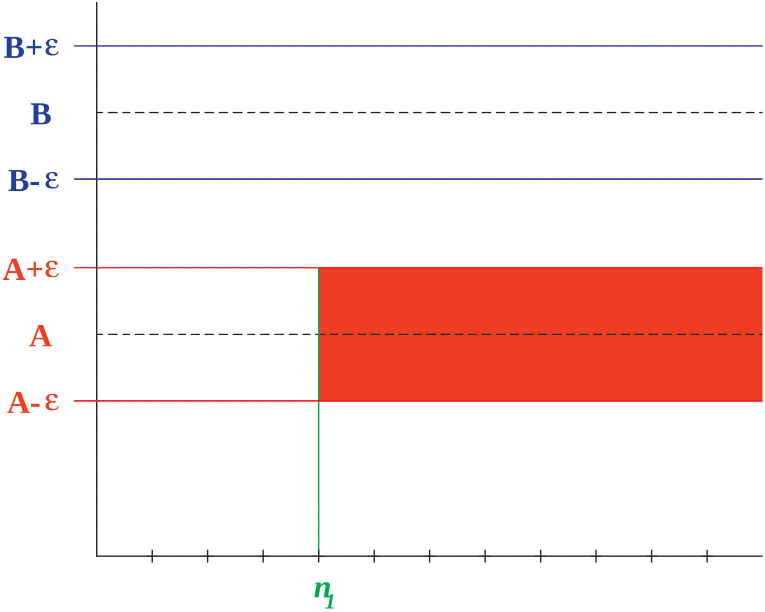
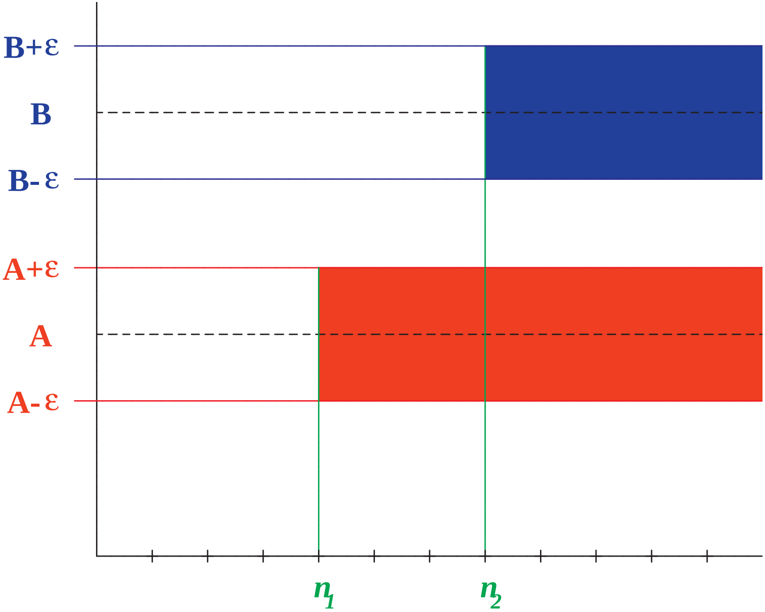
We use the notation $\lim\limits_{n\to\infty} a_n = A$ or simply $\lim a_n = A$.


B
A


B+ε
B
B- ε
A+ε
A
A- ε


B+ε
B
B- ε
A+ε
A
A- ε
n_1


B+ε
B
B- ε
A+ε
A
A- ε
n_1
n_2

Remark

Let $\{a_n\}$ be a sequence of real numbers and $A \in \mathbb{R}$. Then

$$\lim a_n = A \Leftrightarrow \lim(a_n - A) = 0 \Leftrightarrow \lim |a_n - A| = 0.$$

Remark

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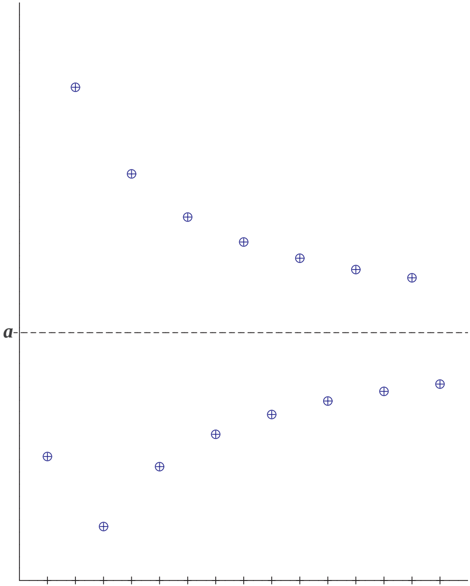
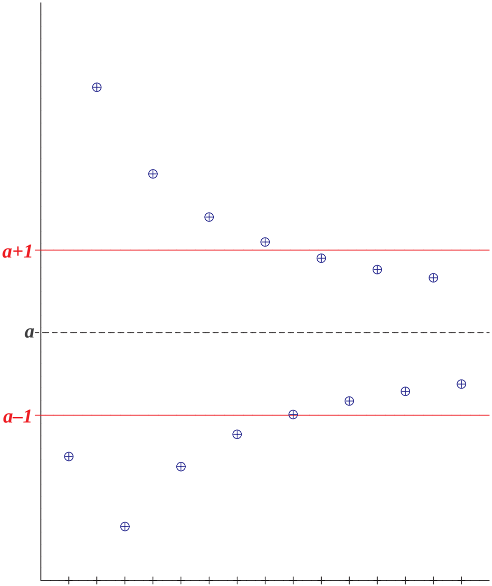
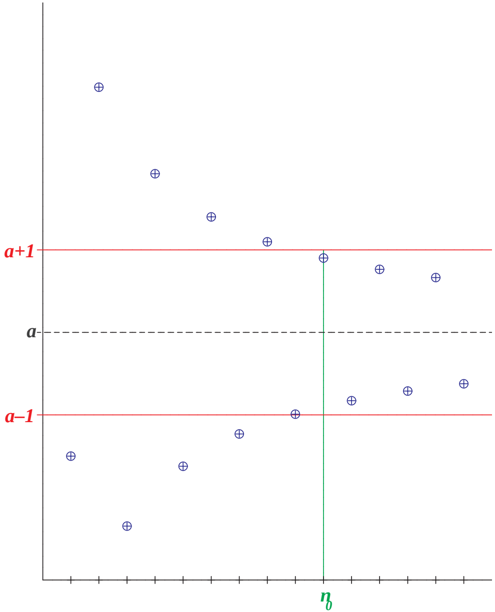
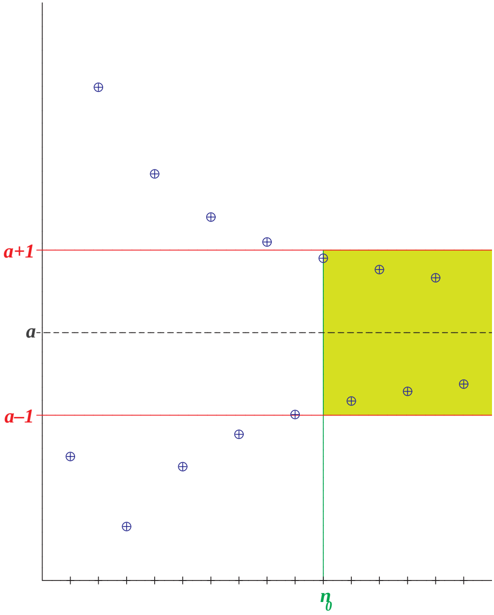
Theorem 8

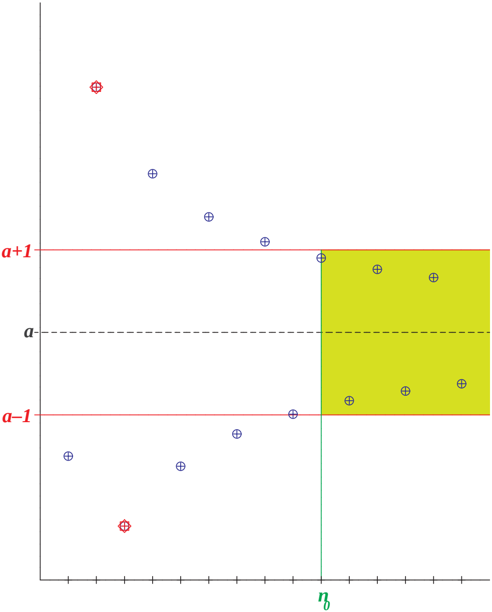
Every convergent sequence is bounded.

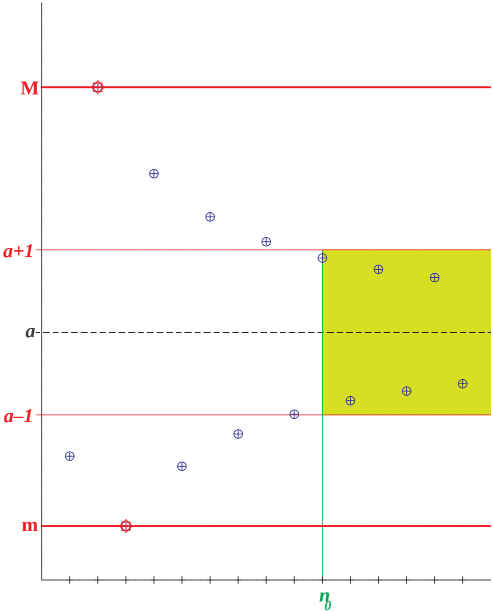

a


a+1
a
a–1


a+1
a
a−1
n0


a+1
a
a–1
n0


a+1
a
a–1
n0


M
a+1
a
a–1
m
n_0

Definition

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that a sequence $\{b_k\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$ if there is an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $b_k = a_{n_k}$ for every $k \in \mathbb{N}$.

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Theorem 9 (limit of a subsequence)

Let $\{b_k\}_{k=1}^{\infty}$ be a subsequence of $\{a_n\}_{n=1}^{\infty}$. If $\lim_{n\to\infty} a_n = A \in \mathbb{R}$, then also $\lim_{k\to\infty} b_k = A$.

Remark

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, $A \in \mathbb{R}$, $K \in \mathbb{R}$, $K > 0$. If

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \in \mathbb{N}, n \geq n_0 \colon \ |a_n - A| < K\varepsilon,$$

then $\lim a_n = A$.

Theorem 10 (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$. *Then*

(i) $\lim(a_n + b_n) = A + B$,

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(i) $\lim(a_n + b_n) = A + B$,

(ii) $\lim(a_n \cdot b_n) = A \cdot B$,

(iii) *if* $B \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $\lim(a_n / b_n) = A/B$.

Theorem 11 (limits and ordering)

Let $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$.

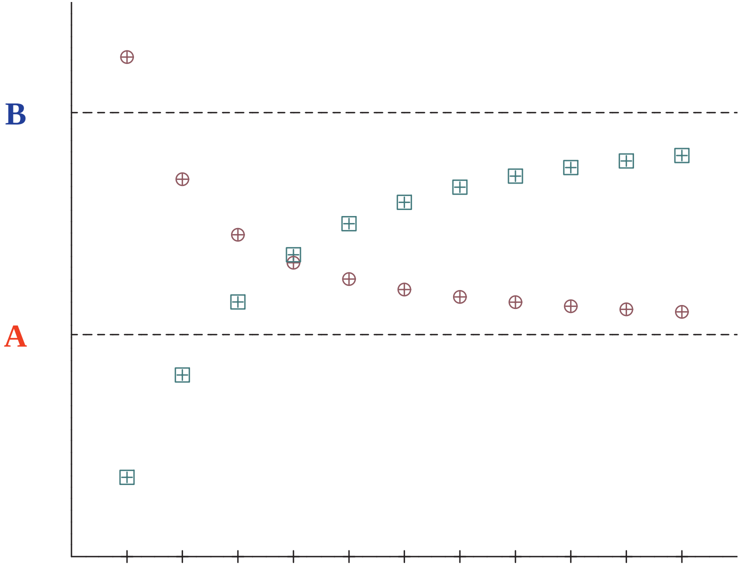
(i) *Suppose that there is* $n_0 \in \mathbb{N}$ *such that* $a_n \geq b_n$ *for every* $n \geq n_0$. *Then* $A \geq B$.

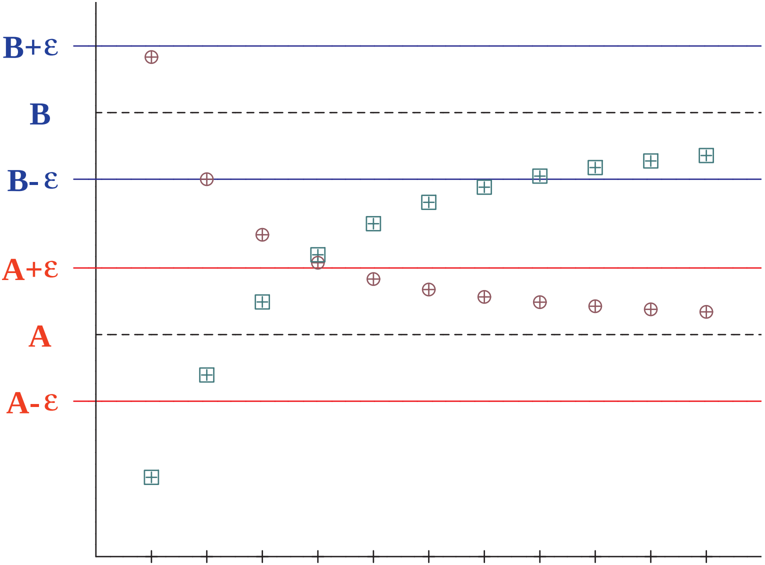
Theorem 11 (limits and ordering)

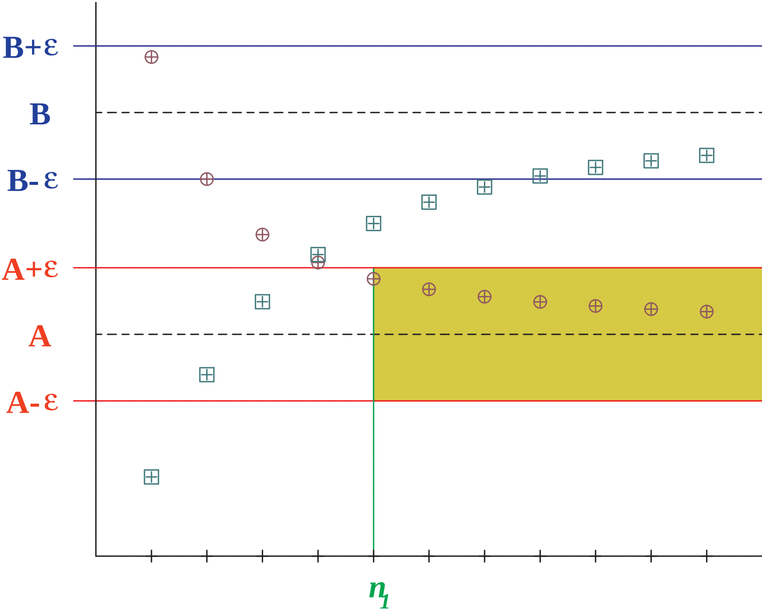
Let $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$*.*

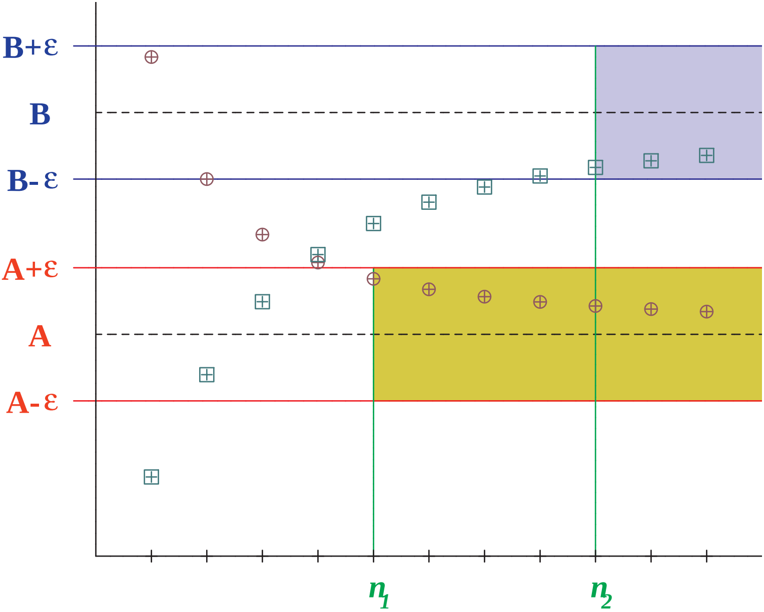
(i) *Suppose that there is* $n_0 \in \mathbb{N}$ *such that* $a_n \geq b_n$ *for every* $n \geq n_0$*. Then* $A \geq B$*.*

(ii) *Suppose that* $A < B$*. Then there is* $n_0 \in \mathbb{N}$ *such that* $a_n < b_n$ *for every* $n \geq n_0$*.*

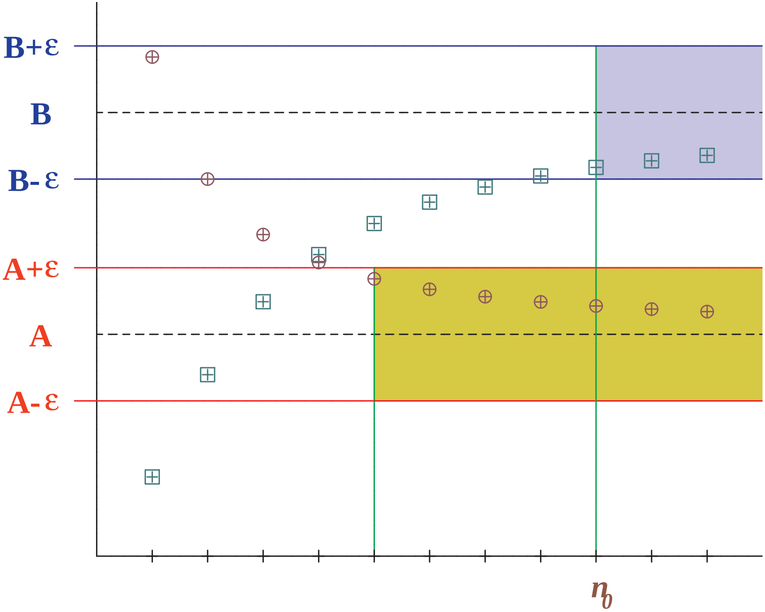

B
A

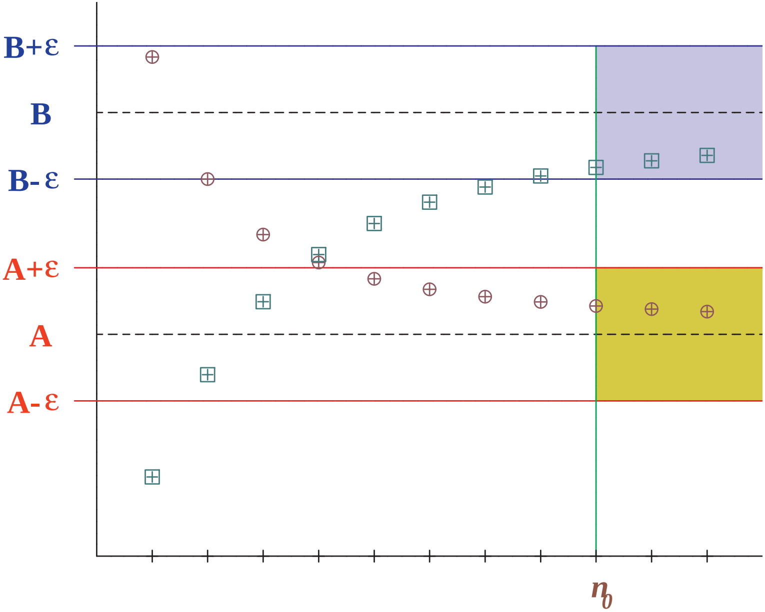

B+ε
B
B- ε
A+ε
A
A- ε


B+ε
B
B- ε
A+ε
A
A- ε
n_1



B+ε
B
B- ε
A+ε
A
A- ε
n1
n2


B+ε
B
B- ε
A+ε
A
A- ε
n_0



B+ε
B
B- ε
A+ε
A
A- ε
n_0

Theorem 12 (two policemen/sandwich theorem)

Let $\{a_n\}$, $\{b_n\}$ *be convergent sequences and let* $\{c_n\}$ *be a sequence such that*

(i) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n \leq c_n \leq b_n$,

(ii) $\lim a_n = \lim b_n$.

Then $\lim c_n$ *exists and* $\lim c_n = \lim a_n$.

Corollary 13

Suppose that $\lim a_n = 0$ *and the sequence* $\{b_n\}$ *is bounded. Then* $\lim a_n b_n = 0$.

Lemma 14 (convergence criterion)

Let $\{a_n\}$ *be a sequence and* $a_n > 0$ *for all* $n \in \mathbb{N}$*. If* $\lim \frac{a_{n+1}}{a_n} < 1$*, then* $\lim a_n = 0$*.*

Lemma 15 ($k-$th root of a sequence)

Let $\{a_n\}$ *be a sequence,* $a_n > 0$ *for all* $n \in \mathbb{N}$*. Let* $k \in \mathbb{N}$*. If* $\lim a_n = A$*, then* $\lim \sqrt[k]{a_n} = \sqrt[k]{A}$*.*

Definition

We say that a sequence $\{a_n\}$ has a limit $+\infty$ (plus infinity) if

$$\forall L \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n > L.$$

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We say that a sequence $\{a_n\}$ has a limit $-\infty$ (minus infinity) if

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Theorem 7 on the uniqueness of a limit holds also for the limits $+\infty$ and $-\infty$. If $\lim a_n = +\infty$, then we say that the sequence $\{a_n\}$ diverges to $+\infty$, similarly for $-\infty$.

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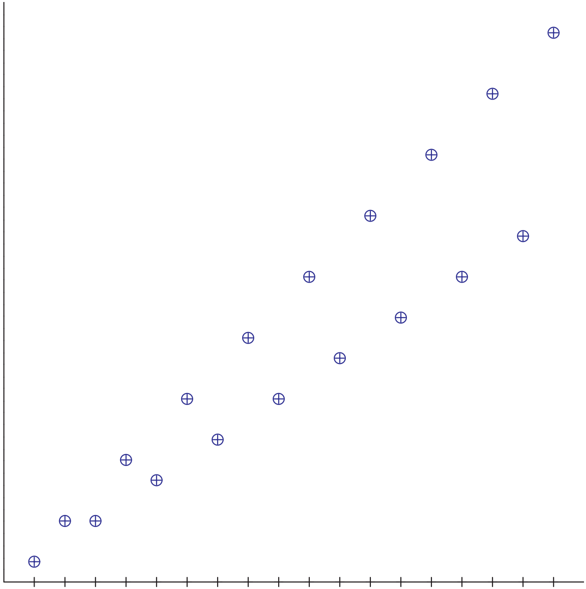
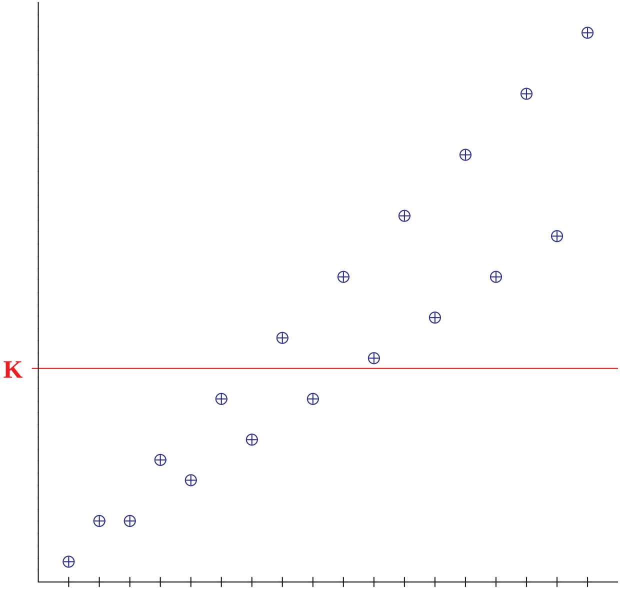
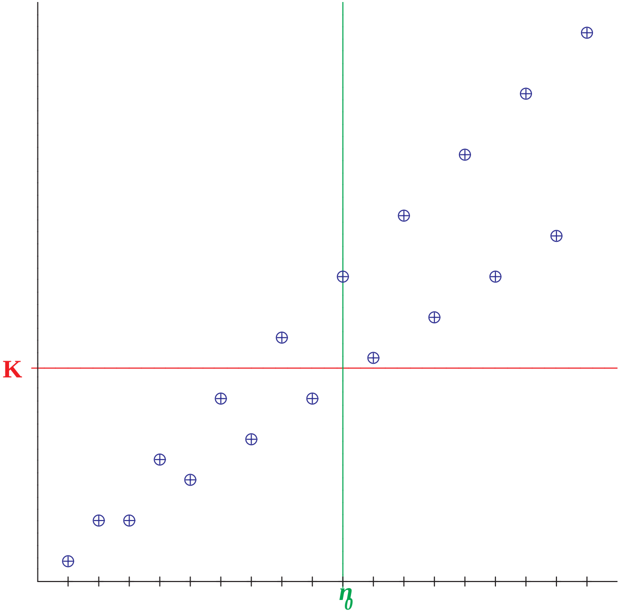
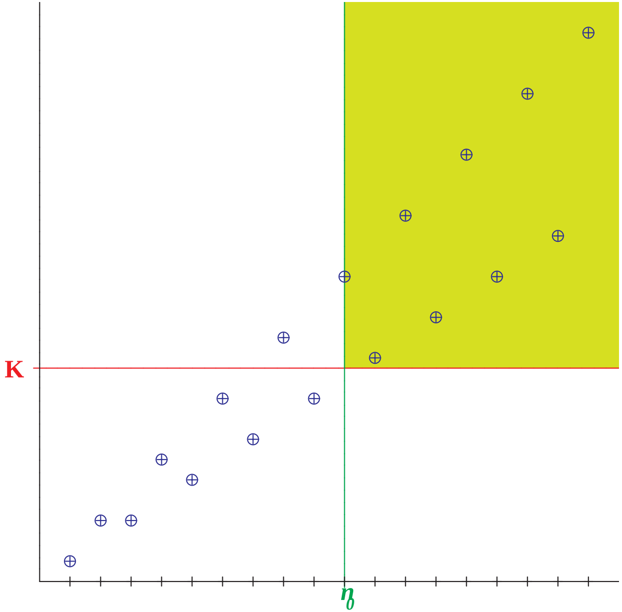
Theorem 7 on the uniqueness of a limit holds also for the limits $+\infty$ and $-\infty$. If $\lim a_n = +\infty$, then we say that the sequence $\{a_n\}$ diverges to $+\infty$, similarly for $-\infty$. If $\lim a_n \in \mathbb{R}$, then we say that the limit is finite, if $\lim a_n = +\infty$ or $\lim a_n = -\infty$, then we say that the limit is infinite.




K


K
n_0


K
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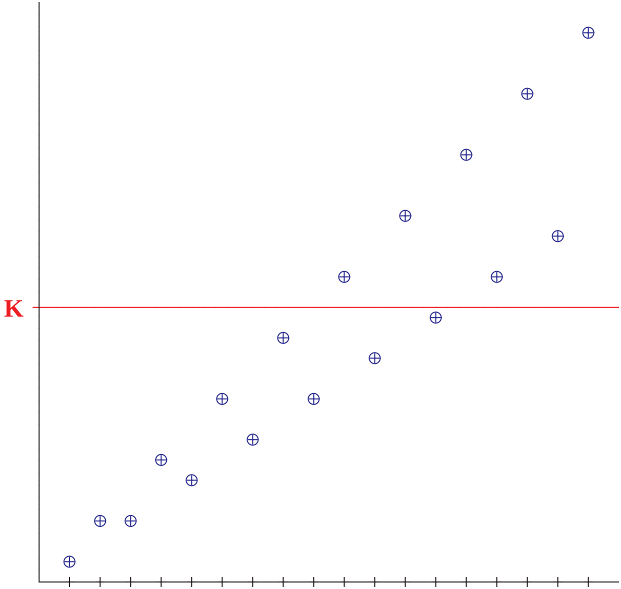
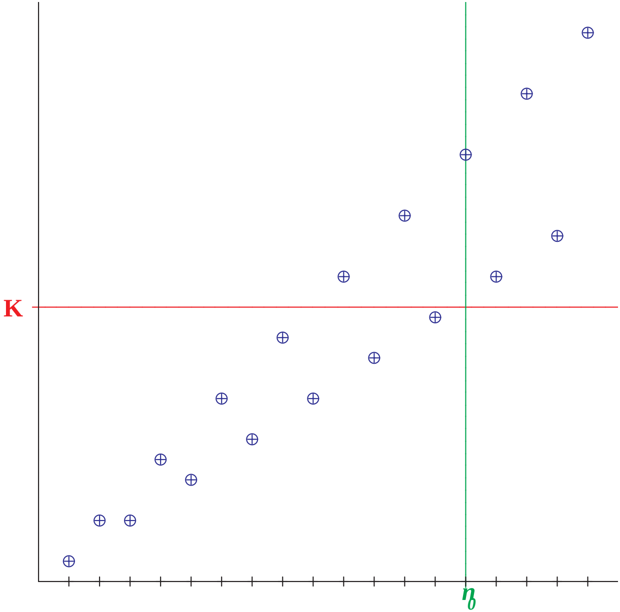
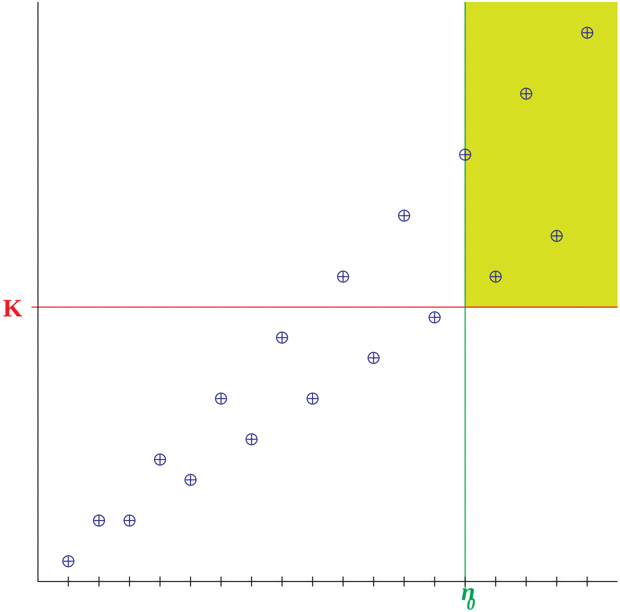
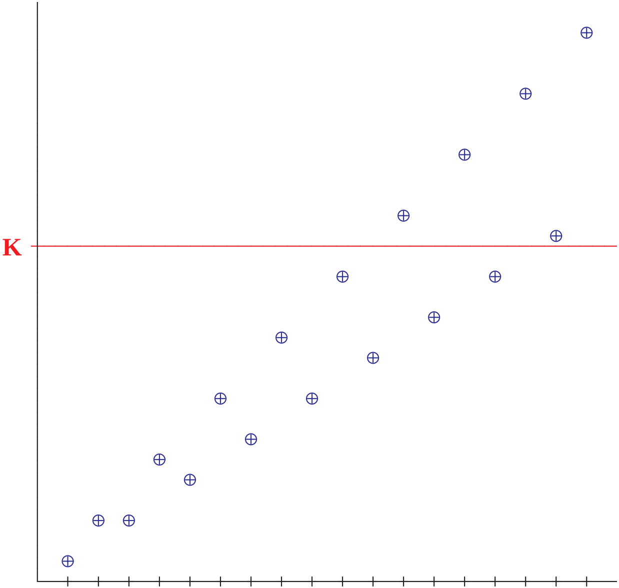

K


K
n_0


K
n_0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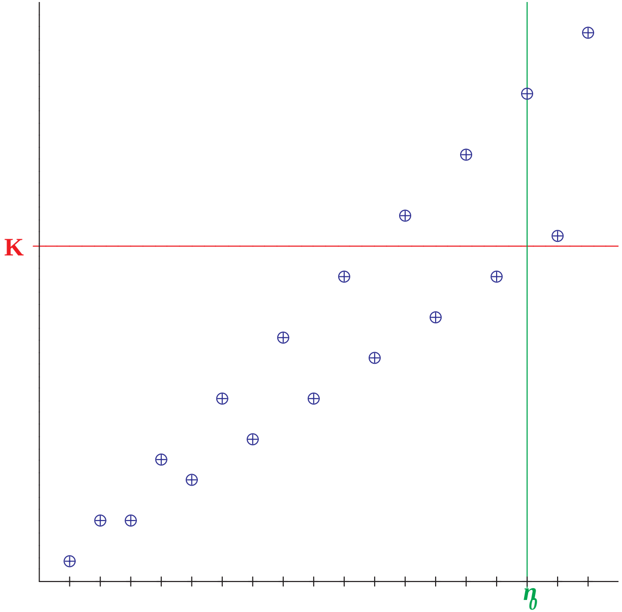
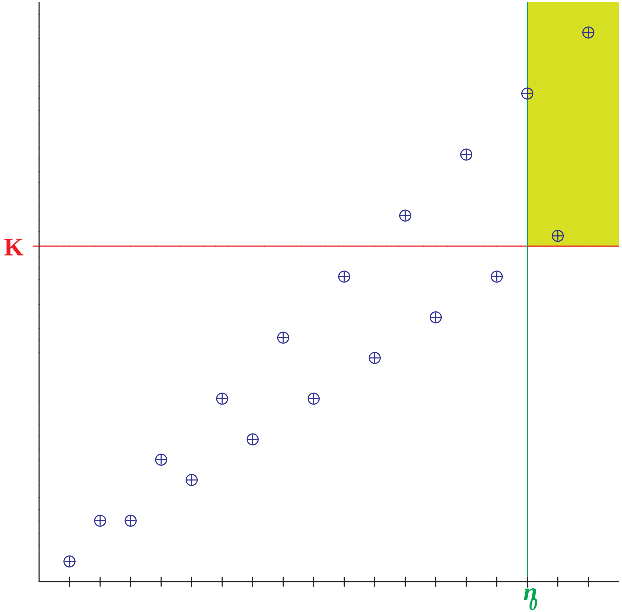
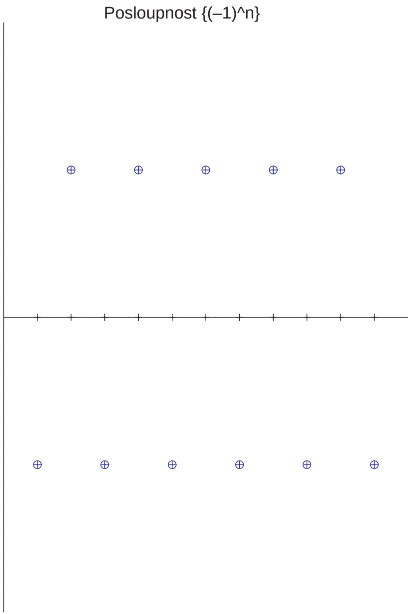
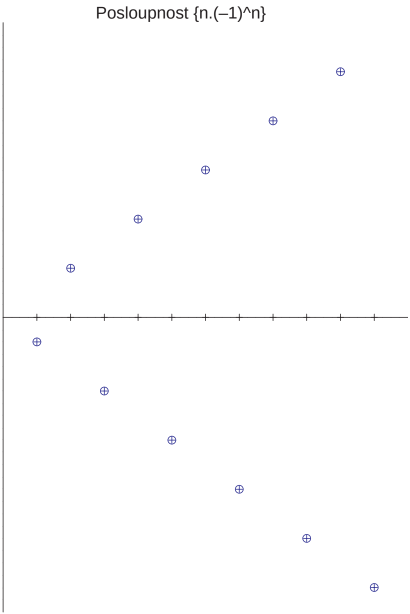
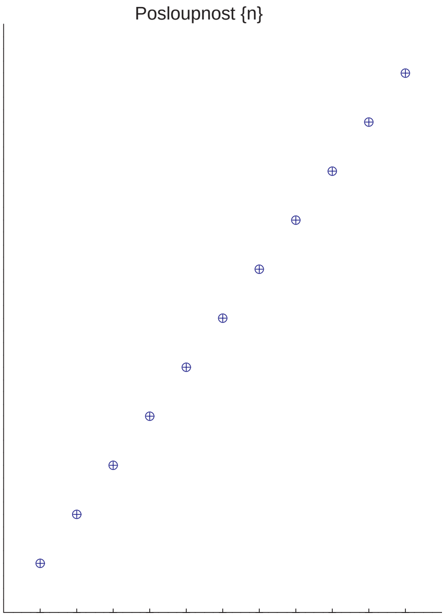
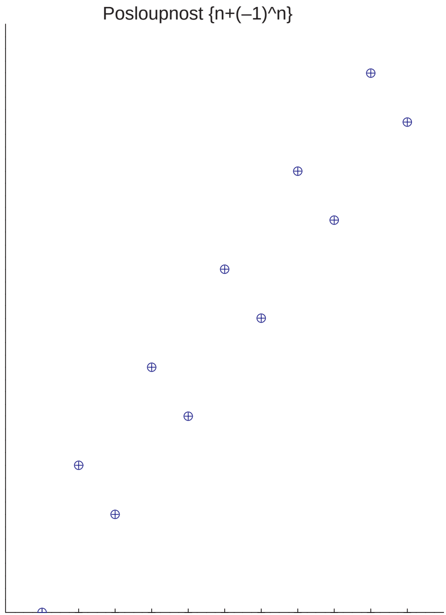

K
n_0

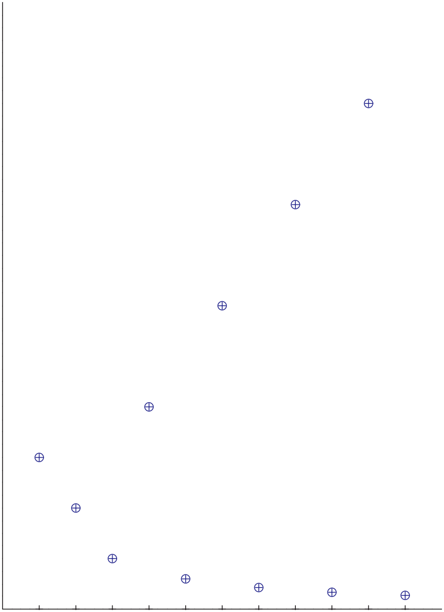

K
n_0


Posloupnost {(–1)^n}


Posloupnost {n.(–1)^n}


Posloupnost {n}


Posloupnost {n+(–1)^n}



Theorem 8 does not hold for infinite limits. But:

Theorem 8'

- *Suppose that* $\lim a_n = +\infty$*. Then the sequence* $\{a_n\}$ *is not bounded from above, but is bounded from below.*
- *Suppose that* $\lim a_n = -\infty$*. Then the sequence* $\{a_n\}$ *is not bounded from below, but is bounded from above.*

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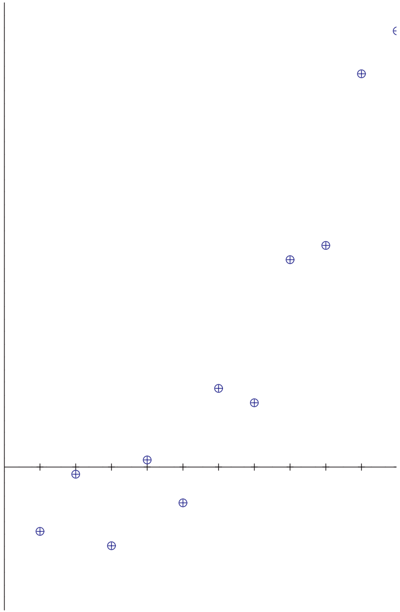
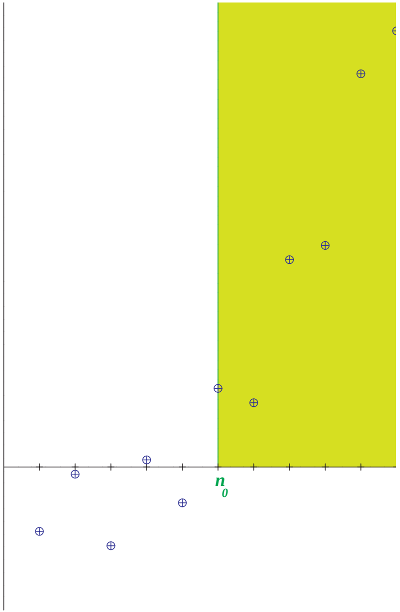
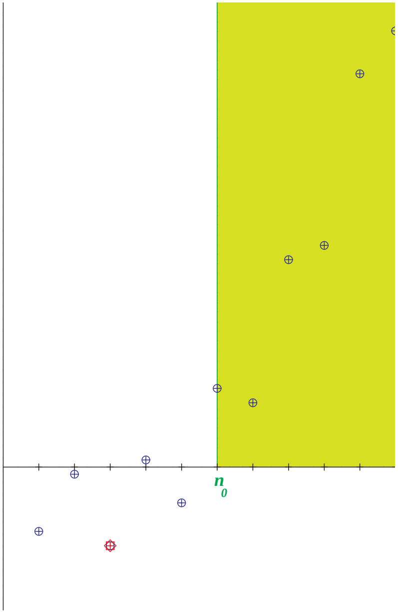
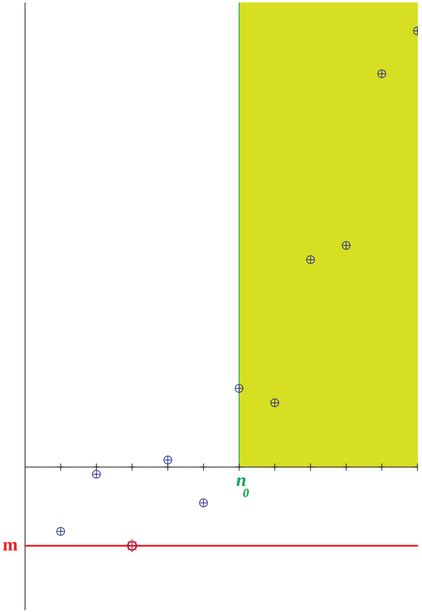
Theorem 9 (limit of a subsequence) holds also for infinite limits.




n_0


n_0



m
n_0

Definition

We define the extended real line by setting $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$ with the following extension of operations and ordering from $\mathbb{R}$:

- $a < +\infty$ and $-\infty < a$ for $a \in \mathbb{R}$, $-\infty < +\infty$,

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- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \pm\infty$ for $a \in \mathbb{R}^*$, $a > 0$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \mp\infty$ for $a \in \mathbb{R}^*$, $a < 0$,
- $\frac{a}{\pm\infty} = 0$ pro $a \in \mathbb{R}$.

The following operations are not defined:

- $(-\infty) + (+\infty)$, $(+\infty) + (-\infty)$, $(+\infty) - (+\infty)$, $(-\infty) - (-\infty)$,

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- $(-\infty) + (+\infty)$, $(+\infty) + (-\infty)$, $(+\infty) - (+\infty)$, $(-\infty) - (-\infty)$,
- $(+\infty) \cdot 0$, $0 \cdot (+\infty)$, $(-\infty) \cdot 0$, $0 \cdot (-\infty)$,
- $\frac{+\infty}{+\infty}$, $\frac{+\infty}{-\infty}$, $\frac{-\infty}{-\infty}$, $\frac{-\infty}{+\infty}$, $\frac{a}{0}$ for $a \in \mathbb{R}^*$.

Theorem 10' (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}^*$ *and* $\lim b_n = B \in \mathbb{R}^*$. *Then*

(i) $\lim(a_n \pm b_n) = A \pm B$ *if the right-hand side is defined,*

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(iii) $\lim a_n/b_n = A/B$ *if the right-hand side is defined.*

Theorem 16

Suppose that $\lim a_n = A \in \mathbb{R}^*$*,* $A > 0$*,* $\lim b_n = 0$ *and there is* $n_0 \in \mathbb{N}$ *such that we have* $b_n > 0$ *for every* $n \in \mathbb{N}$*,* $n \geq n_0$*. Then* $\lim a_n/b_n = +\infty$*.*

Theorem 11 (limits and ordering) and Theorem 12 (sandwich theorem) hold also for infinite limits. Even the following modification holds:

Theorem 12' (one policeman)

Let $\{a_n\}$ and $\{b_n\}$ be two sequences.

- *If $\lim a_n = +\infty$ and there is $n_0 \in \mathbb{N}$ such that $b_n \geq a_n$ for every $n \in \mathbb{N}$, $n \geq n_0$, then $\lim b_n = +\infty$.*
- *If $\lim a_n = -\infty$ and there is $n_0 \in \mathbb{N}$ such that $b_n \leq a_n$ for every $n \in \mathbb{N}$, $n \geq n_0$, then $\lim b_n = -\infty$.*

Definition

Let $A \subset \mathbb{R}$ be non-empty. If A is not bounded from above, then we define $\sup A = +\infty$. If A is not bounded from below, then we define $\inf A = -\infty$.

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Lemma 17

Let $M \subset \mathbb{R}$ be non-empty and $G \in \mathbb{R}^*$. Then the following statements are equivalent:

(1) $G = \sup M$.

(2) *The number G is an upper bound of M and there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of members of M such that $\lim x_n = G$.*

The connection between sequences and $\mathbb{R}$

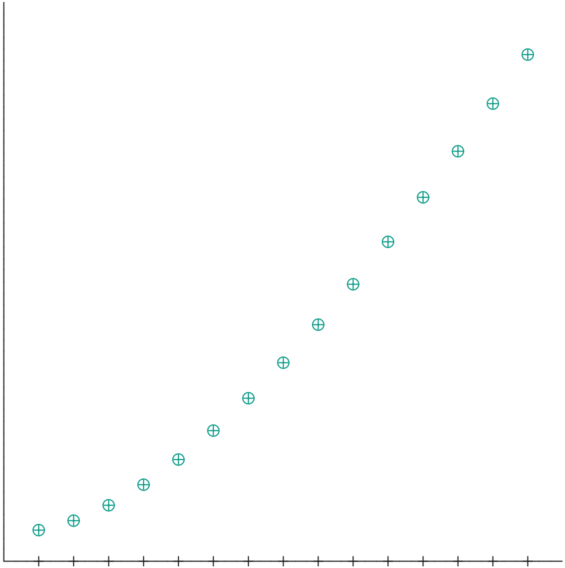
Theorem 18

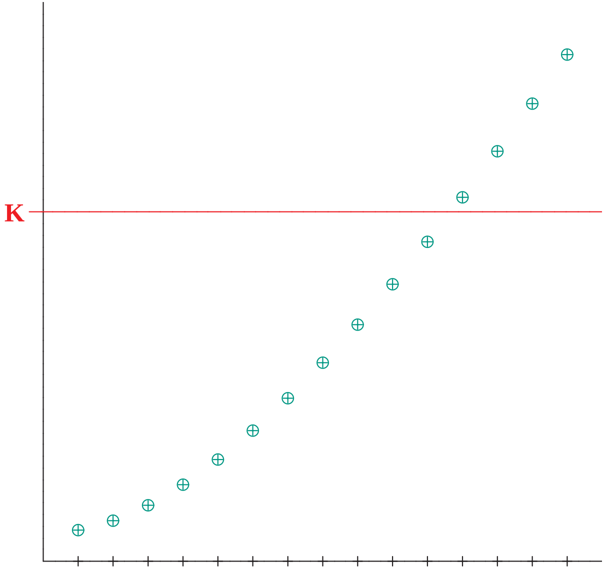
For all $x \in \mathbb{R}$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$, such that $x_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$ **and** $\lim x_n = x$.

Famous examples: $\sqrt{2}$, π and e.

Theorem 19 (limit of a monotone sequence)

Every monotone sequence has a limit. If $\{a_n\}$ *is non-decreasing, then* $\lim a_n = \sup\{a_n;\ n \in \mathbb{N}\}$*. If* $\{a_n\}$ *is non-increasing, then* $\lim a_n = \inf\{a_n;\ n \in \mathbb{N}\}$*.*

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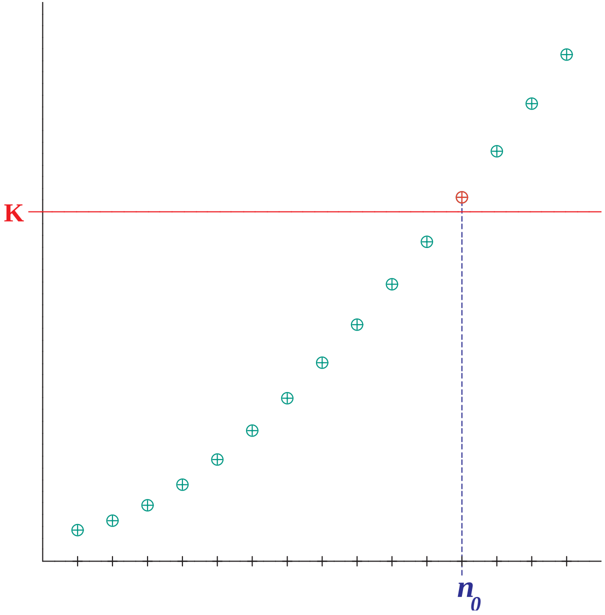
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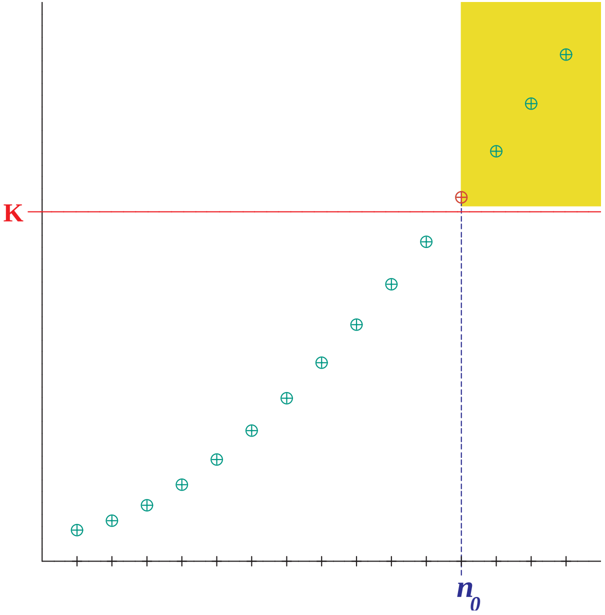
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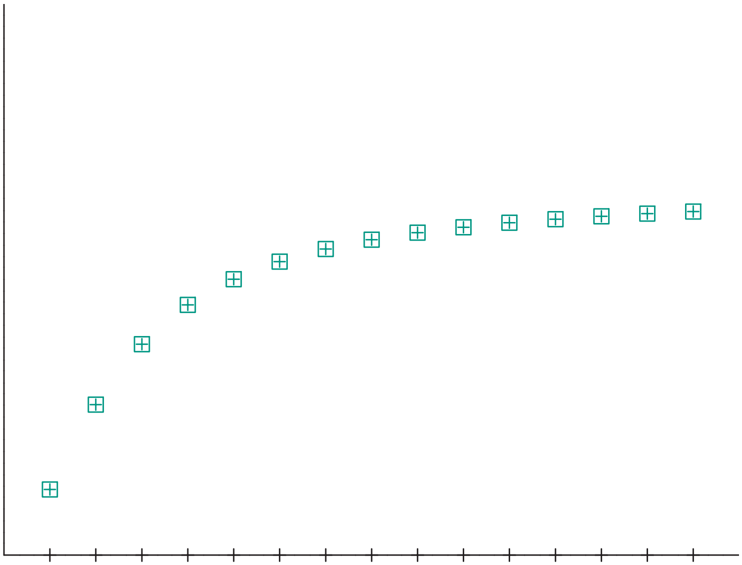
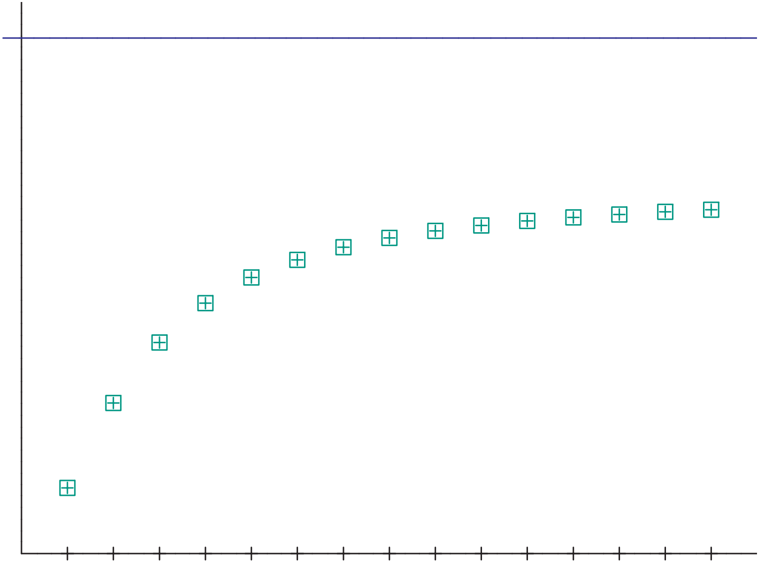
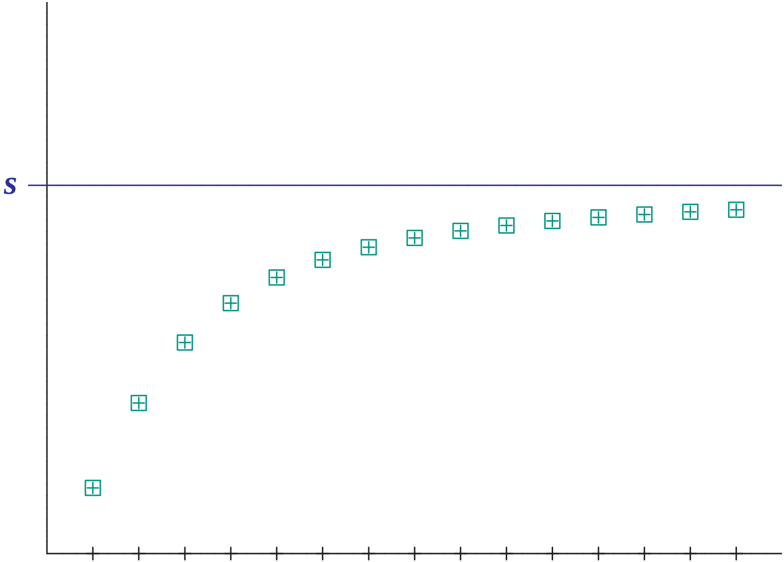
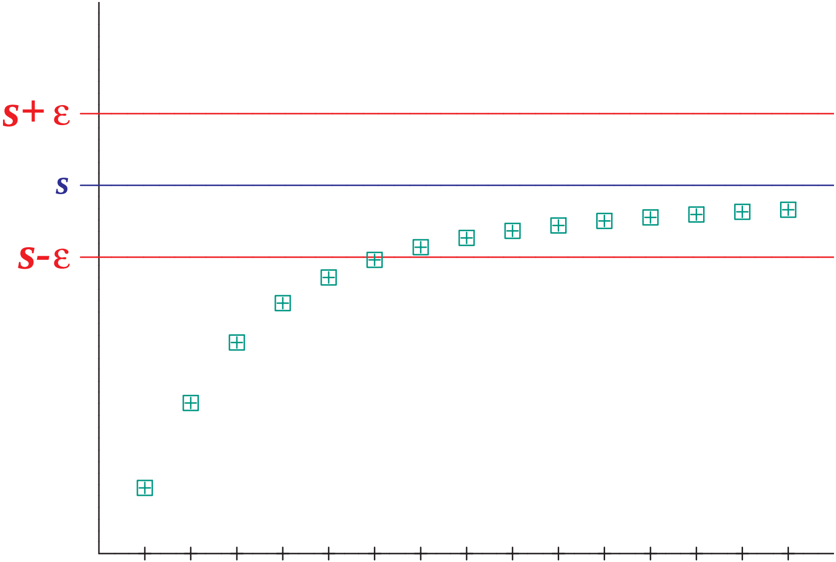
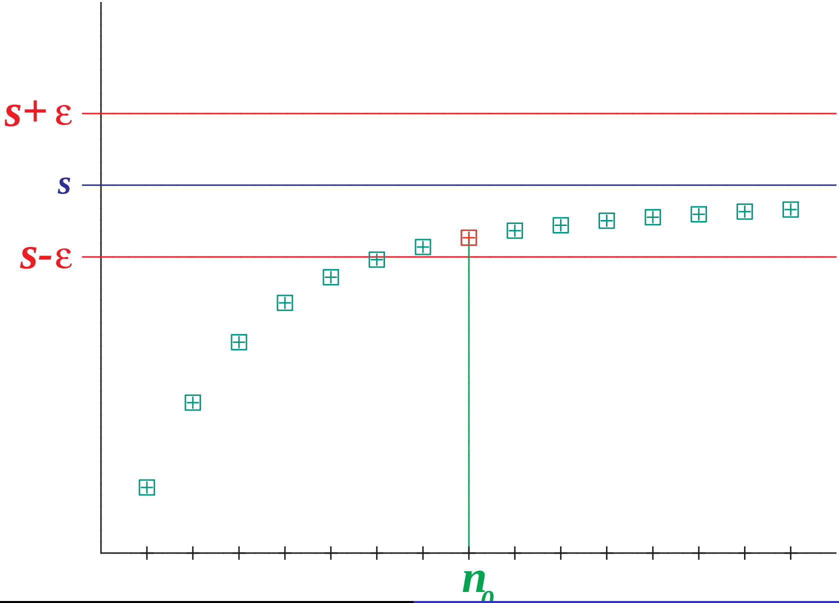
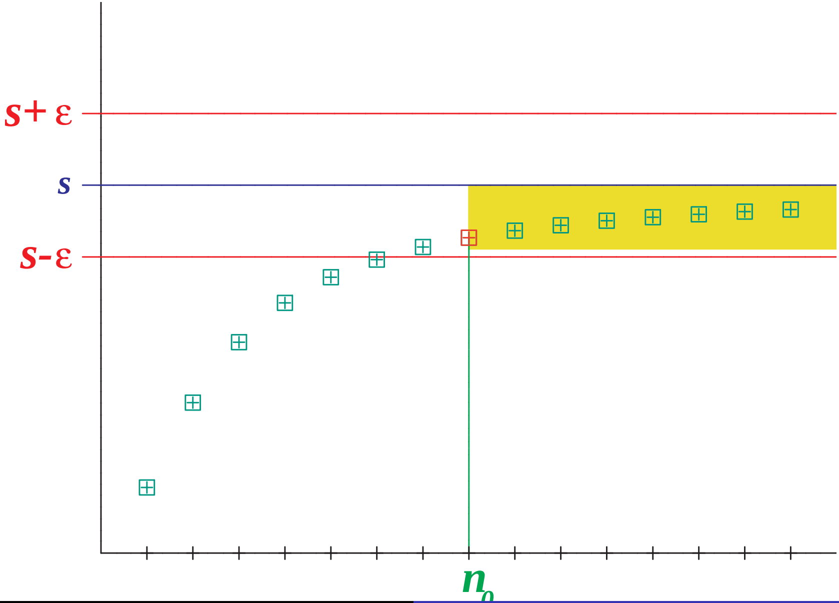





s


s+ ε
s
s-ε

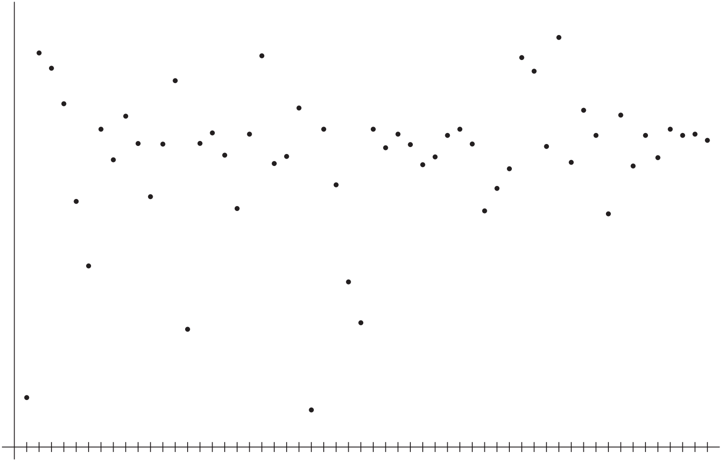

s+ ε
s
s-ε
n0

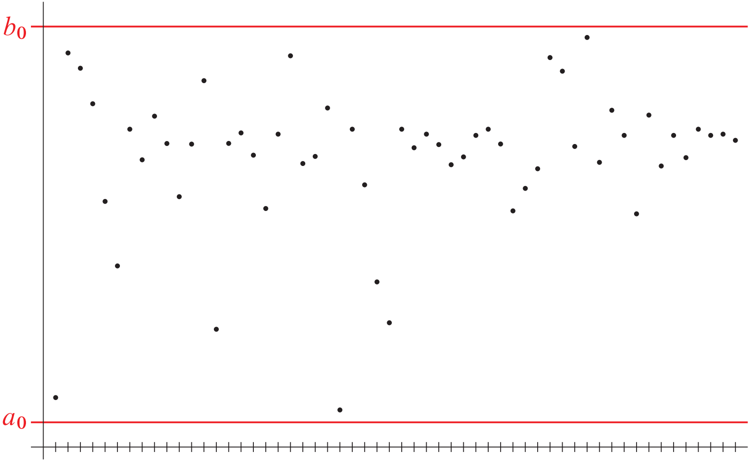


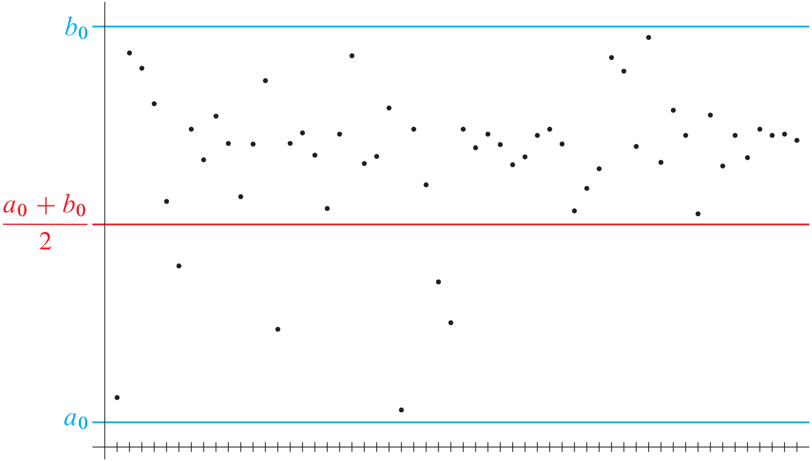
$s+\varepsilon$
s
$s-\varepsilon$
n_0

Theorem 20 (Bolzano-Weierstraß)

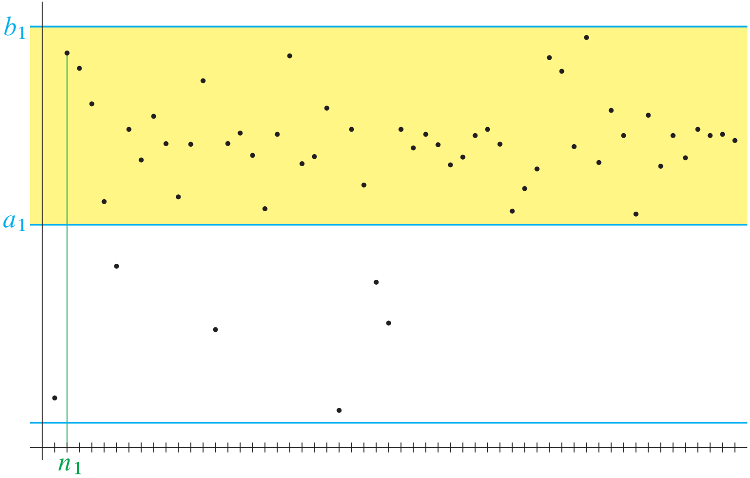
Every bounded sequence contains a convergent subsequence.

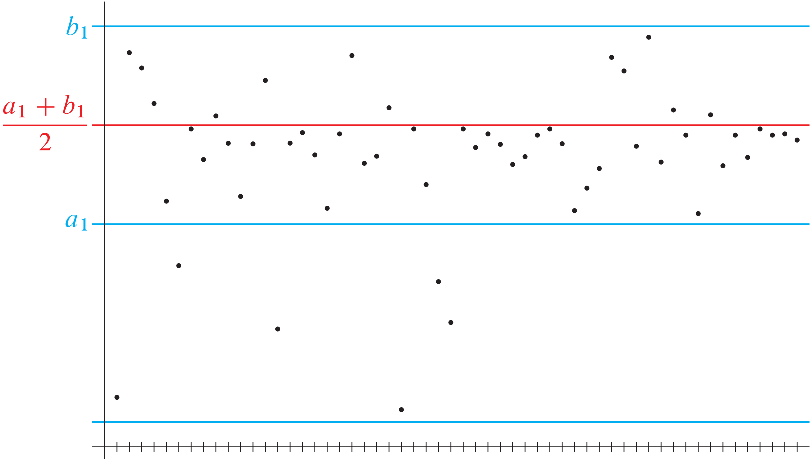


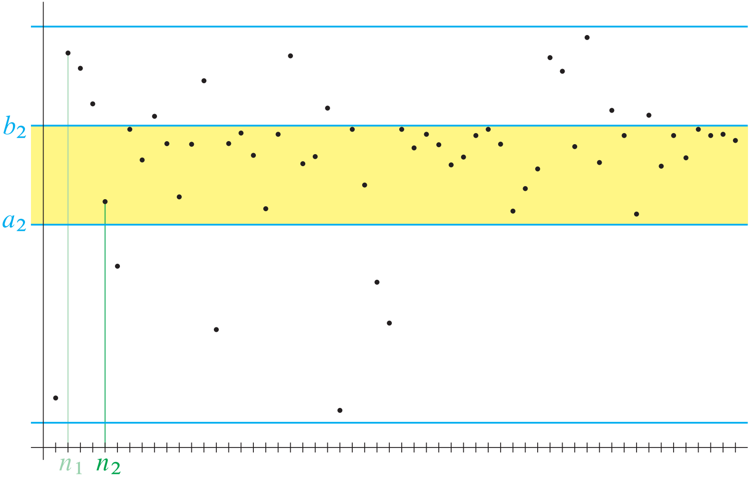

b_0
a_0

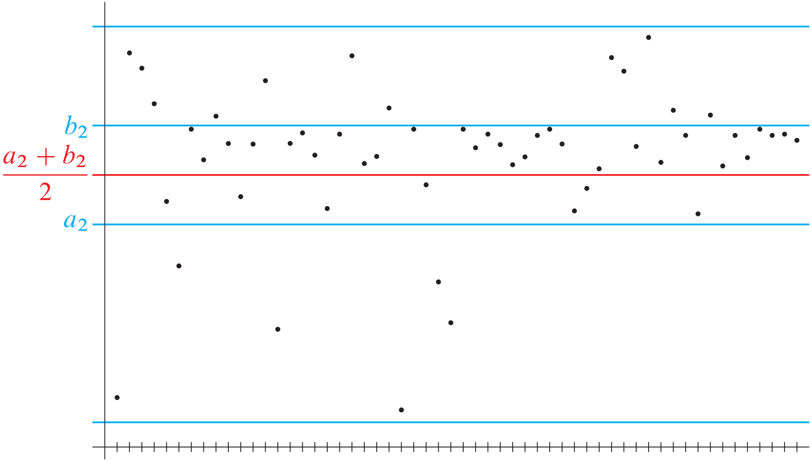


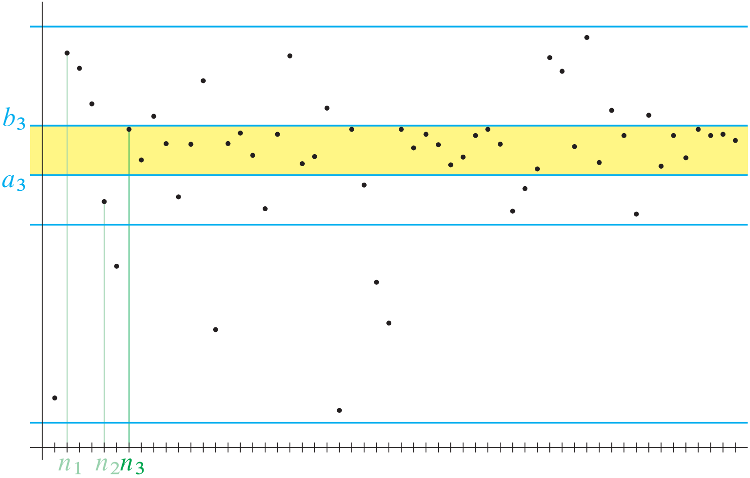
b_0
$\frac{a_0 + b_0}{2}$
a_0

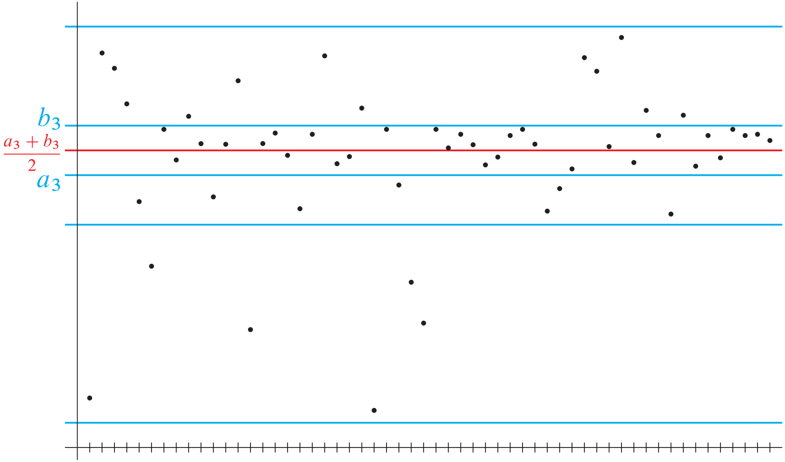

b_1
a_1
n_1

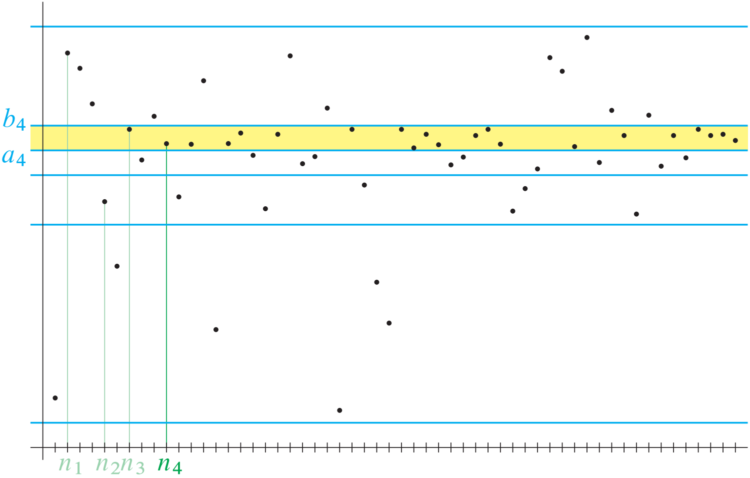

b_1
$\frac{a_1 + b_1}{2}$
a_1

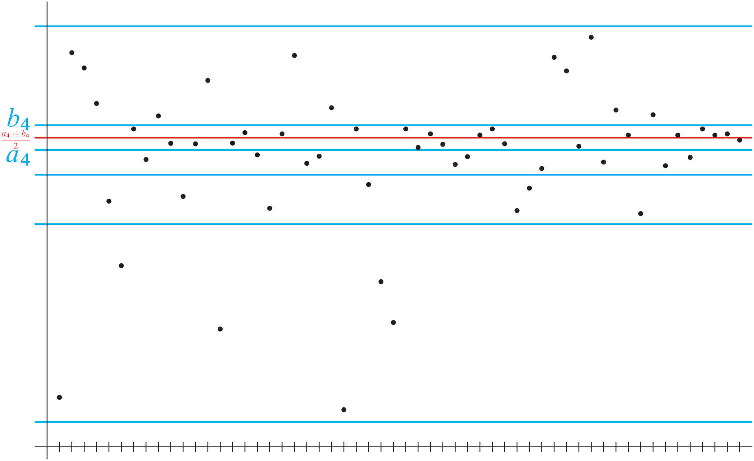

b_2
a_2
n_1
n_2

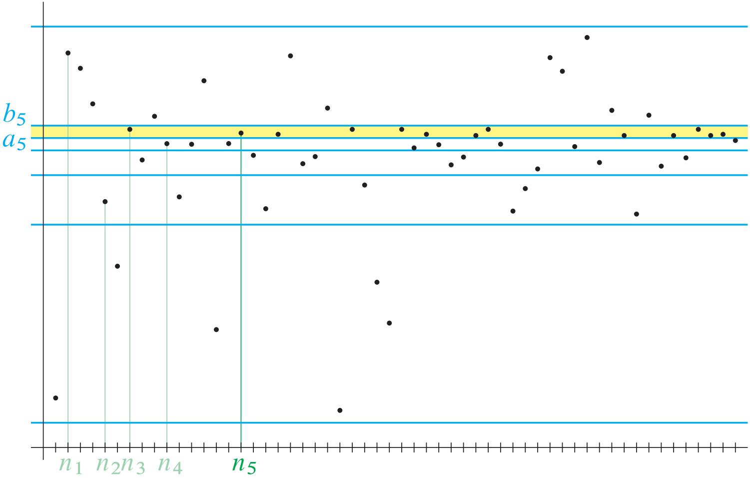

b_2
$\frac{a_2+b_2}{2}$
a_2

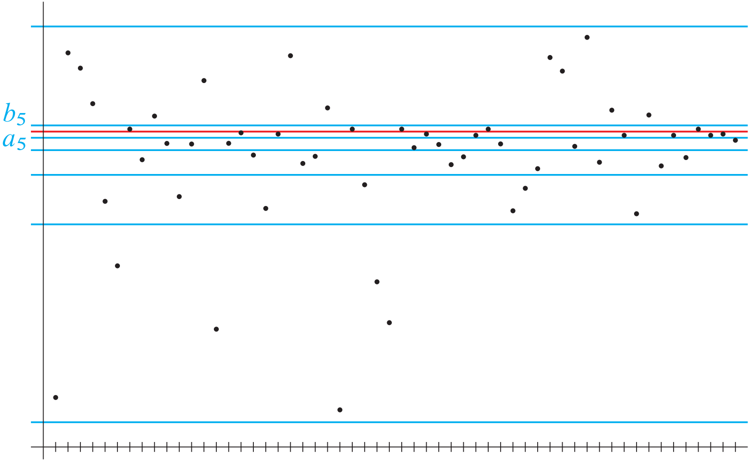

b_3
a_3
n_1
n_2
n_3

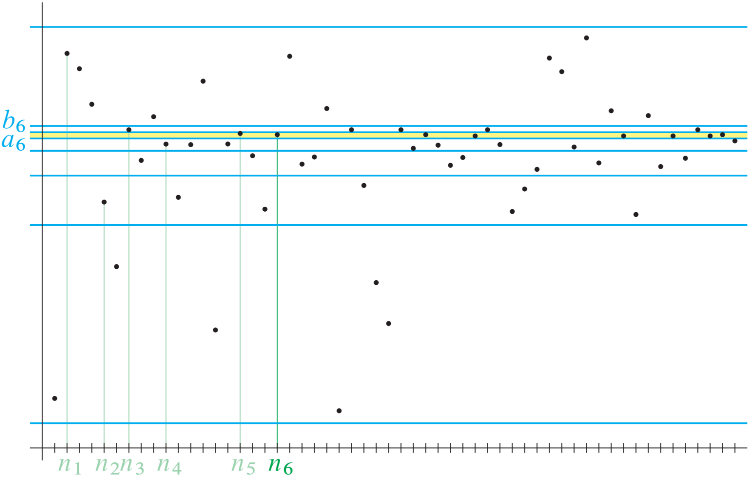

b_3
$\frac{a_3+b_3}{2}$
a_3


b_4
a_4
n_1
n_2
n_3
n_4


b_4
$\frac{a_4+b_4}{2}$
a_4


b_5
a_5
n_1
n_2
n_3
n_4
n_5


b_5
a_5



b_6
a_6
n_1
n_2
n_3
n_4
n_5
n_6

Definition

Let A and B be sets. A mapping f from A to B is a rule which assigns to each member x of the set A a unique member y of the set B. This element y is denoted by the symbol $f(x)$.

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- By $f\colon A \to B$ we denote the fact that f is a mapping from A to B.
- By $f\colon x \mapsto f(x)$ we denote the fact that the mapping f assigns $f(x)$ to an element x.
- The set A from the definition of the mapping f is called the domain of f and it is denoted by D_f.

Definition

Let $f\colon A \to B$ be a mapping.

- The subset $G_f = \{[x, y] \in A \times B;\ x \in A, y = f(x)\}$ of the Cartesian product $A \times B$ is called the graph of the mapping f.

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- The image of the set $M \subset A$ under the mapping f is the set

$$f(M) = \{y \in B;\ \exists x \in M\colon f(x) = y\} \quad (= \{f(x);\ x \in M\}).$$

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- The set $f(A)$ is called the range of the mapping f, it is denoted by R_f.
- The pre-image of the set $W \subset B$ under the mapping f is the set

$$f_{-1}(W) = \{x \in A;\ f(x) \in W\}.$$

Remark

Let $f\colon A \to B$, $X, Y \subset A$, $U, V \subset B$. Then

- $f_{-1}(U \cup V) = f_{-1}(U) \cup f_{-1}(V)$,

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- $f(X \cup Y) = f(X) \cup f(Y)$,
- $f(X \cap Y) \subset f(X) \cap f(Y)$.

Definition

Let A, B, C be sets, $C \subset A$ and $f\colon A \to B$. The mapping $\tilde{f}\colon C \to B$ given by the formula $\tilde{f}(x) = f(x)$ for each $x \in C$ is called the **restriction of the mapping f to the set C**. It is denoted by $f|_C$.

Definition

Let $f\colon A \to B$ and $g\colon B \to C$ be two mappings. The symbol $g \circ f$ denotes a mapping from A to C defined by

$$(g \circ f)(x) = g(f(x)).$$

This mapping is called a compound mapping or a composition of the mapping f and the mapping g.

Definition

We say that a mapping $f\colon A \to B$

- maps the set A onto the set B if $f(A) = B$, i.e. if to each $y \in B$ there exist $x \in A$ such that $f(x) = y$;

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$$\forall x_1, x_2 \in A\colon\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2),$$

- is a bijection of A onto B (or a bijective mapping), if it is at the same time one-to-one and maps A onto B.

Definition

Let $f\colon A \to B$ be bijective (i.e. one-to-one and onto). An inverse mapping $f^{-1}\colon B \to A$ is a mapping that to each $y \in B$ assigns a (uniquely determined) element $x \in A$ satisfying $f(x) = y$.

IV. Functions of one real variable

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Definition

A function f of one real variable (or a function for short) is a mapping $f\colon M \to \mathbb{R}$, where M is a subset of real numbers.

Definition

A function $f\colon J \to \mathbb{R}$ is increasing on an interval J, if for each pair $x_1, x_2 \in J$, $x_1 < x_2$ the inequality $f(x_1) < f(x_2)$ holds. Analogously we define a function decreasing (non-decreasing, non-increasing) on an interval J.

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Definition

A monotone function on an interval J is a function which is non-decreasing or non-increasing on J.

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A function $f\colon J \to \mathbb{R}$ is increasing on an interval J, if for each pair $x_1, x_2 \in J$, $x_1 < x_2$ the inequality $f(x_1) < f(x_2)$ holds. Analogously we define a function decreasing (non-decreasing, non-increasing) on an interval J.

Definition

A monotone function on an interval J is a function which is non-decreasing or non-increasing on J. A strictly monotone function on an interval J is a function which is increasing or decreasing on J.

Definition

Let f be a function and $M \subset D_f$. We say that f is

- bounded from above on M if there is $K \in \mathbb{R}$ such that $f(x) \leq K$ for all $x \in M$,

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- odd if for each $x \in D_f$ we have $-x \in D_f$ and $f(-x) = -f(x)$,
- even if for each $x \in D_f$ we have $-x \in D_f$ and $f(-x) = f(x)$,

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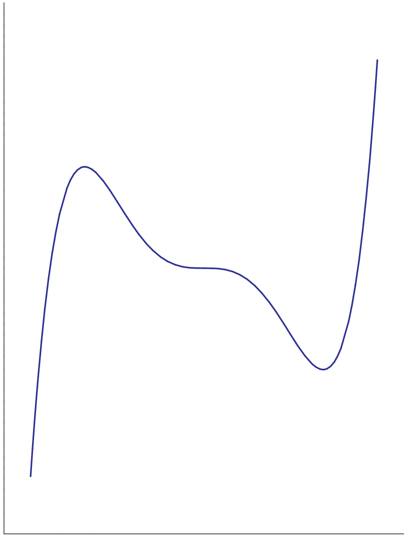
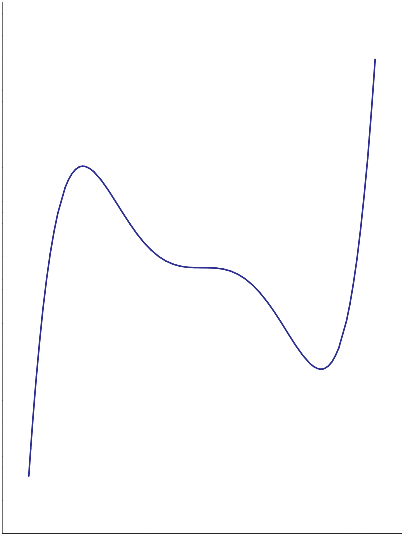
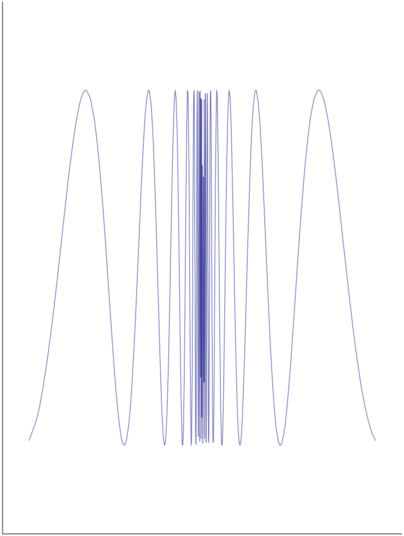
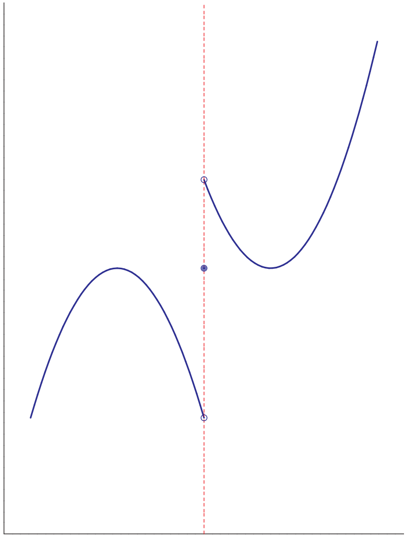
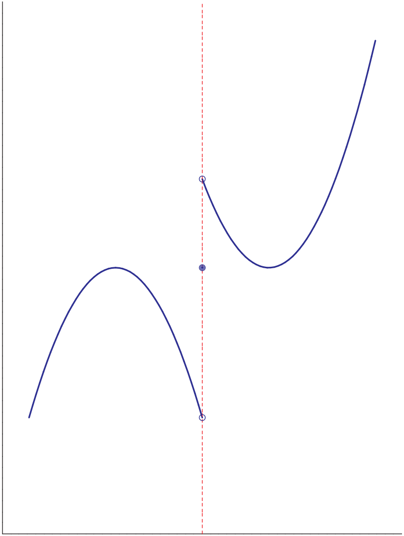
- even if for each $x \in D_f$ we have $-x \in D_f$ and $f(-x) = f(x)$,
- periodic with a period a, where $a \in \mathbb{R}$, $a > 0$, if for each $x \in D_f$ we have $x + a \in D_f$, $x - a \in D_f$ and $f(x + a) = f(x - a) = f(x)$.

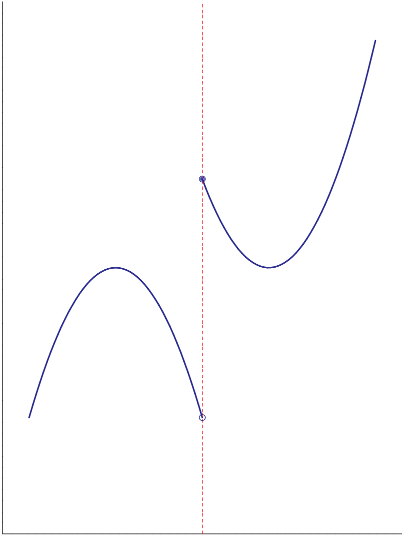


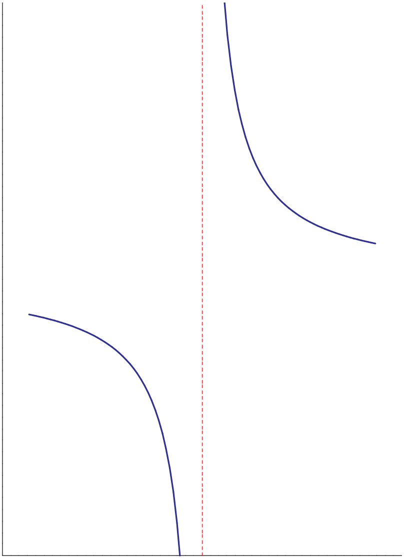


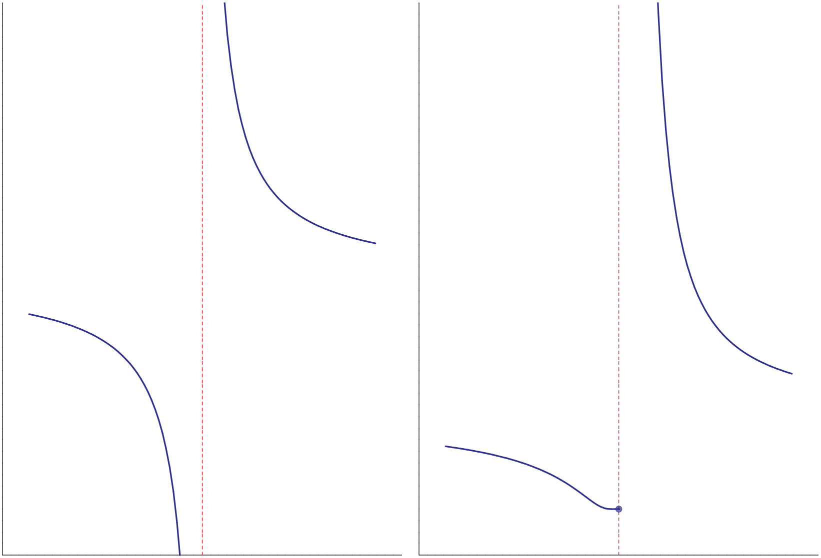












Definition

Let $c \in \mathbb{R}$ and $\varepsilon > 0$. We define

- a neighbourhood of a point c with radius ε by $B(c, \varepsilon) = (c - \varepsilon, c + \varepsilon)$,

Definition

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Definition

We say that $A \in \mathbb{R}$ is a limit of a function f at a point $c \in \mathbb{R}$ if

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Theorem 21 (uniqueness of a limit)

Let f be a function and $c \in \mathbb{R}$. Then f has a most one limit $A \in \mathbb{R}$ at c.

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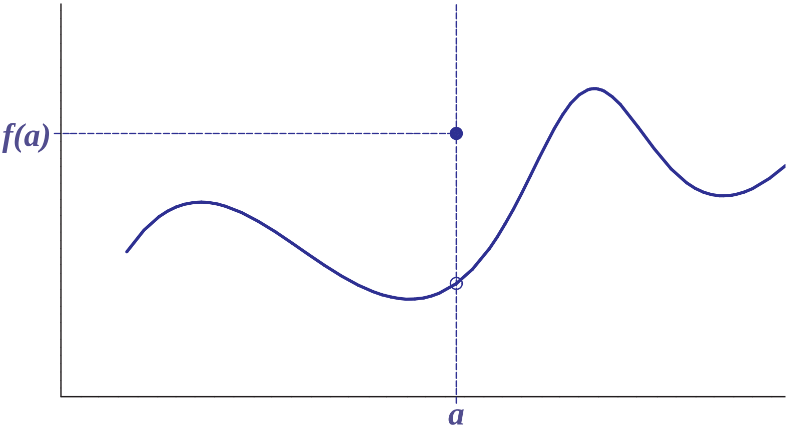
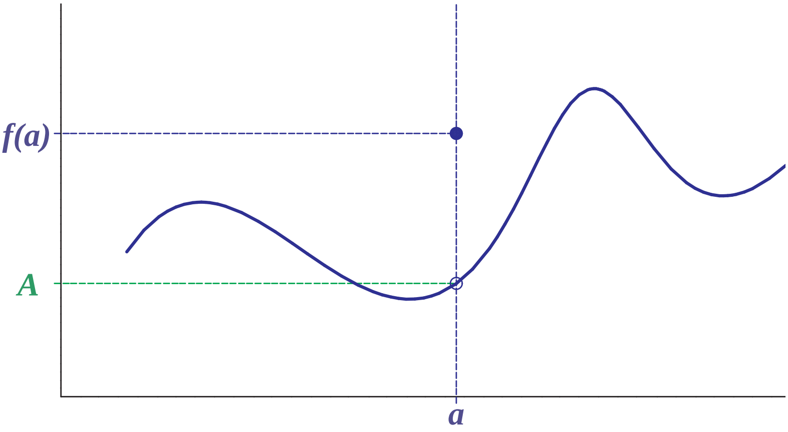
Theorem 21 (uniqueness of a limit)

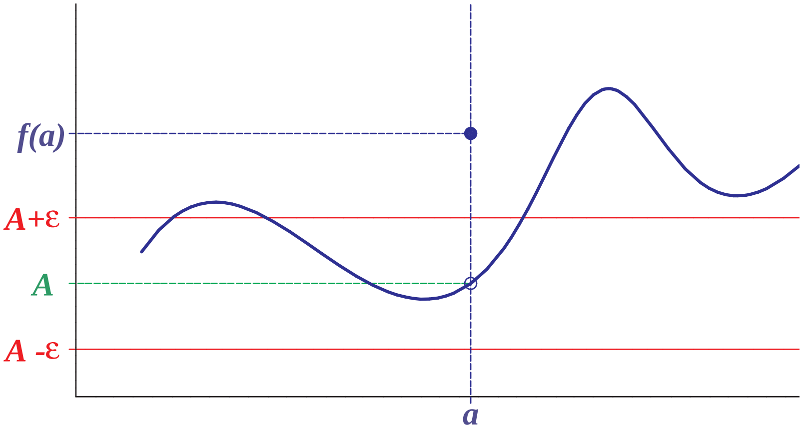
Let f be a function and $c \in \mathbb{R}$. Then f has a most one limit $A \in \mathbb{R}$ at c.

The fact that f has a limit $A \in \mathbb{R}$ at $c \in \mathbb{R}$ is denoted by $\lim\limits_{x \to c} f(x) = A$.

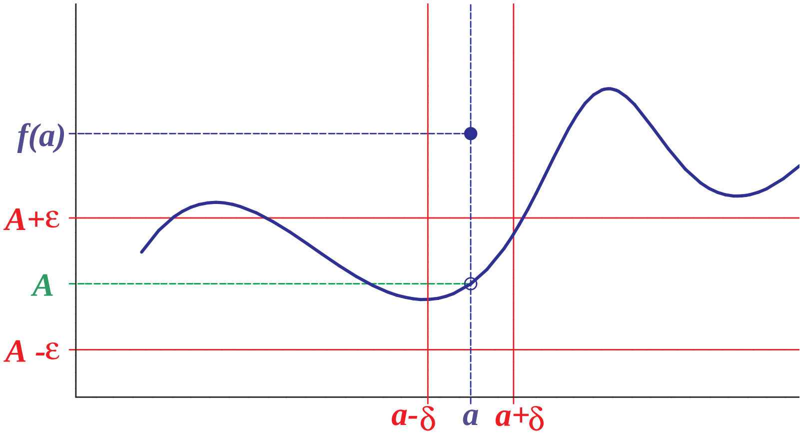


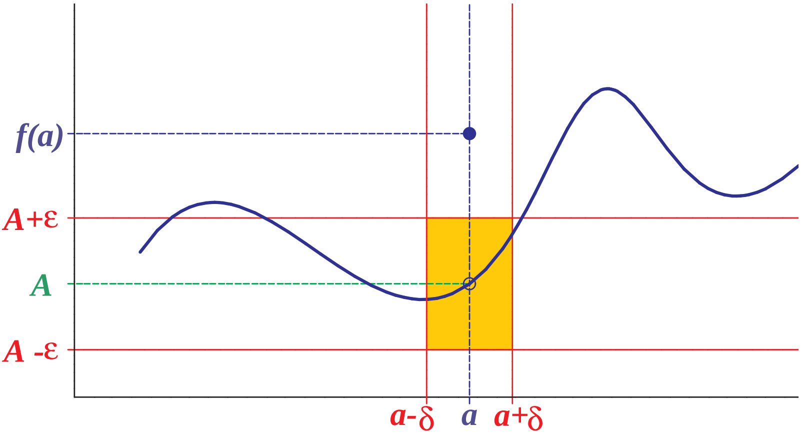
f(a)
a


f(a)
A
a

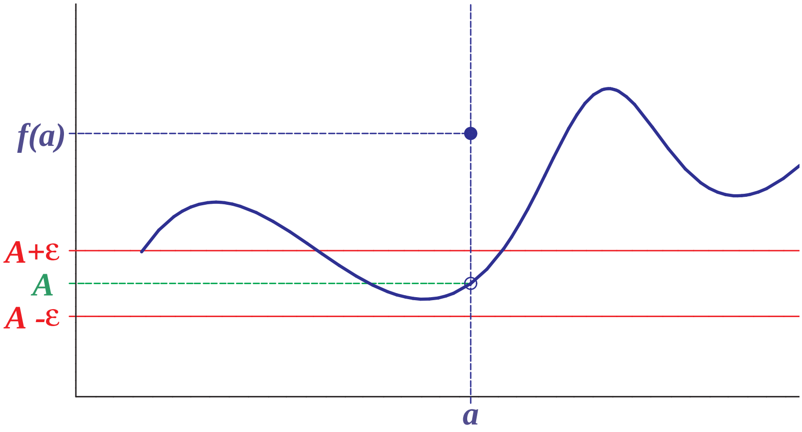


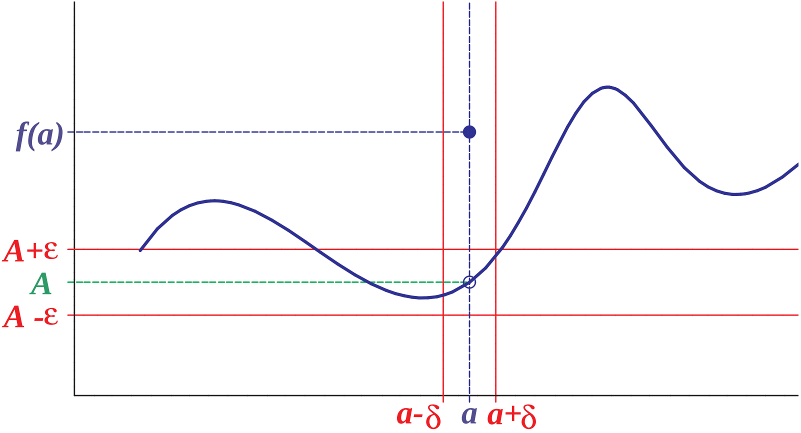
f(a)
A+ε
A
A -ε
a

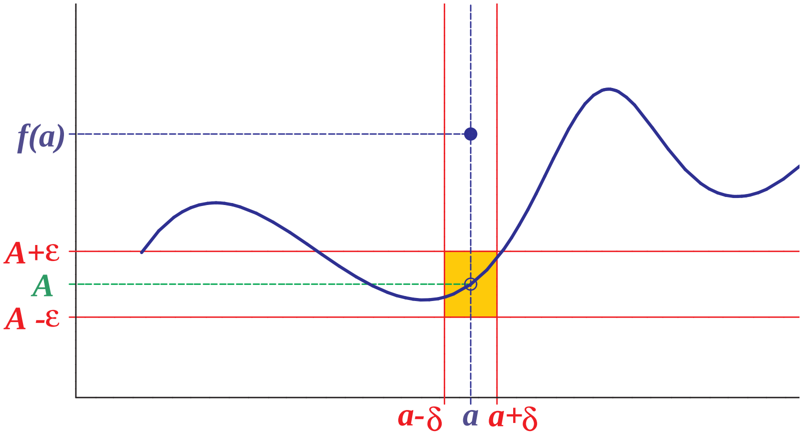

f(a)
A+ε
A
A -ε
a-δ
a
a+δ



f(a)
A+ε
A
A -ε
a-δ
a
a+δ


f(a)
A+ε
A
A -ε
a


f(a)
A+ε
A
A -ε
a-δ
a
a+δ


f(a)
A+ε
A
A -ε
a-δ
a
a+δ

Definition

We say that a function f is continuous at a point $c \in \mathbb{R}$ if

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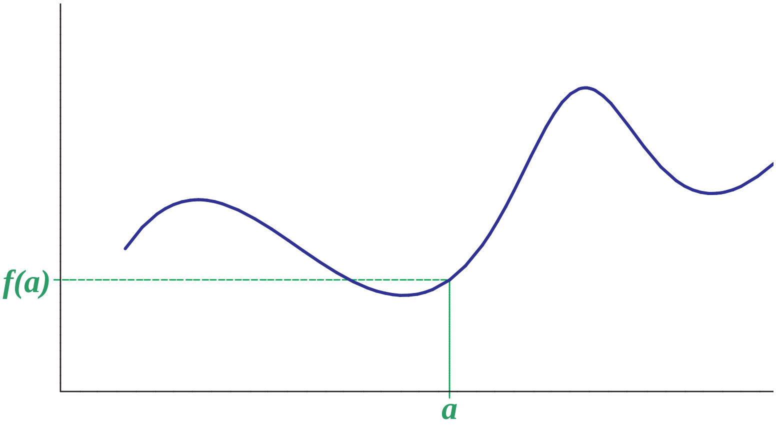
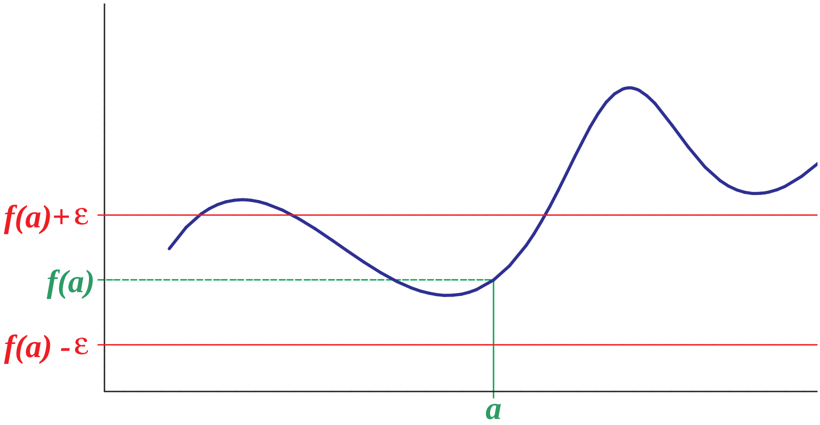
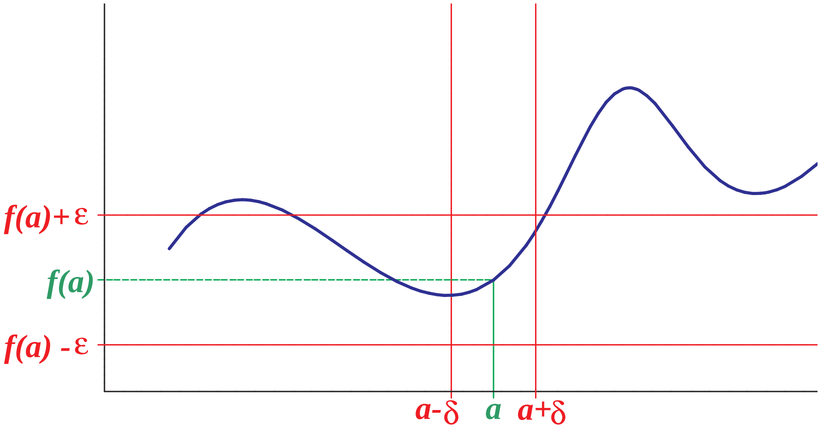
Remark

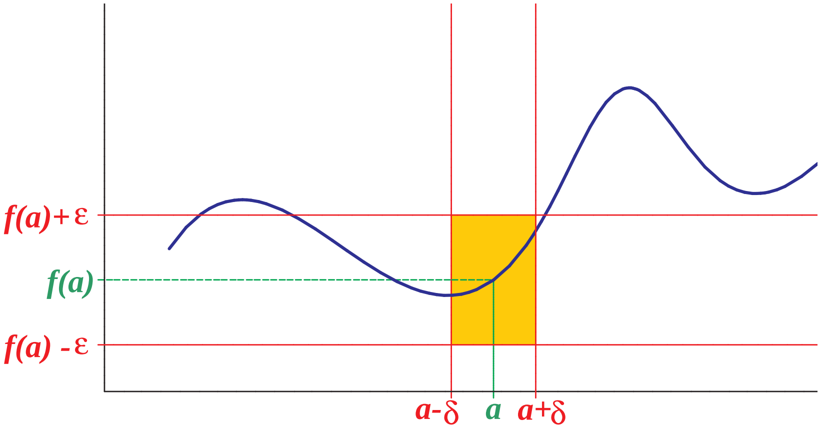
A function f is continuous at a point c if and only if

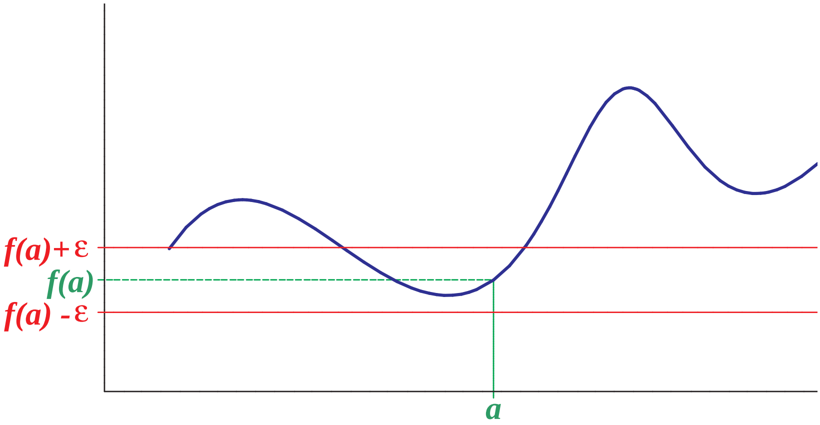
$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists \delta \in \mathbb{R}, \delta > 0 \; \forall x \in B(c, \delta) \colon \; f(x) \in B(f(c), \varepsilon).$$

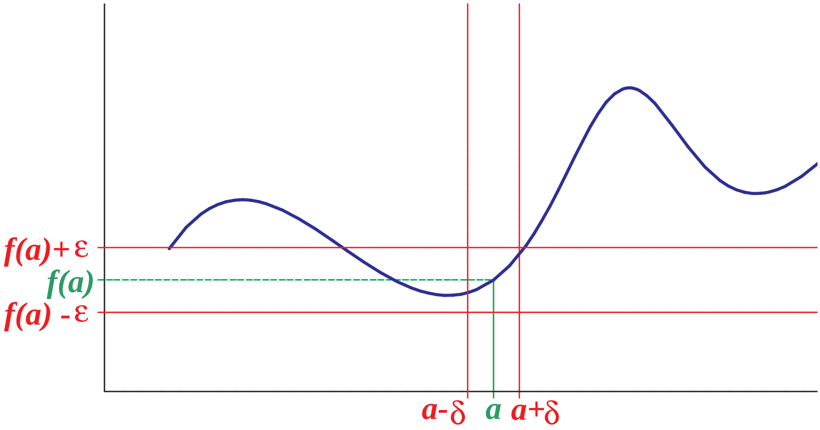

f(a)
a

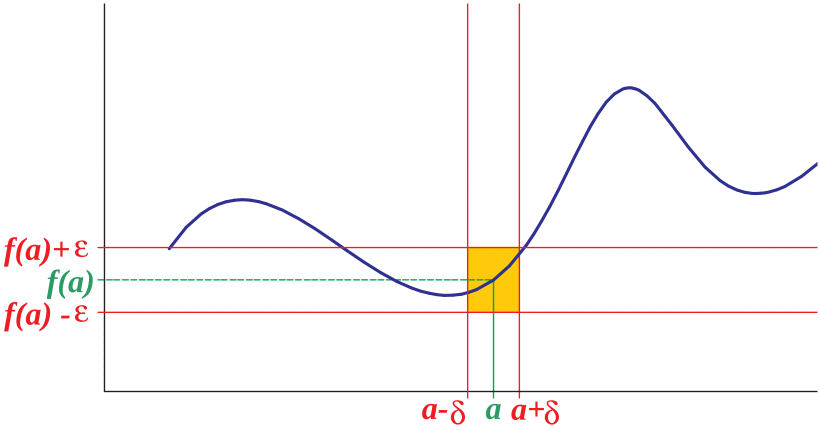

f(a)+ε
f(a)
f(a) -ε
a

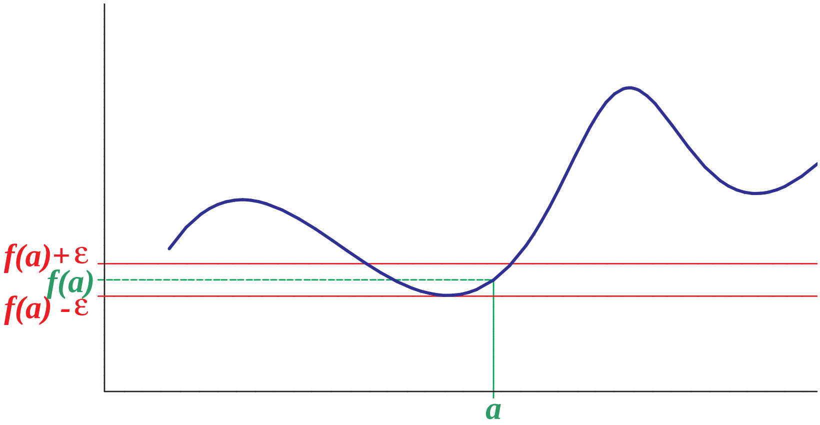

f(a)+ε
f(a)
f(a) -ε
a-δ
a
a+δ

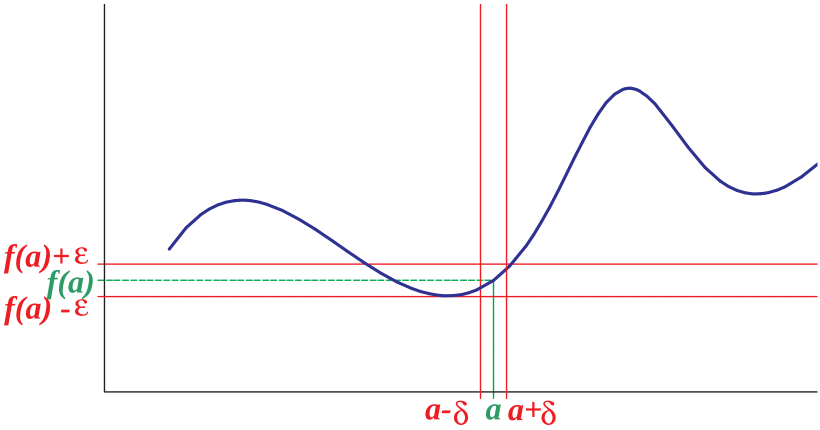

f(a)+ε
f(a)
f(a) -ε
a-δ
a
a+δ

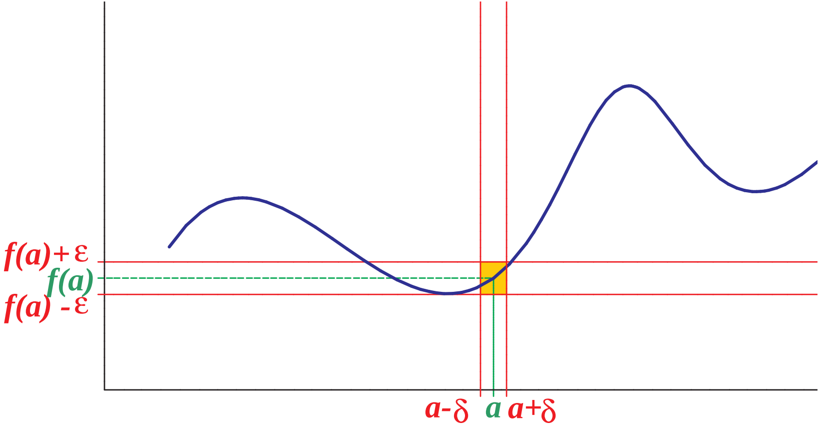

f(a)+ε
f(a)
f(a) -ε
a


f(a)+ε
f(a)
f(a) -ε
a-δ
a
a+δ


f(a)+ε
f(a)
f(a) -ε
a-δ
a
a+δ


f(a)+ε
f(a)
f(a) -ε
a


f(a)+ε
f(a)
f(a) -ε
a-δ
a
a+δ


f(a)+ε
f(a)
f(a) -ε
a-δ
a
a+δ

Definition

Let $\varepsilon > 0$. A neighbourhood and a punctured neighbourhood of $+\infty$ (resp. $-\infty$) is defined as follows:

$$P(+\infty, \varepsilon) = B(+\infty, \varepsilon) = (1/\varepsilon, +\infty),$$
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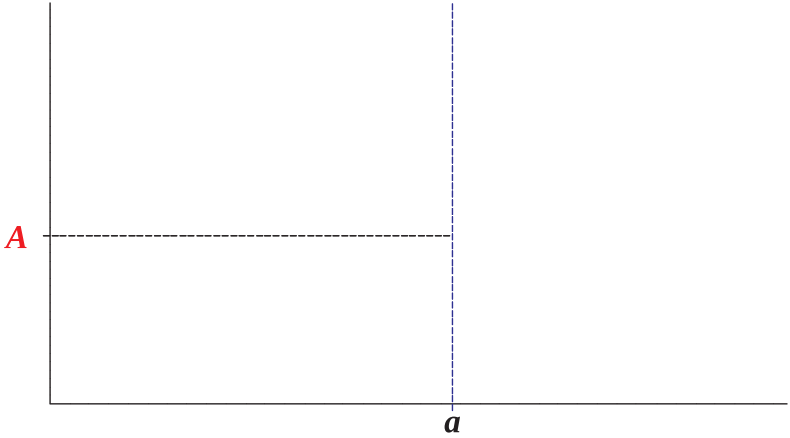
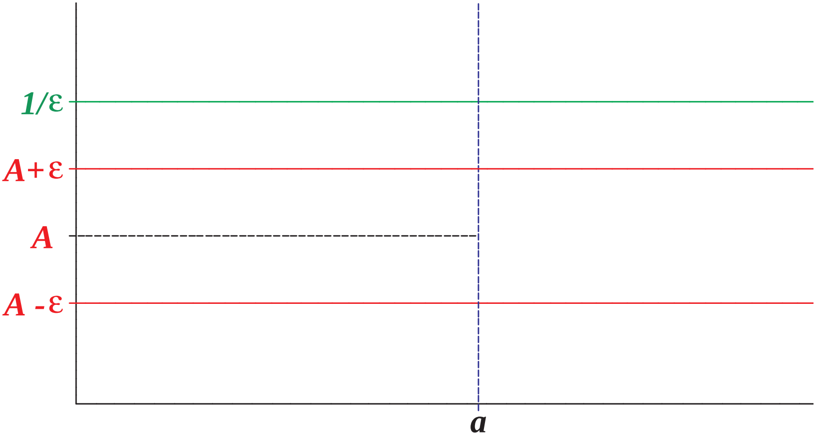
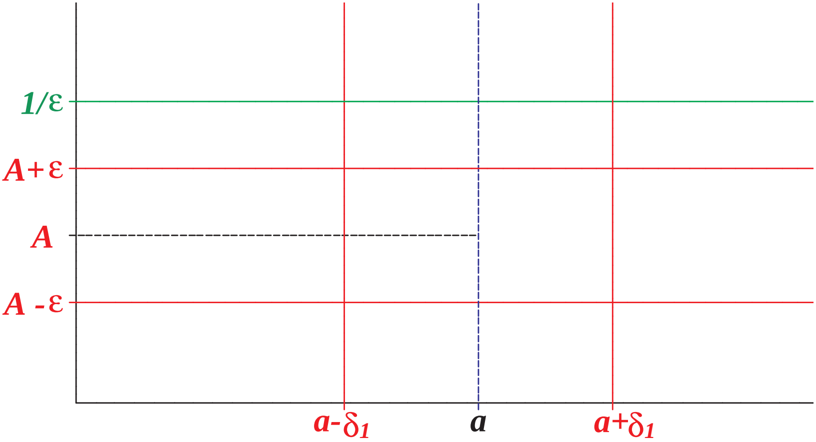
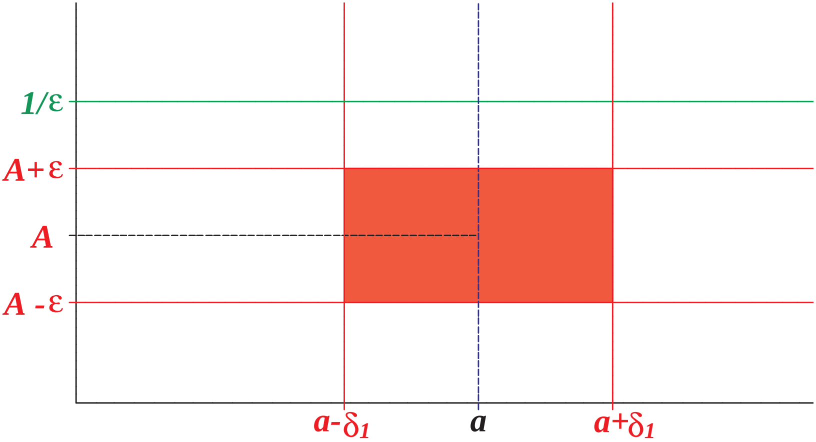
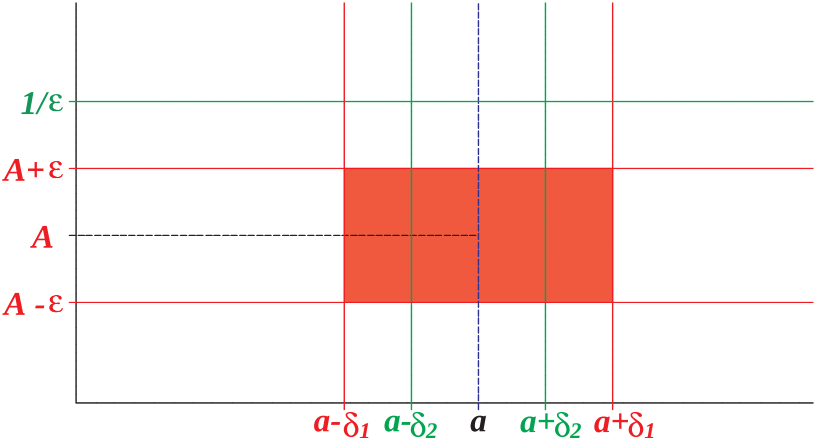
Theorem 21 holds also for $c \in \mathbb{R}^*$, $A \in \mathbb{R}^*$, so we can again use the notation $\lim_{x \to c} f(x) = A$.

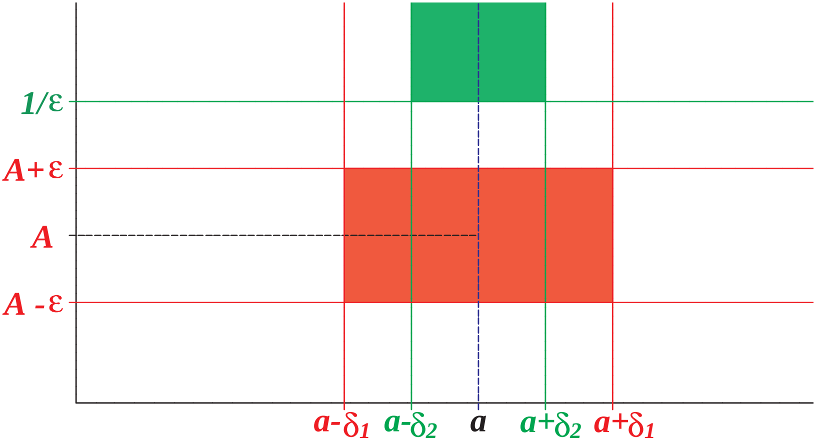

A
a

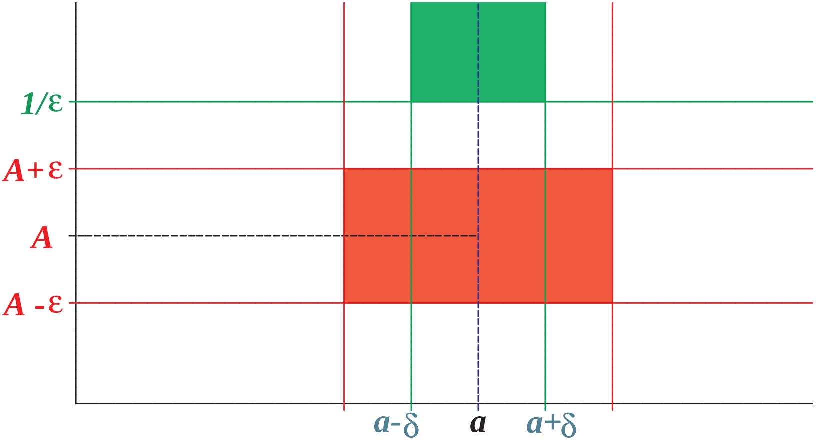

1/ε
A+ε
A
A -ε
a

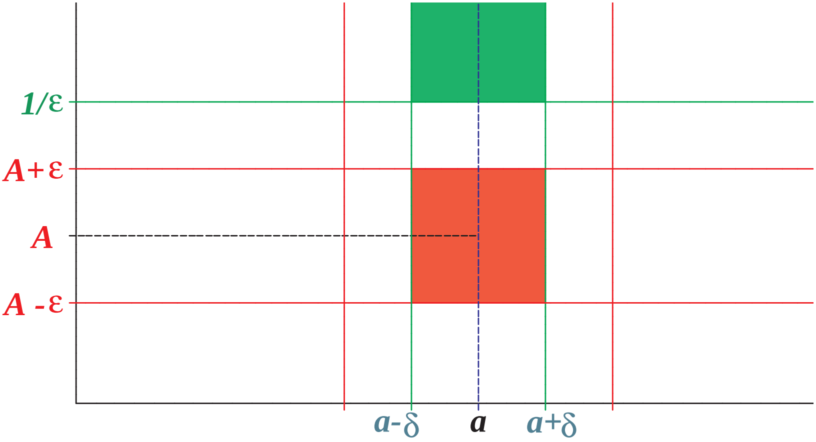

1/ε
A+ε
A
A -ε
a-δ1
a
a+δ1

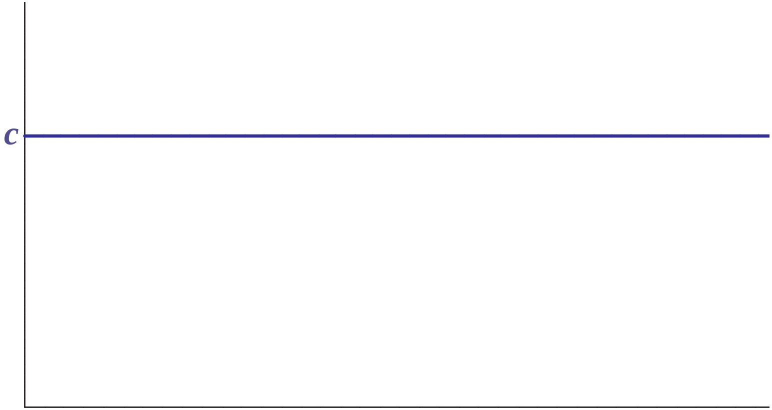

1/ε
A+ε
A
A -ε
a-δ1
a
a+δ1

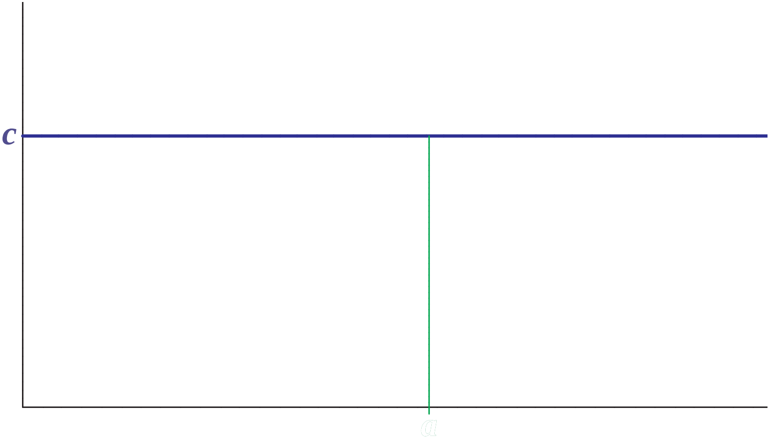

1/ε
A+ε
A
A -ε
a-δ1
a-δ2
a
a+δ2
a+δ1

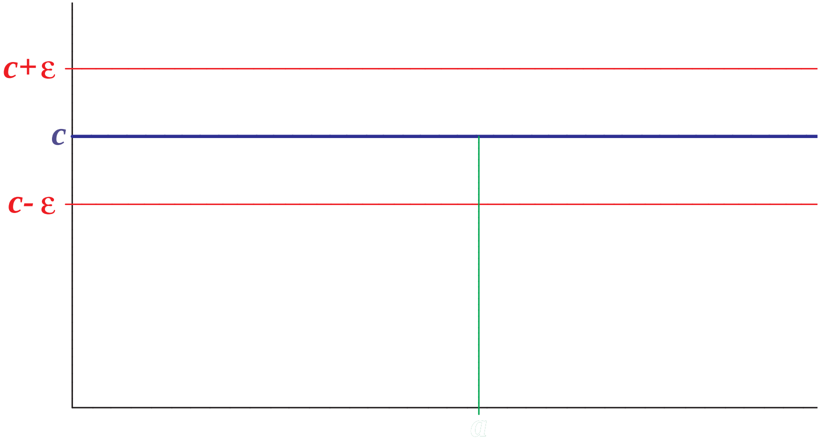

1/ε
A+ε
A
A -ε
a-δ1
a-δ2
a
a+δ2
a+δ1

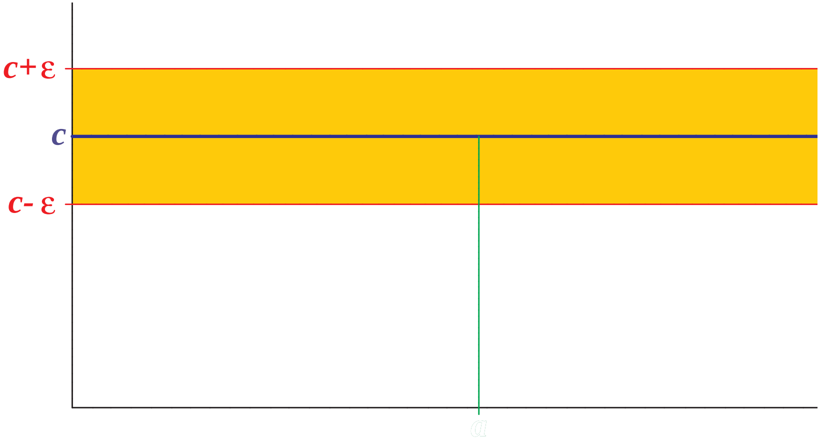

1/ε
A+ε
A
A -ε
a-δ
a
a+δ

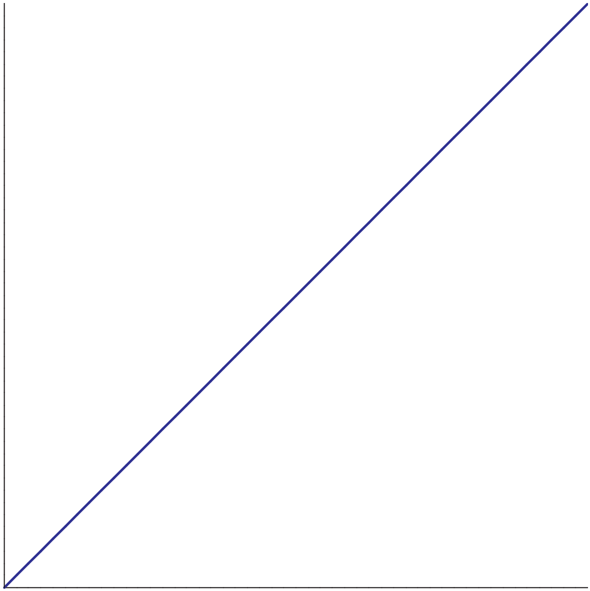

1/ε
A+ε
A
A -ε
a-δ
a
a+δ

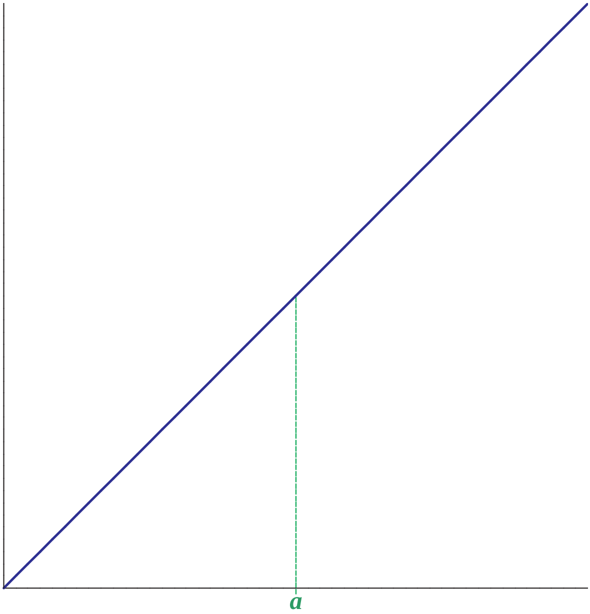

c

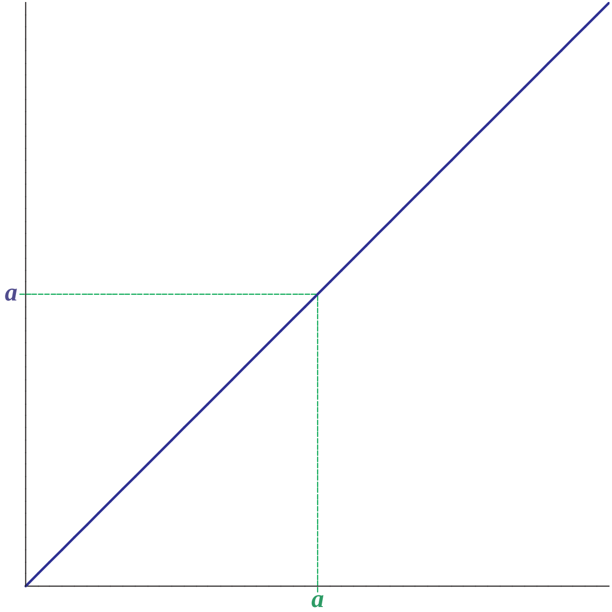

c
a

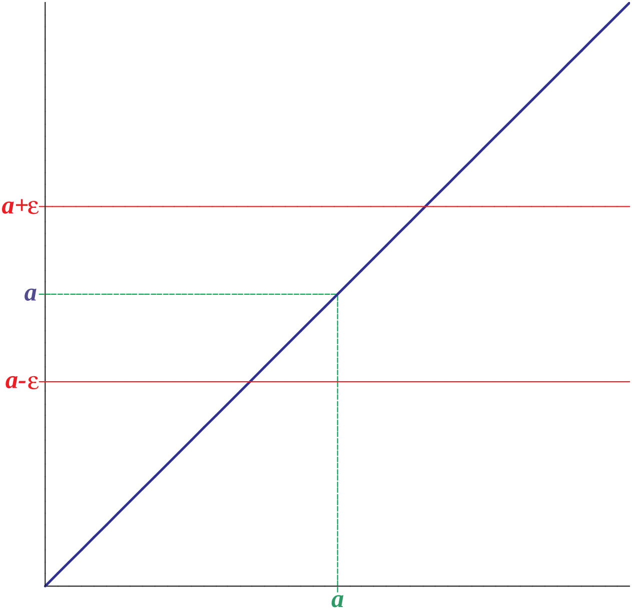

c+ε
c
c- ε
a


c+ε
c
c- ε
a

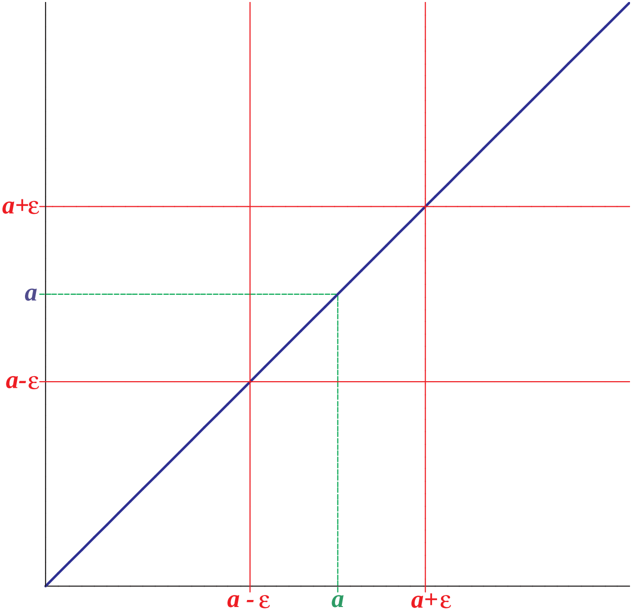


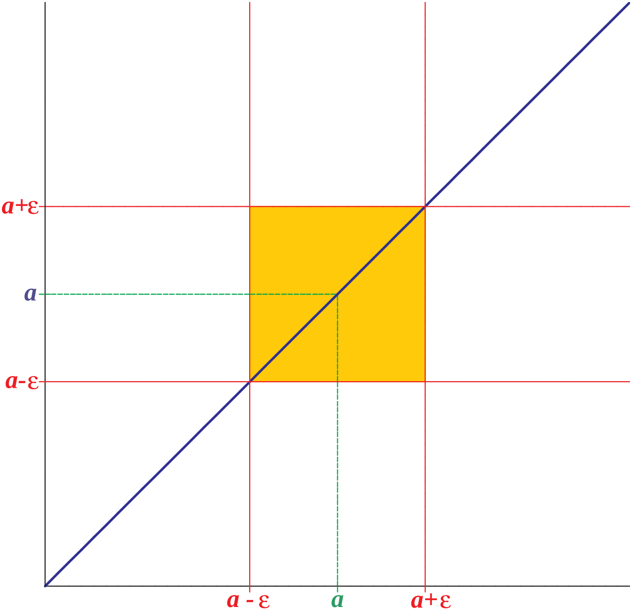

a


a
a



$a+\varepsilon$
a
$a-\varepsilon$
a


a+ε
a
a-ε
a - ε
a
a+ε


a+ε
a
a-ε
a - ε
a
a+ε

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- a left neighbourhood of c by $B^-(c,\varepsilon) = (c-\varepsilon, c]$,
- a right punctured neighbourhood of c by $P^+(c,\varepsilon) = (c, c+\varepsilon)$,

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Definition

Let $c \in \mathbb{R}$ and $\varepsilon > 0$. We define

- a right neighbourhood of c by $B^+(c, \varepsilon) = [c, c + \varepsilon)$,
- a left neighbourhood of c by $B^-(c, \varepsilon) = (c - \varepsilon, c]$,
- a right punctured neighbourhood of c by $P^+(c, \varepsilon) = (c, c + \varepsilon)$,
- a left punctured neighbourhood of c by $P^-(c, \varepsilon) = (c - \varepsilon, c)$,
- a left neighbourhood and left punctured neighbourhood of $+\infty$ by $B^-(+\infty, \varepsilon) = P^-(+\infty, \varepsilon) = (1/\varepsilon, +\infty)$,

Definition

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- a right punctured neighbourhood of c by $P^+(c, \varepsilon) = (c, c + \varepsilon)$,
- a left punctured neighbourhood of c by $P^-(c, \varepsilon) = (c - \varepsilon, c)$,
- a left neighbourhood and left punctured neighbourhood of $+\infty$ by $B^-(+\infty, \varepsilon) = P^-(+\infty, \varepsilon) = (1/\varepsilon, +\infty)$,
- a right neighbourhood and right punctured neighbourhood of $-\infty$ by $B^+(-\infty, \varepsilon) = P^+(-\infty, \varepsilon) = (-\infty, -1/\varepsilon)$.

Definition

Let $A \in \mathbb{R}^*$, $c \in \mathbb{R} \cup \{-\infty\}$. We say that a function f has a limit from the right at c equal to $A \in \mathbb{R}^*$ (denoted by $\lim\limits_{x \to c+} f(x) = A$) if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists \delta \in \mathbb{R}, \delta > 0\ \forall x \in P^+(c, \delta) \colon f(x) \in B(A, \varepsilon).$$

Analogously we define the notion of limit from the left at $c \in \mathbb{R} \cup \{+\infty\}$ and we use the notation $\lim\limits_{x \to c-} f(x)$.

Definition

Let $A \in \mathbb{R}^*$, $c \in \mathbb{R} \cup \{-\infty\}$. We say that a function f has a limit from the right at c equal to $A \in \mathbb{R}^*$ (denoted by $\lim\limits_{x \to c+} f(x) = A$) if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists \delta \in \mathbb{R}, \delta > 0 \; \forall x \in P^+(c, \delta) \colon f(x) \in B(A, \varepsilon).$$

Analogously we define the notion of limit from the left at $c \in \mathbb{R} \cup \{+\infty\}$ and we use the notation $\lim\limits_{x \to c-} f(x)$.

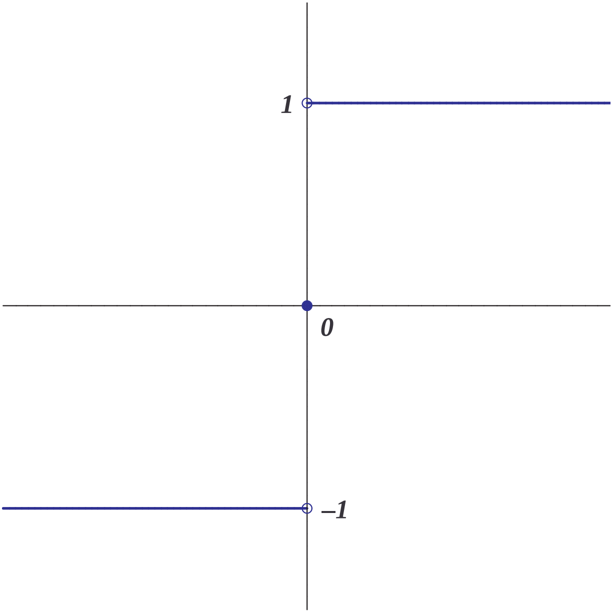
Remark

Let $c \in \mathbb{R}$, $A \in \mathbb{R}^*$. Then

$$\lim_{x \to c} f(x) = A \Leftrightarrow \left(\lim_{x \to c+} f(x) = A \;\; \& \;\; \lim_{x \to c-} f(x) = A \right).$$

Definition

Let $c \in \mathbb{R}$. We say that a function f is continuous at c from the right (from the left, resp.) if $\lim_{x \to c+} f(x) = f(c)$ ($\lim_{x \to c-} f(x) = f(c)$, resp.).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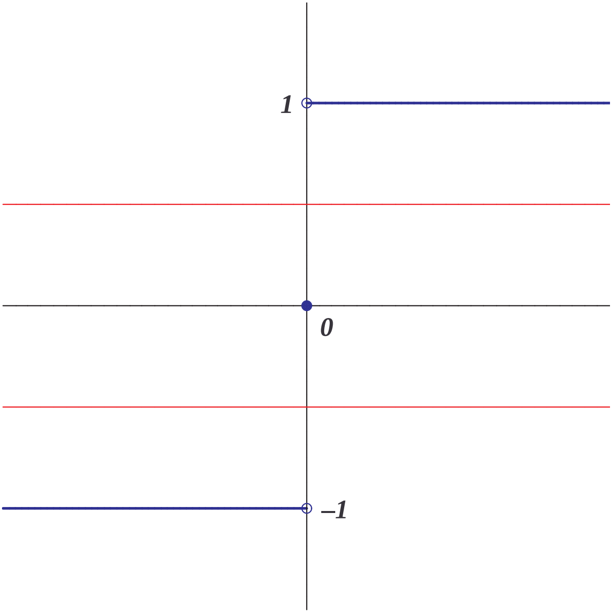
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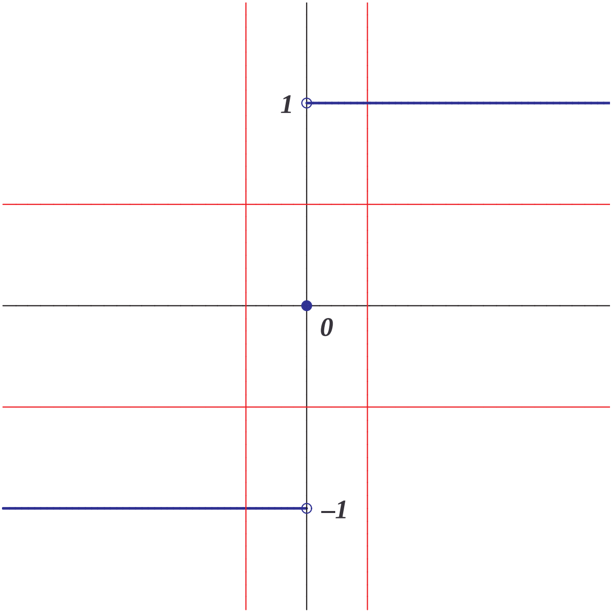
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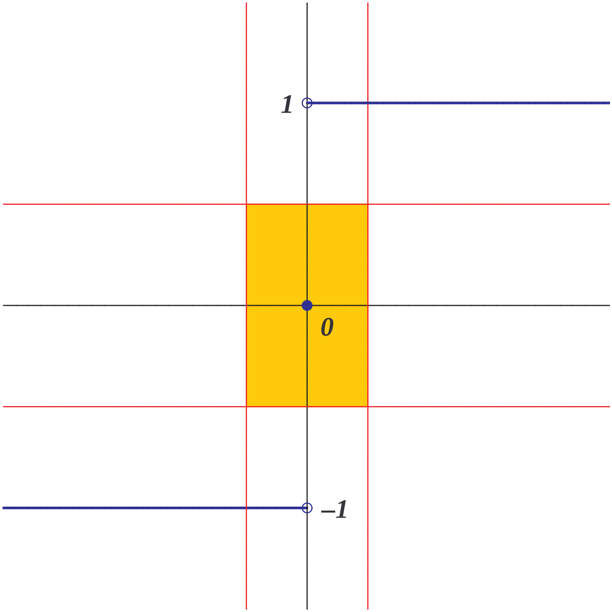
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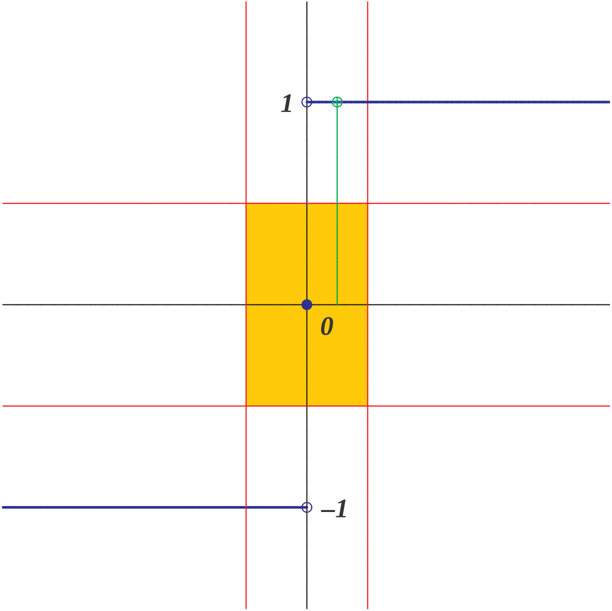
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0
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Theorem 22

Let f has a finite limit at $c \in \mathbb{R}^*$. Then there exists $\delta > 0$ such that f is bounded on $P(c, \delta)$.

Theorem 23 (arithmetics of limits)

Let $c \in \mathbb{R}^*$, $\lim_{x \to c} f(x) = A \in \mathbb{R}^*$ *and* $\lim_{x \to c} g(x) = B \in \mathbb{R}^*$. *Then*

(i) $\lim_{x \to c}(f(x) + g(x)) = A + B$ *if the expression* $A + B$ *is defined,*

(ii) $\lim_{x \to c} f(x)g(x) = AB$ *if the expression* AB *is defined,*

(iii) $\lim_{x \to c} f(x)/g(x) = A/B$ *if the expression* A/B *is defined.*

Theorem 23 (arithmetics of limits)

Let $c \in \mathbb{R}^*$, $\lim_{x \to c} f(x) = A \in \mathbb{R}^*$ *and* $\lim_{x \to c} g(x) = B \in \mathbb{R}^*$. *Then*

(i) $\lim_{x \to c}(f(x) + g(x)) = A + B$ *if the expression* $A + B$ *is defined,*

(ii) $\lim_{x \to c} f(x)g(x) = AB$ *if the expression* AB *is defined,*

(iii) $\lim_{x \to c} f(x)/g(x) = A/B$ *if the expression* A/B *is defined.*

Corollary

Suppose that the functions f *and* g *are continuous at* $c \in \mathbb{R}$. *Then also the functions* $f + g$ *and* fg *are continuous at* c. *If moreover* $g(c) \neq 0$, *then also the function* f/g *is continuous at* c.

Theorem 24

Let $c \in \mathbb{R}^*$, $\lim_{x \to c} g(x) = 0$, $\lim_{x \to c} f(x) = A \in \mathbb{R}^*$ *and* $A > 0$*. If there exists* $\eta > 0$ *such that the function* g *is positive on* $P(c, \eta)$*, then* $\lim_{x \to c}\big(f(x)/g(x)\big) = +\infty$*.*

Theorem 25 (limits and inequalities)

Suppose that $c \in \mathbb{R}^*$ *and* $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ *exist.*
(i) If $\lim_{x\to c} f(x) > \lim_{x\to c} g(x)$, *then there exists* $\delta > 0$ *such that*

$$\forall x \in P(c,\delta)\colon\ f(x) > g(x).$$

Theorem 25 (limits and inequalities)

Suppose that $c \in \mathbb{R}^*$ *and* $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ *exist.*
(i) If $\lim_{x\to c} f(x) > \lim_{x\to c} g(x)$, *then there exists* $\delta > 0$ *such that*

$$\forall x \in P(c,\delta)\colon\ f(x) > g(x).$$

(ii) If there exists $\delta > 0$ *such that* $\forall x \in P(c,\delta)\colon\ f(x) \le g(x)$, *then*

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x).$$

Theorem 25 (limits and inequalities)

Suppose that $c \in \mathbb{R}^*$ *and* $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ *exist.*
(i) If $\lim_{x\to c} f(x) > \lim_{x\to c} g(x)$, *then there exists* $\delta > 0$ *such that*

$$\forall x \in P(c,\delta)\colon\ f(x) > g(x).$$

(ii) If there exists $\delta > 0$ *such that* $\forall x \in P(c,\delta)\colon\ f(x) \leq g(x)$, *then*

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

(iii) (two policemen/sandwich theorem) Suppose that there exists $\eta > 0$ *such that*

$$\forall x \in P(c,\eta)\colon\ f(x) \leq h(x) \leq g(x).$$

If moreover $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = A \in \mathbb{R}^*$, *then the limit* $\lim_{x\to c} h(x)$ *also exists and equals* A.

Corollary

Let $c \in \mathbb{R}^*$, $\lim_{x \to c} f(x) = 0$ and suppose there exists $\eta > 0$ such that g is bounded on $P(c, \eta)$. Then $\lim_{x \to c} \big(f(x)g(x)\big) = 0$.

Theorem 26 (limit of a composition)

Let $c, A, B \in \mathbb{R}^*$, $\lim_{x \to c} g(x) = A$, $\lim_{y \to A} f(y) = B$ *and at least one of the following conditions is satisfied:*

(I) $\exists \eta \in \mathbb{R}, \eta > 0\ \forall x \in P(c, \eta) \colon g(x) \neq A$,

(C) *the function* f *is continuous at* A.

Then

$$\lim_{x \to c} f\big(g(x)\big) = B.$$

Theorem 26 (limit of a composition)

Let $c, A, B \in \mathbb{R}^*$, $\lim_{x\to c} g(x) = A$, $\lim_{y\to A} f(y) = B$ *and at least one of the following conditions is satisfied:*

(I) $\exists \eta \in \mathbb{R}, \eta > 0 \ \forall x \in P(c,\eta)\colon g(x) \neq A$,

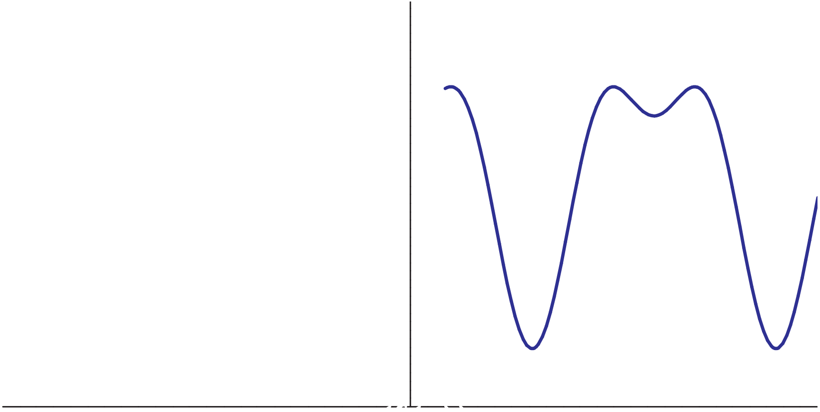
(C) *the function* f *is continuous at* A*.*

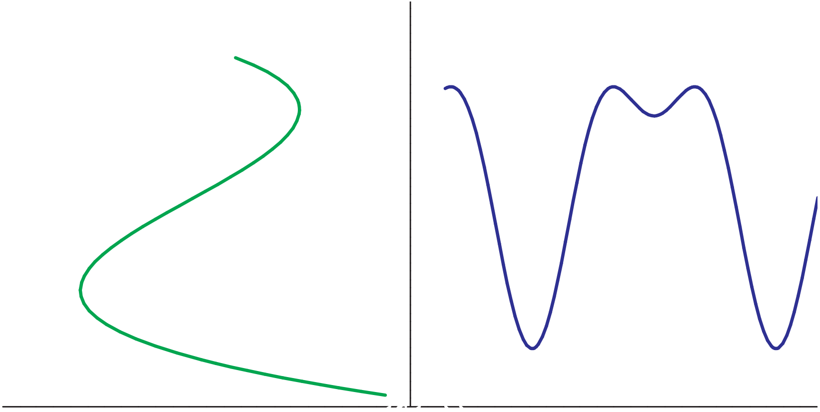
Then

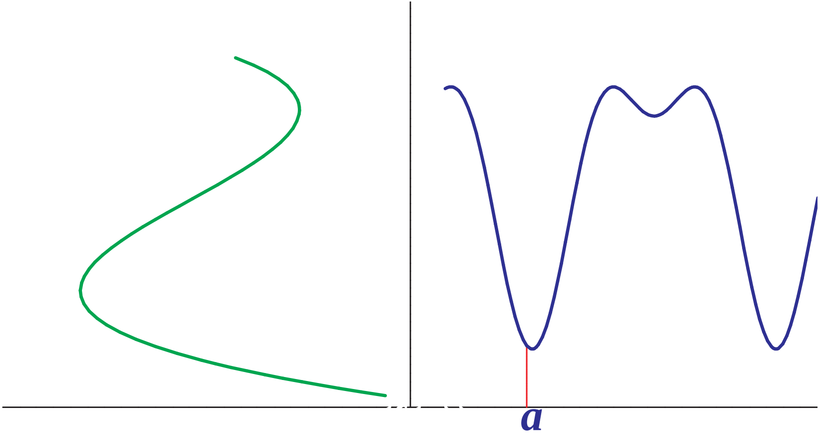
$$\lim_{x\to c} f\big(g(x)\big) = B.$$

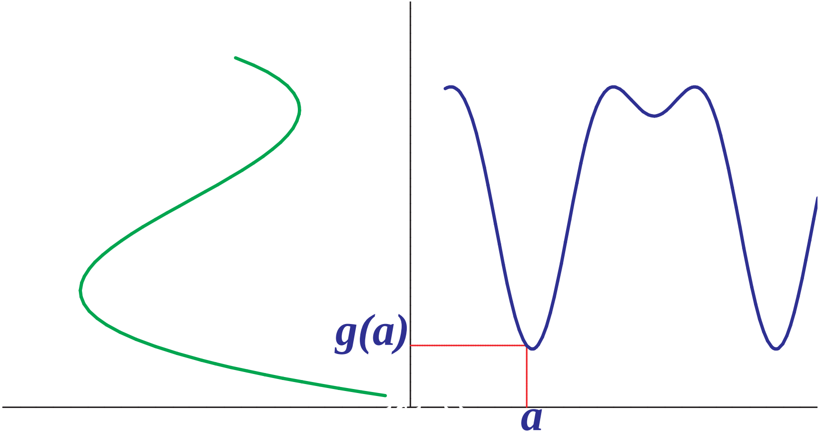
Corollary

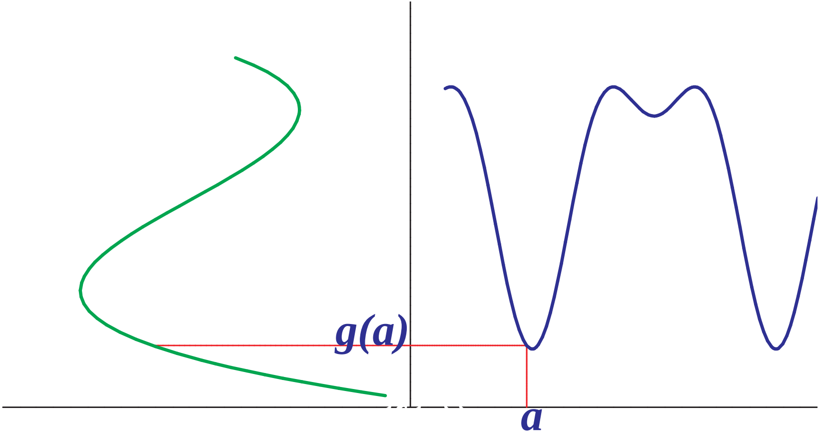
Suppose that the function g *is continuous at* $c \in \mathbb{R}$ *and the function* f *is continuous at* $g(c)$*. Then the function* $f \circ g$ *is continuous at* c*.*

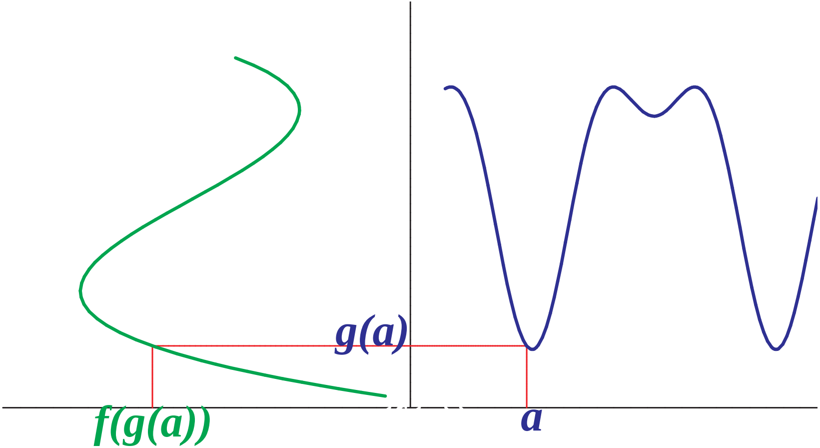


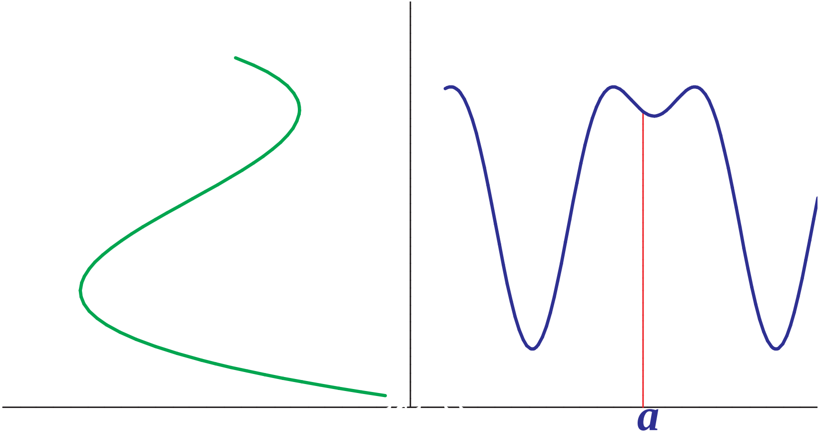


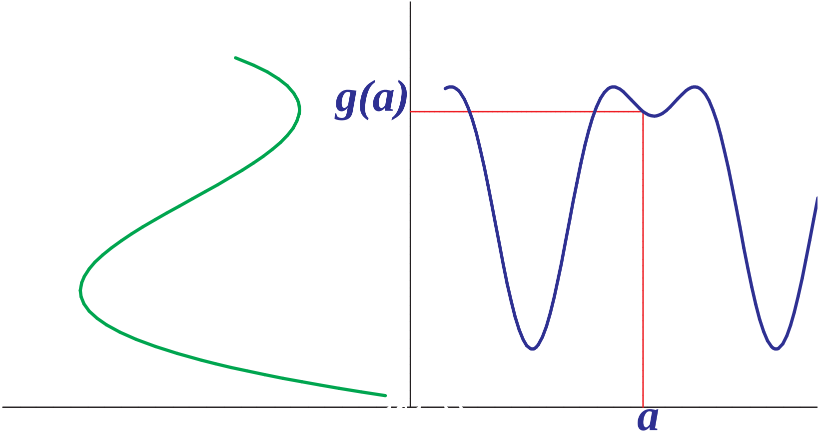

a

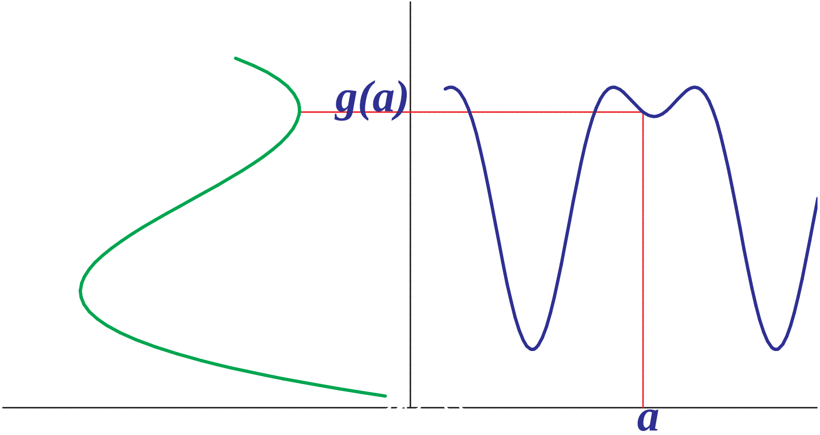

g(a)
a

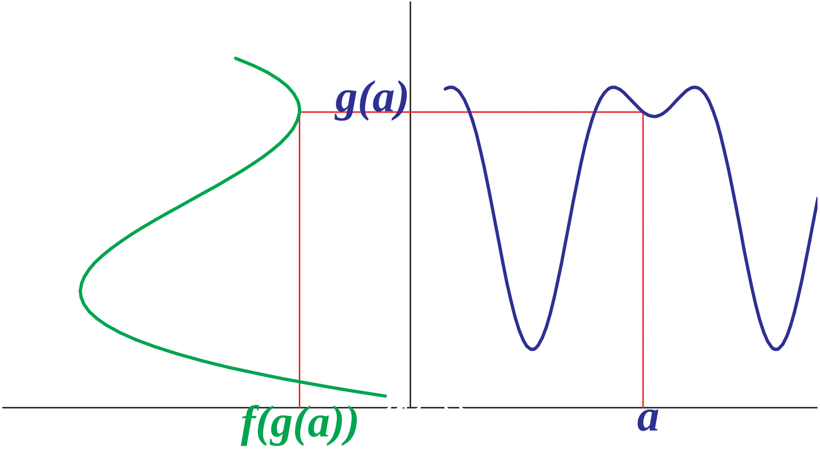

g(a)
a

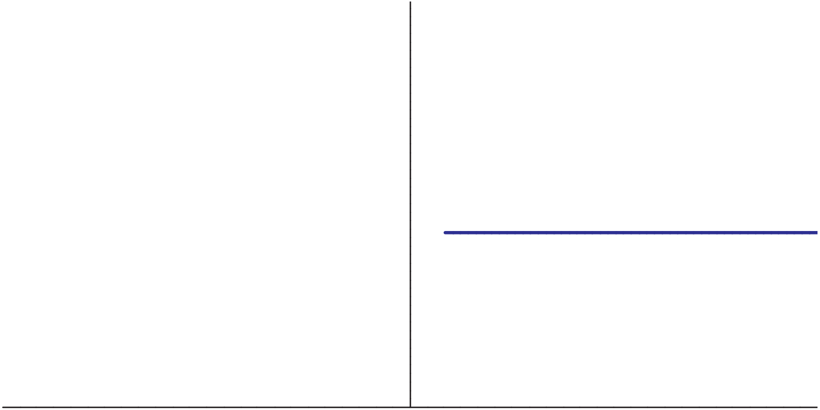

g(a)
f(g(a))
a

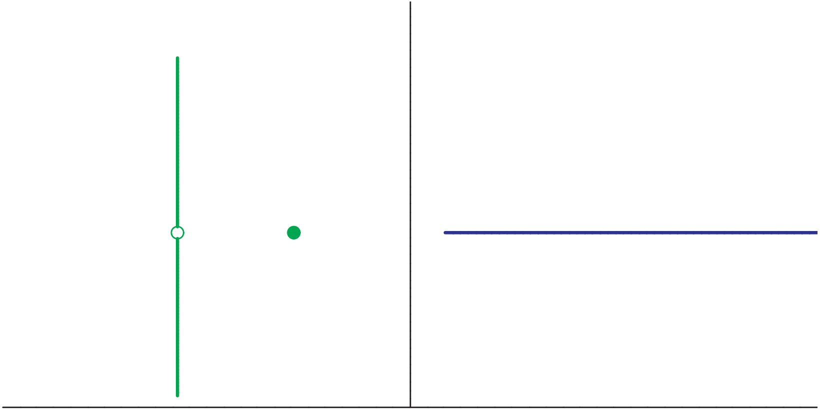

a

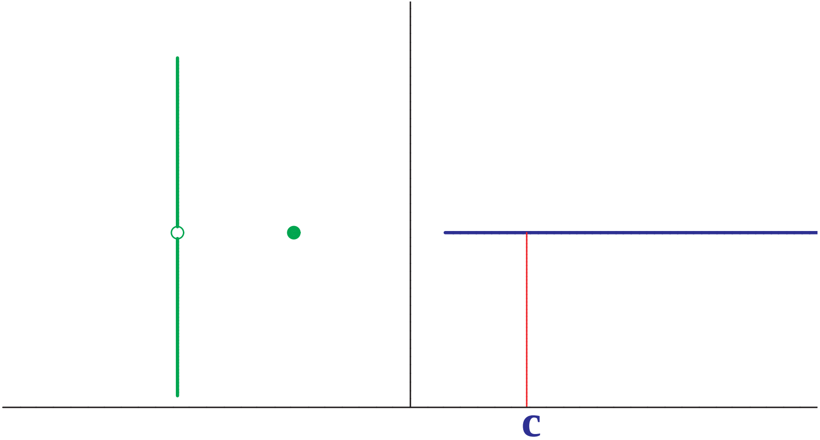

g(a)
a

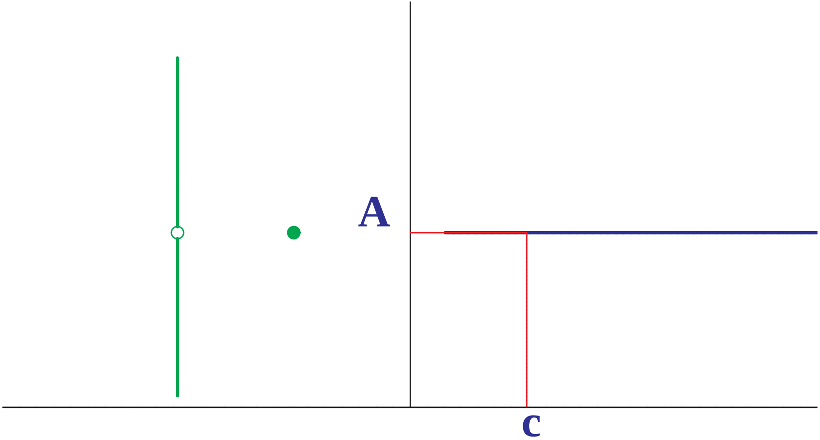

g(a)
a

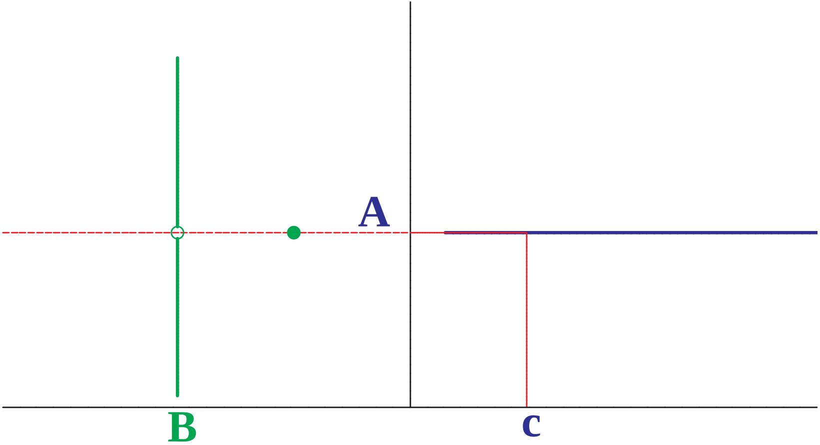

g(a)
f(g(a))
a

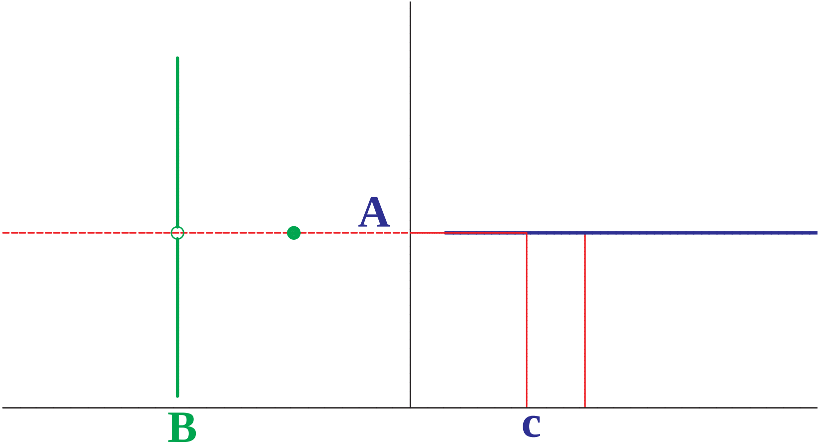


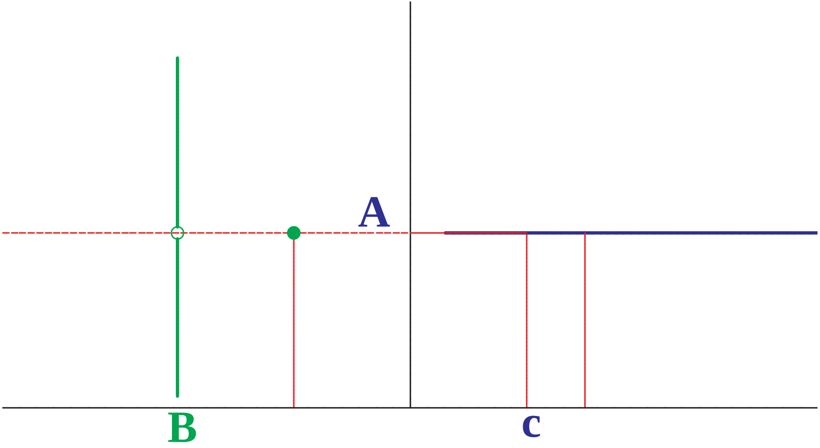


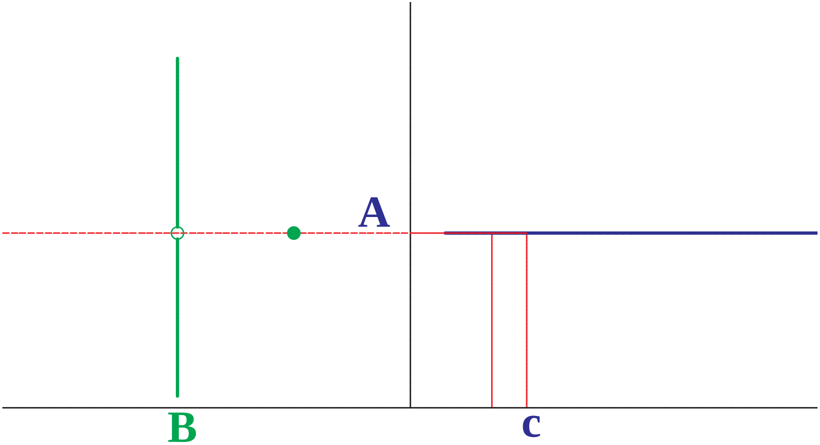

c

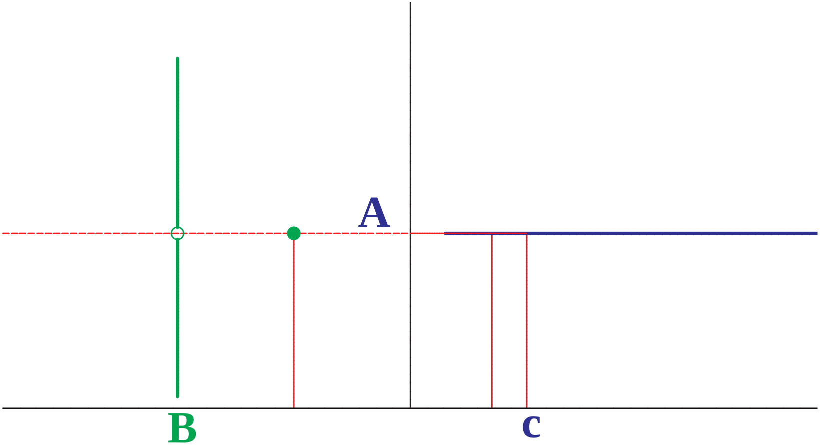

A
c

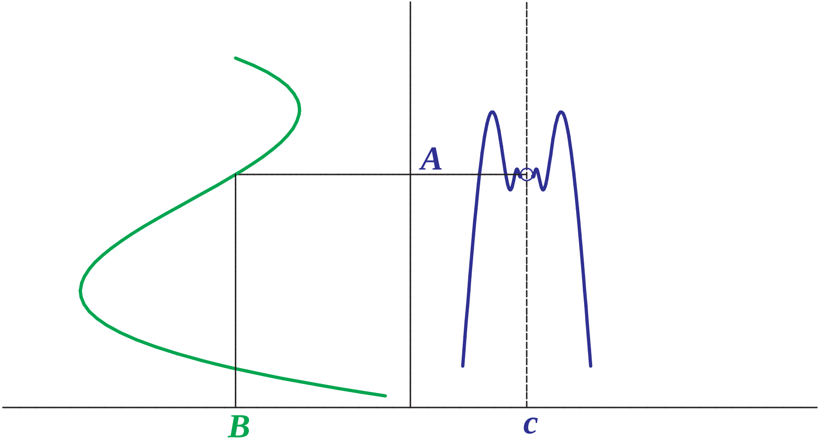

A
B
c

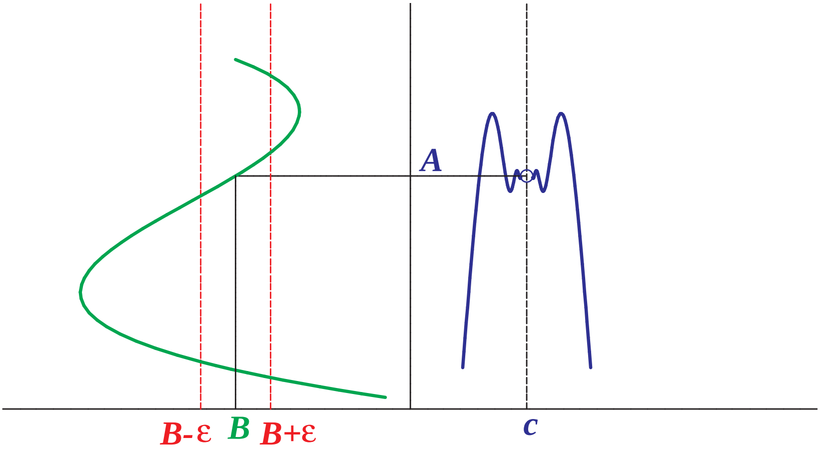

A
B
c

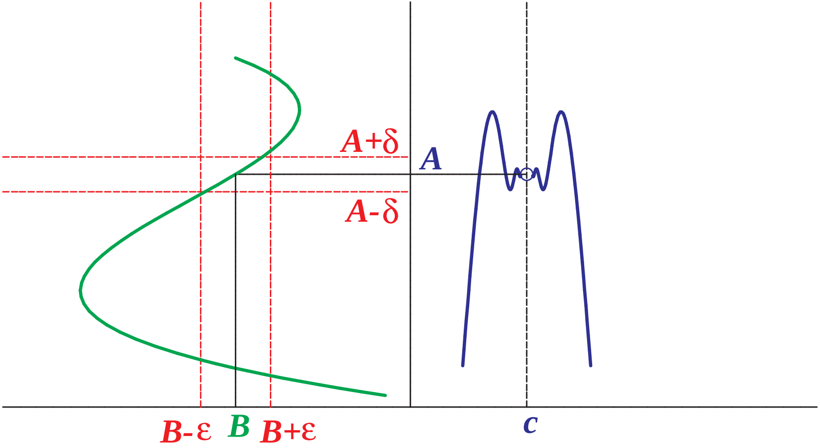

A
B
c

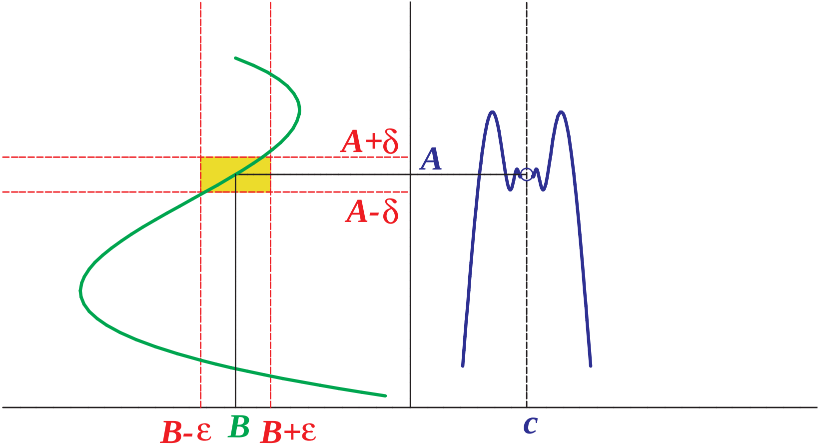

A
B
c

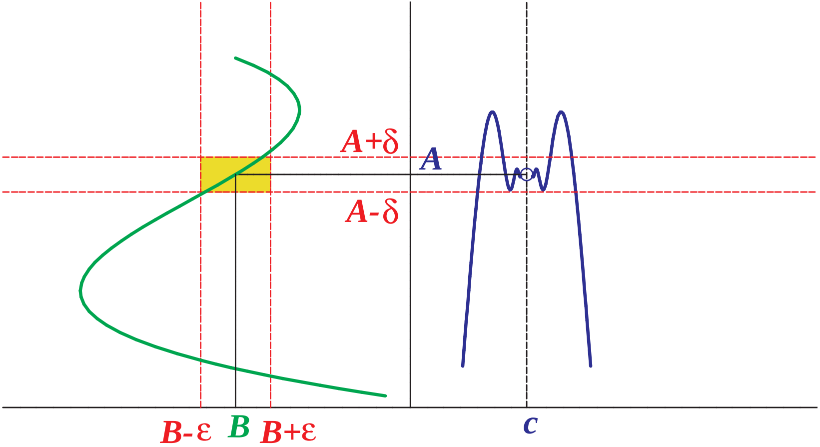

A
B
c

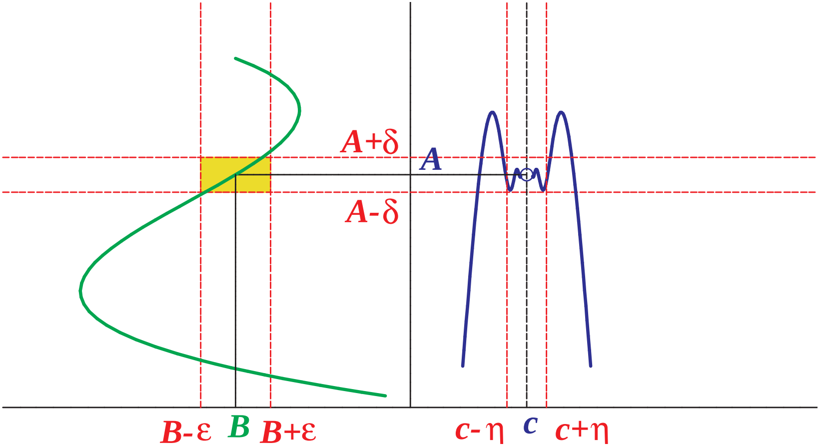

A
B
c

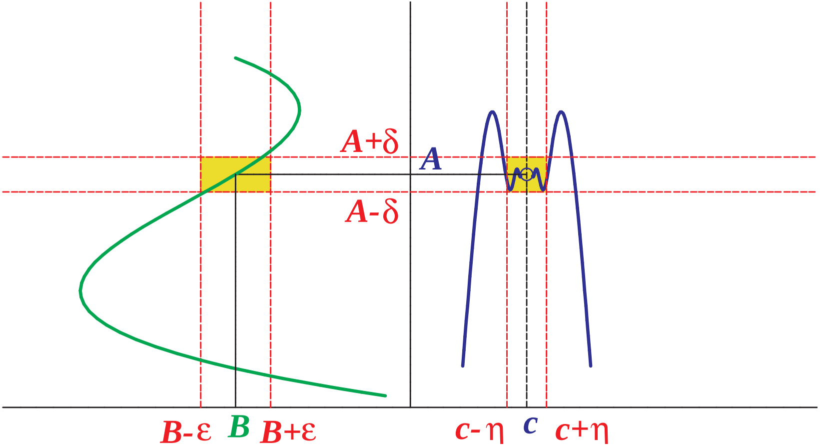

A
B-ε
B
B+ε
c

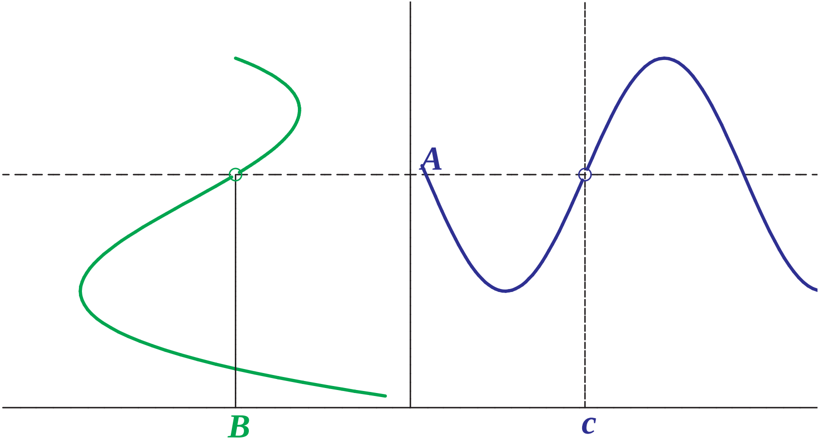

A+δ
A
A-δ
B-ε
B
B+ε
c

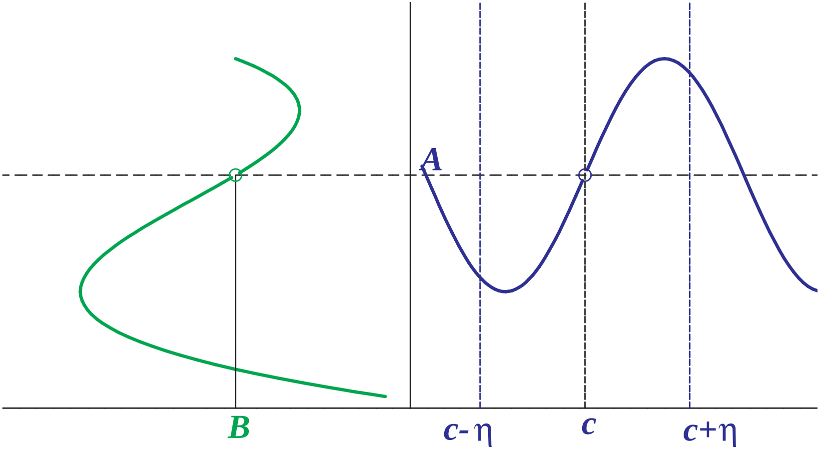

A+δ
A
A-δ
B-ε
B
B+ε
c

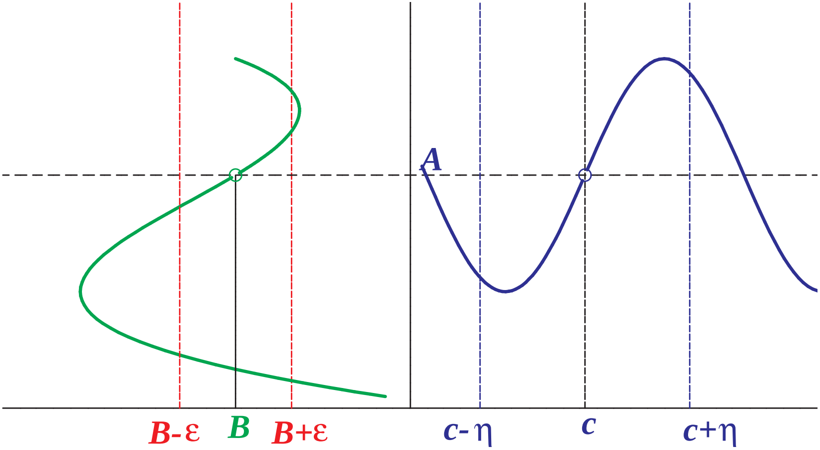

A+δ
A
A-δ
B-ε
B
B+ε
c

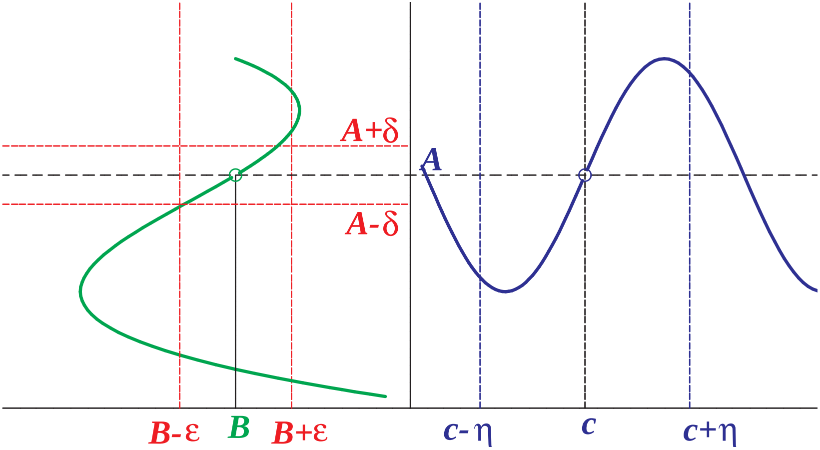

A+δ
A
A-δ
B-ε
B
B+ε
c- η
c
c+η

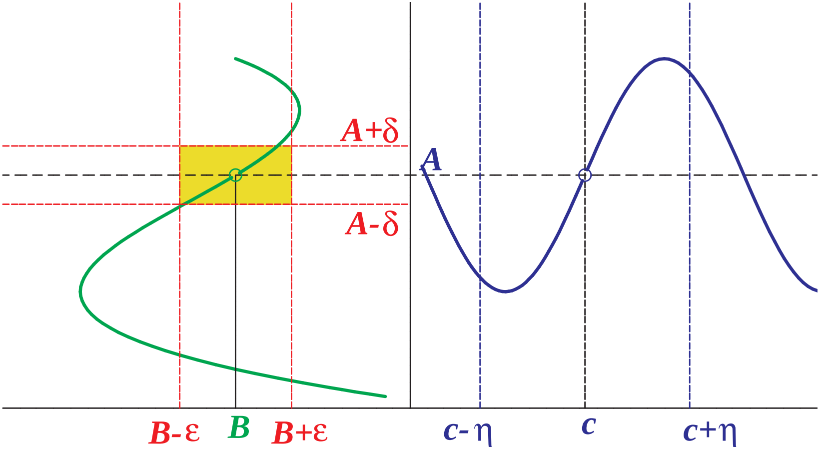

$A+\delta$
A
$A-\delta$
$B-\varepsilon$
B
$B+\varepsilon$
$c-\eta$
c
$c+\eta$

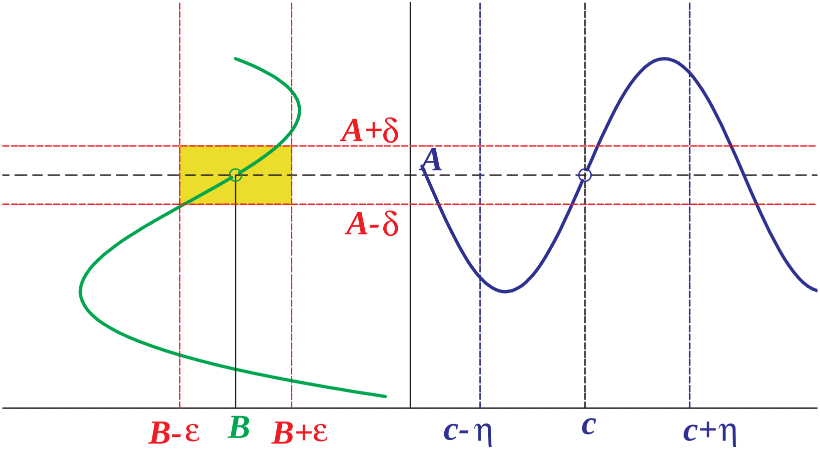

A
B
c

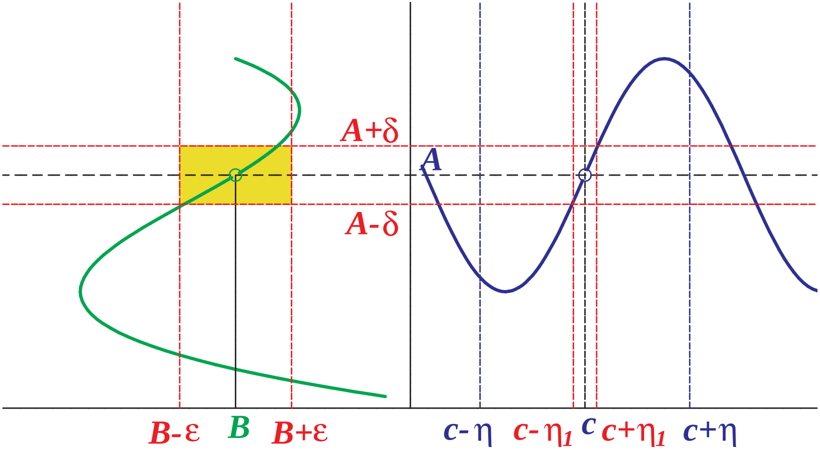

A
B
c- η
c
c+η

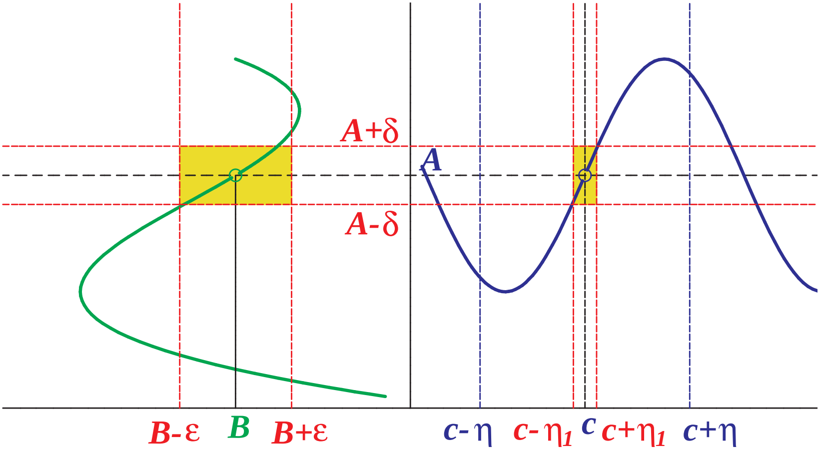

A
B-ε
B
B+ε
c- η
c
c+η


A+δ
A
A-δ
B-ε
B
B+ε
c- η
c
c+η

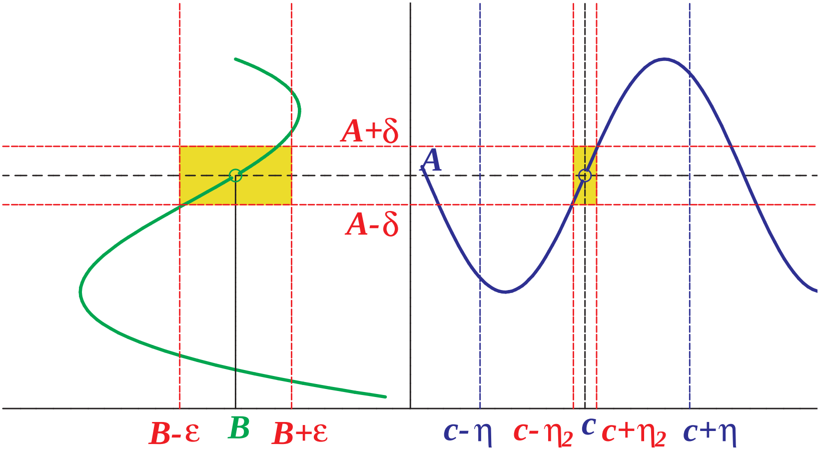

A+δ
A
A-δ
B-ε
B
B+ε
c- η
c
c+η


A+δ
A
A-δ
B-ε
B
B+ε
c- η
c
c+η


A+δ
A
A-δ
B-ε
B
B+ε
c- η
c- η1
c
c+η1
c+η



$A+\delta$
A
$A-\delta$
$B-\varepsilon$
B
$B+\varepsilon$
$c-\eta$
$c-\eta_1$
c
$c+\eta_1$
$c+\eta$



$A+\delta$
A
$A-\delta$
$B-\varepsilon$
B
$B+\varepsilon$
$c-\eta$
$c-\eta_2$
c
$c+\eta_2$
$c+\eta$

Theorem 27 (Heine)

Let $c \in \mathbb{R}^*$*,* $A \in \mathbb{R}^*$ *and the function* f *satisfies* $\lim_{x\to c} f(x) = A$*. If the sequence* $\{x_n\}$ *satisfies* $x_n \in D_f$*,* $x_n \neq c$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} x_n = c$*, then* $\lim_{n\to\infty} f(x_n) = A$*.*

Theorem 28 (limit of a monotone function)

Let $a, b \in \mathbb{R}^*$, $a < b$. Suppose that f is a function monotone on an interval (a, b). Then the limits $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ exist. Moreover,

- *if f is non-decreasing on (a, b), then $\lim_{x\to a+} f(x) = \inf f\bigl((a, b)\bigr)$ and $\lim_{x\to b-} f(x) = \sup f\bigl((a, b)\bigr)$;*
- *if f is non-increasing on (a, b), then $\lim_{x\to a+} f(x) = \sup f\bigl((a, b)\bigr)$ and $\lim_{x\to b-} f(x) = \inf f\bigl((a, b)\bigr)$.*

Definition

A polynomial is a function P of the form

$$P(x) = a_0 + a_1 x + \dots + a_n x^n, \quad x \in \mathbb{R},$$

where $n \in \mathbb{N} \cup \{0\}$ and $a_0, a_1, \dots, a_n \in \mathbb{R}$. The numbers $a_0, \dots, a_n$ are called the coefficients of the polynomial P.

Definition

A polynomial is a function P of the form

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad x \in \mathbb{R},$$

where $n \in \mathbb{N} \cup \{0\}$ and $a_0, a_1, \ldots, a_n \in \mathbb{R}$. The numbers $a_0, \ldots, a_n$ are called the coefficients of the polynomial P.

Remark

Let $n, m \in \mathbb{N} \cup \{0\}$ and

$$\begin{aligned} P(x) &= a_0 + a_1 x + \cdots + a_n x^n, \quad x \in \mathbb{R}, \\ Q(x) &= b_0 + b_1 x + \cdots + b_m x^m, \quad x \in \mathbb{R}, \end{aligned}$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$, $b_0, b_1, \ldots, b_m \in \mathbb{R}$, $b_m \neq 0$. If the polynomials P and Q are equal (i.e. $P(x) = Q(x)$ for each $x \in \mathbb{R}$), then $n = m$ and $a_0 = b_0, \ldots, a_n = b_n$.

Definition

Let P be a polynomial of the form

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad x \in \mathbb{R}.$$

We say that P is a polynomial of degree n if $a_n \neq 0$. The degree of a zero polynomial (i.e. a constant zero function defined on $\mathbb{R}$) is defined as -1.

Definition

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. If $\lim_{n\to\infty}(a_0 + a_1 + \cdots + a_n)$ exists, we denote it by

$$\sum_{k=0}^{\infty} a_k \quad \text{or} \quad a_1 + a_2 + a_3 + \dots$$

Definition

The exponential function (denoted by $\exp$) is defined by

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$$

for every $x \in \mathbb{R}$. The number $\exp(1)$ is denoted by e (and it is called Euler's number).

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Theorem 29 (existence of the exponential)

For every $x \in \mathbb{R}$ the limit $\lim\limits_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}$ exists and is finite.

Properties of the exponential function

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- $D_{\exp} = \mathbb{R}$, $R_{\exp} = (0, +\infty)$,

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- $\forall r \in \mathbb{Q}\colon\ \exp r = e^r$.

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- $\lim\limits_{x \to 1} \frac{\log x}{x-1} = 1$.

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Let $a, b \in (0, +\infty)$, $a \neq 1$. The general logarithm to base a is defined by

$$\log_a b = \frac{\log b}{\log a}.$$

Definition

The sine and cosine functions (denoted by $\sin$ and $\cos$) are defined by

$$\sin x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \qquad \cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

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Theorem 30 (existence of sine and cosine functions)

For every $x \in \mathbb{R}$ *the limits* $\lim\limits_{n\to\infty} \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!}$, $\lim\limits_{n\to\infty} \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}$ *exist and they are finite.*

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- $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.

Definition

The function tangent is denoted by tg and defined by

$$\mathrm{tg}\, x = \frac{\sin x}{\cos x}$$

for every $x \in \mathbb{R}$ for which the fraction is defined, i.e.

$$D_{\mathrm{tg}} = \{x \in \mathbb{R};\ x \neq \pi/2 + k\pi, k \in \mathbb{Z}\}.$$

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The function cotangent is denoted by cotg and defined on a set $D_{\mathrm{cotg}} = \{x \in \mathbb{R};\ x \neq k\pi, k \in \mathbb{Z}\}$ by

$$\mathrm{cotg}\, x = \frac{\cos x}{\sin x}.$$

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- The functions tg and cotg are continuous at every point of their domains.
- The functions tg and cotg are odd.
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- $R_{\operatorname{tg}} = R_{\operatorname{cotg}} = \mathbb{R}$

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- The function arccotangent (denoted by $\mathrm{arccotg}$) is an inverse function to the function $\mathrm{cotg}|_{(0,\pi)}$.

Properties of inverse trigonometric functions

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- $\operatorname{arctg} 0 = 0$, $\operatorname{arctg} 1 = \frac{\pi}{4}$, $\operatorname{arccotg} 0 = \frac{\pi}{2}$
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- $\forall x \in [-1, 1]\colon\ \arcsin x + \arccos x = \frac{\pi}{2}$,
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- $\lim\limits_{x \to +\infty} \mathrm{arctg}\, x = \frac{\pi}{2}$, $\lim\limits_{x \to -\infty} \mathrm{arctg}\, x = -\frac{\pi}{2}$
  $\lim\limits_{x \to +\infty} \mathrm{arccotg}\, x = 0$, $\lim\limits_{x \to -\infty} \mathrm{arccotg}\, x = \pi$

Definition

Let $J \subset \mathbb{R}$ be a non-degenerate interval (i.e. it contains infinitely many points). A function $f\colon J \to \mathbb{R}$ is continuous on the interval J if

- f is continuous at every inner point J,
- f is continuous from the right at the left endpoint of J if this point belongs to J,
- f is continuous from the left at the right endpoint of J if this point belongs to J.

Theorem 31 (continuity of the compound function on an interval)

Let I and J be intervals, $g\colon I \to J$, $f\colon J \to \mathbb{R}$, let g be continuous on I and let f be continuous on J. Then the function $f \circ g$ is continuous on I.

Theorem 32 (Bolzano, intermediate value theorem)

Let f be a function continuous on an interval $[a, b]$ and suppose that $f(a) < f(b)$. Then for each $C \in (f(a), f(b))$ there exists $\xi \in (a, b)$ satisfying $f(\xi) = C$.

Theorem 33 (an image of an interval under a continuous function)

Let J be an interval and let $f\colon J \to \mathbb{R}$ be a function continuous on J. Then $f(J)$ is an interval.

Definition

Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that f attains its maximum (resp. minimum) on M at $x \in M$ if

$$\forall y \in M \colon f(y) \leq f(x) \quad \text{(resp. } \forall y \in M \colon f(y) \geq f(x)\text{)}.$$

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Definition

Let $M \subset \mathbb{R}$, $x \in M$ and a function f is defined at least on M (i.e. $M \subset D_f$). We say that the function f has at x

- a local maximum with respect to M if there exists $\delta > 0$ such that $\forall y \in B(x, \delta) \cap M\colon f(y) \leq f(x)$,

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The points of local maxima or minima are collectively called the points of local extrema.

Theorem 34 (Heine theorem for continuity on an interval)

Let f be a function continuous on an interval J and $c \in J$. Then $\lim f(x_n) = f(c)$ for each sequence $\{x_n\}_{n=1}^{\infty}$ of points in the interval J satisfying $\lim x_n = c$.

Theorem 35 (extrema of continuous functions)

Let f be a function continuous on an interval $[a, b]$. Then f attains its maximum and minimum on $[a, b]$.

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Corollary 36 (boundedness of a continuous function)

Let f be a function continuous on an interval $[a, b]$. Then f is bounded on $[a, b]$.

Theorem 37 (continuity of an inverse function)

Let f be a continuous function that is increasing (resp. decreasing) on an interval J. Then the function f^{-1} is continuous and increasing (resp. decreasing) on the interval $f(J)$.

Corollary 38

Functions nth root, exponential, general power, arcsin, arccos, arctg, arccotg are continuous on their domains.

Definition

Let f be a function and $a \in \mathbb{R}$. Then

- the derivative of the function f at the point a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

- the derivative of f at a from the right is defined by

$$f'_{+}(a) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h},$$

- the derivative of f at a from the left is defined by

$$f'_{-}(a) = \lim_{h \to 0-} \frac{f(a+h) - f(a)}{h},$$

Definition

Suppose that the function f has a finite derivative at a point $a \in \mathbb{R}$. The line

$$T_a = \left\{[x, y] \in \mathbb{R}^2;\ y = f(a) + f'(a)(x - a)\right\}.$$

is called the tangent to the graph of f at the point $[a, f(a)]$.

Definition

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Theorem 39

Suppose that the function f has a finite derivative at a point $a \in \mathbb{R}$. Then f is continuous at a.

Theorem 40 (arithmetics of derivatives)

Suppose that the functions f and g have finite derivatives at $a \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$. Then

(i) $(f+g)'(a) = f'(a) + g'(a)$,

Theorem 40 (arithmetics of derivatives)

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(ii) $(\alpha f)'(a) = \alpha \cdot f'(a)$,

(iii) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$,

(iv) *if* $g(a) \neq 0$, *then*

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$

Theorem 41 (derivative of a compound function)

Suppose that the function f has a finite derivative at $y_0 \in \mathbb{R}$, the function g has a finite derivative at $x_0 \in \mathbb{R}$, and $y_0 = g(x_0)$. Then

$$(f \circ g)'(x_0) = f'(y_0) \cdot g'(x_0).$$

Theorem 41 (derivative of a compound function)

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Theorem 42 (derivative of an inverse function)

Let f be a function continuous and strictly monotone on an interval (a, b) and suppose that it has a finite and non-zero derivative $f'(x_0)$ at $x_0 \in (a, b)$. Then the function f^{-1} has a derivative at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$

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Derivatives of elementary functions

- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,
- $(\exp x)' = \exp x$ for $x \in \mathbb{R}$,
- $(x^a)' = ax^{a-1}$ for $x \in (0, +\infty)$, $a \in \mathbb{R}$,
- $(a^x)' = a^x \log a$ for $x \in \mathbb{R}$, $a \in \mathbb{R}$, $a > 0$,
- $(\sin x)' = \cos x$ for $x \in \mathbb{R}$,
- $(\cos x)' = -\sin x$ for $x \in \mathbb{R}$,
- $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$ for $x \in D_{\operatorname{tg}}$,
- $(\operatorname{cotg} x)' = -\frac{1}{\sin^2 x}$ for $x \in D_{\operatorname{cotg}}$,
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$,
- $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$,
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- $(\operatorname{arccotg} x)' = -\frac{1}{1+x^2}$ for $x \in \mathbb{R}$.

Theorem 43 (necessary condition for a local extremum)

Suppose that a function f has a local extremum at $x_0 \in \mathbb{R}$. If $f'(x_0)$ exists, then $f'(x_0) = 0$.

Theorem 44 (Rolle)

Suppose that $a, b \in \mathbb{R}$, $a < b$, and a function f has the following properties:

(i) *it is continuous on the interval $[a, b]$,*

(ii) *it has a derivative (finite or infinite) at every point of the open interval (a, b),*

(iii) $f(a) = f(b)$.

Then there exists $\xi \in (a, b)$ satisfying $f'(\xi) = 0$.

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Then there exists $\xi \in (a, b)$ satisfying $f'(\xi) = 0$.

Theorem 45 (Lagrange, mean value theorem)

Suppose that $a, b \in \mathbb{R}$, $a < b$, a function f is continuous on an interval $[a, b]$ and has a derivative (finite or infinite) at every point of the interval (a, b). Then there is $\xi \in (a, b)$ satisfying

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 46 (sign of the derivative and monotonicity)

Let $J \subset \mathbb{R}$ be a non-degenerate interval. Suppose that a function f is continuous on J and it has a derivative at every inner point of J (the set of all inner points of J is denoted by $\operatorname{Int} J$).

(i) *If $f'(x) > 0$ for all $x \in \operatorname{Int} J$, then f is increasing on J.*

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(iv) *If $f'(x) \leq 0$ for all $x \in \operatorname{Int} J$, then f is non-increasing on J.*

Theorem 47 (computation of a one-sided derivative)

Suppose that a function f is continuous from the right at $a \in \mathbb{R}$ and the limit $\lim_{x \to a+} f'(x)$ exists. Then the derivative $f'_+(a)$ exists and

$$f'_+(a) = \lim_{x \to a+} f'(x).$$

Theorem 48 (l'Hospital's rule)

Suppose that functions f and g have finite derivatives on some punctured neighbourhood of $a \in \mathbb{R}^*$ and the limit $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ exist. Suppose further that one of the following conditions hold:

(i) $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$,

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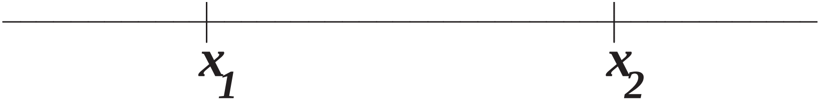
Then the limit $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Convex combination

Convex combination



$$1 \cdot x_1 + 0 \cdot x_2 = x_1 + 0 \cdot (x_2 - x_1) = x_1$$

Convex combination

$$0 \cdot x_1 + 1 \cdot x_2 = x_1 + 1 \cdot (x_2 - x_1) = x_2$$

Convex combination

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 = x_1 + \frac{1}{2}(x_2 - x_1)$$

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$$\frac{3}{4}x_1 + \frac{1}{4}x_2 = x_1 + \frac{1}{4}(x_2 - x_1)$$

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Convex combination

$$\lambda x_1 + (1-\lambda)x_2 = x_1 + (1-\lambda)(x_2 - x_1), \quad \lambda \in [0,1]$$

Definition

We say that a function f is

- convex on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2),$$
for each $x_1, x_2 \in I$ and each $\lambda \in [0,1]$;

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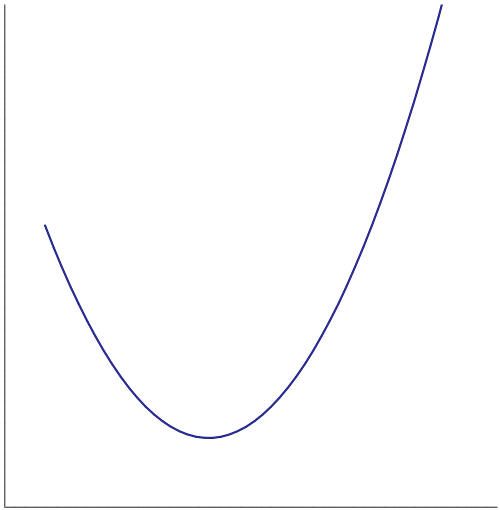
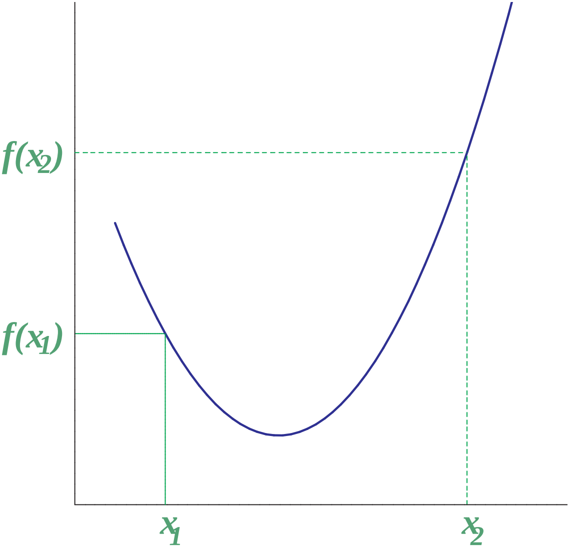
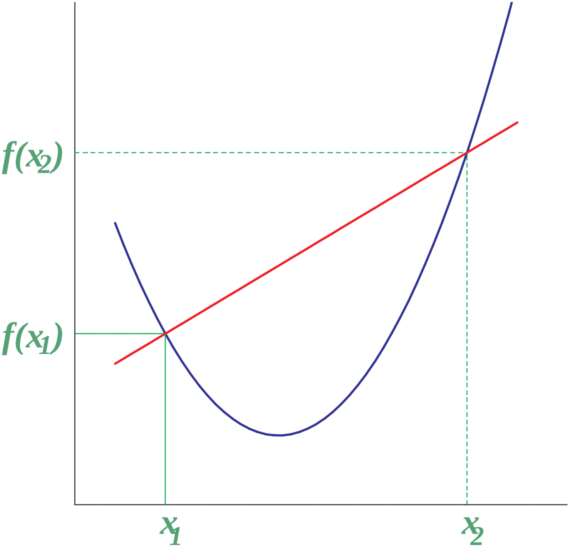



f(x2)
f(x1)
x1
x2



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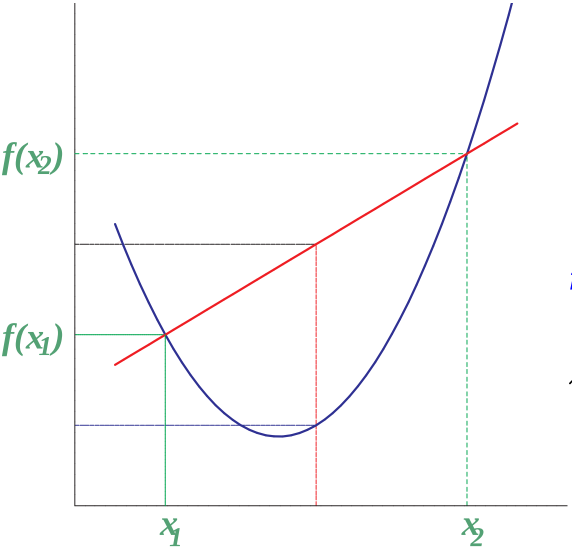

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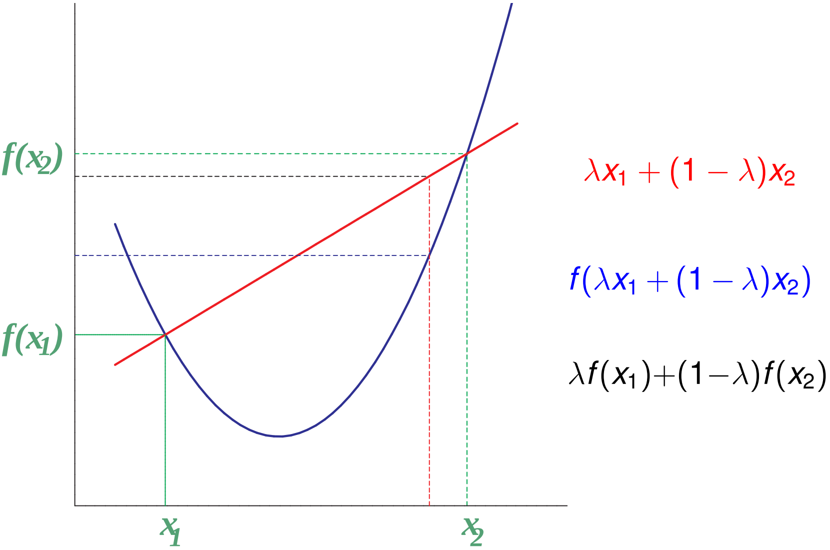
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Lemma 49

A function f is convex on an interval I if and only if

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

for each three points $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$.

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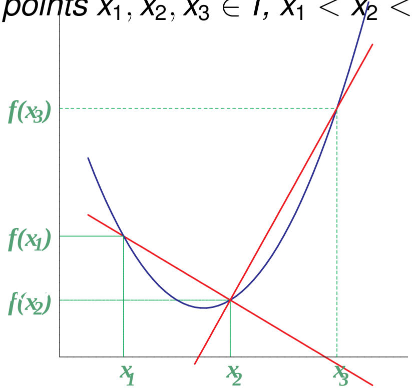

Definition

Suppose that a function f has a finite derivative on some neighbourhood of $a \in \mathbb{R}$. The second derivative of f at a is defined by

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h}$$

if the limit exists.

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Let $n \in \mathbb{N}$ and suppose that f has a finite nth derivative (denoted by $f^{(n)}$) on some neighbourhood of $a \in \mathbb{R}$. Then the $(n+1)$th derivative of f at a is defined by

$$f^{(n+1)}(a) = \lim_{h \to 0} \frac{f^{(n)}(a+h) - f^{(n)}(a)}{h}$$

if the limit exists.

Theorem 50 (second derivative and convexity)

Let $a, b \in \mathbb{R}^*$, $a < b$, and suppose that a function f has a finite second derivative on the interval (a, b).

(i) *If $f''(x) > 0$ for each $x \in (a, b)$, then f is strictly convex on (a, b).*

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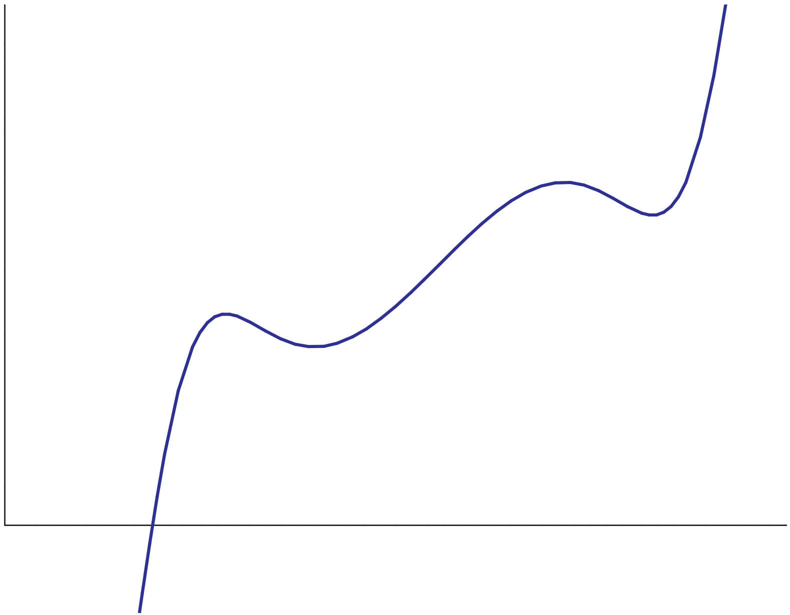

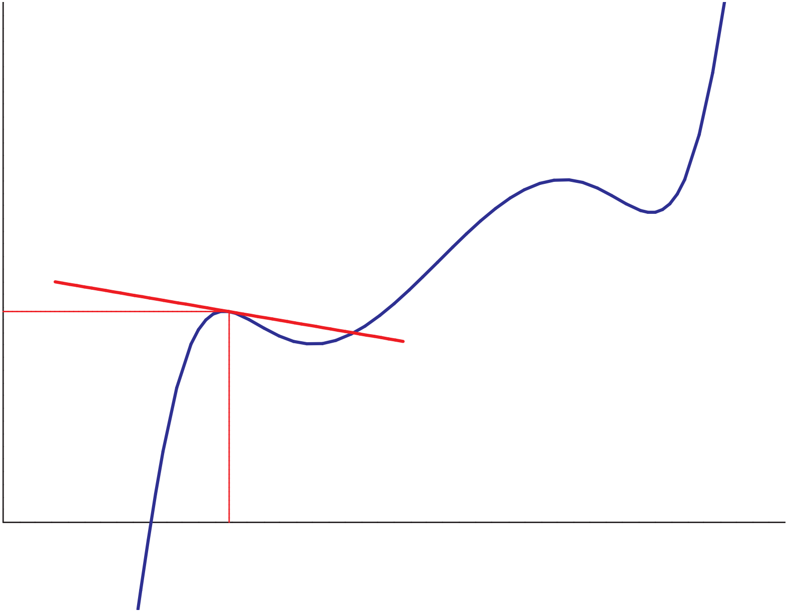

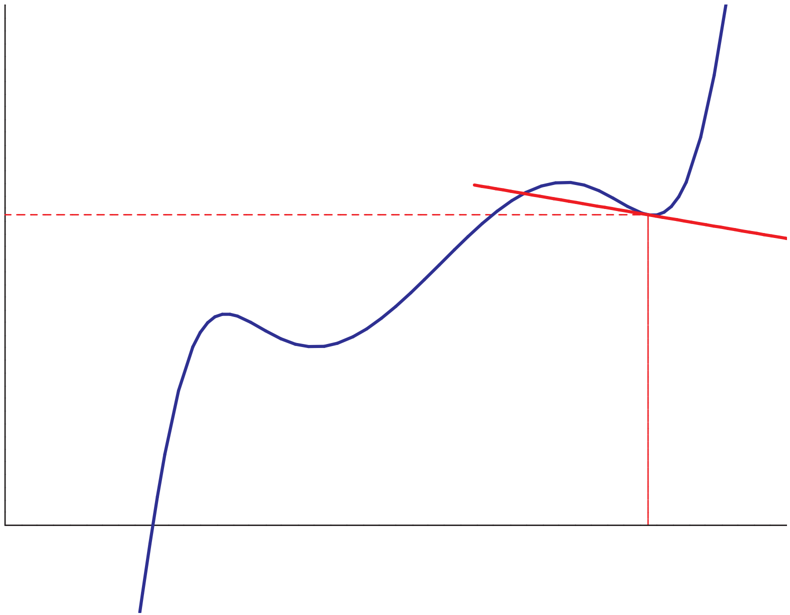

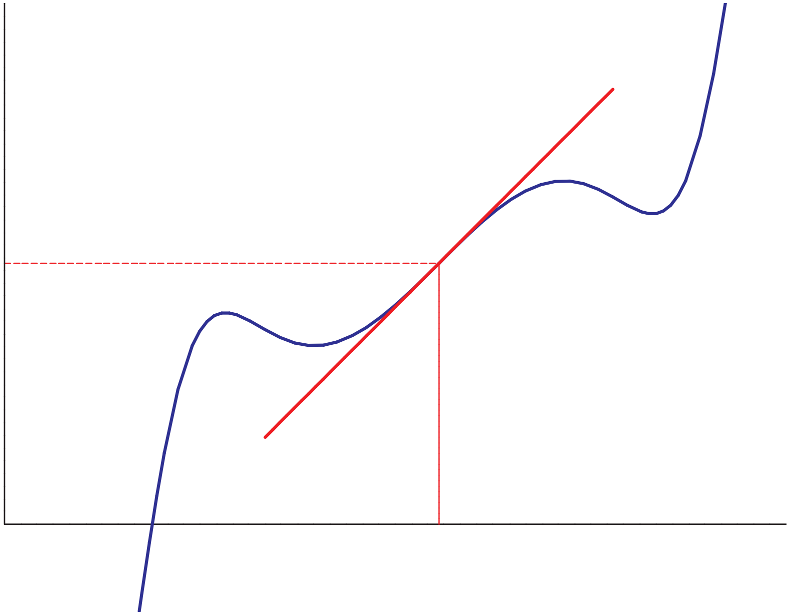

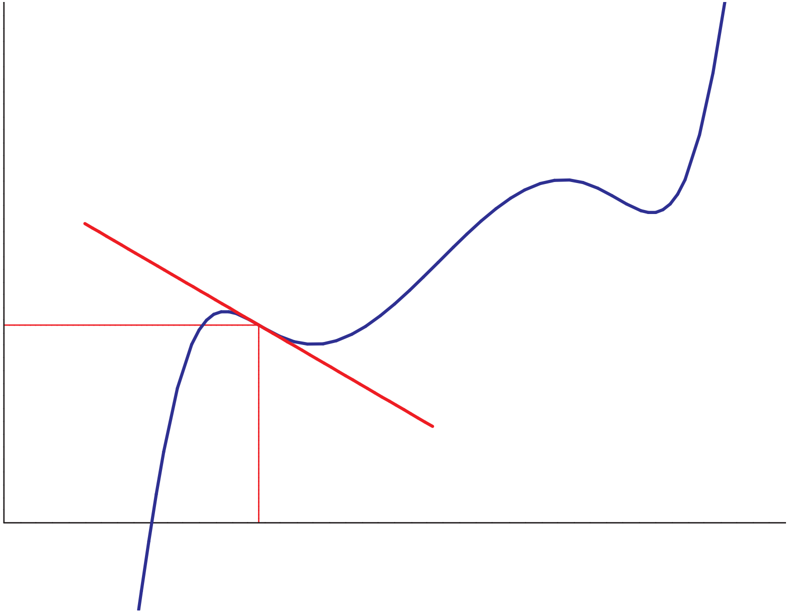

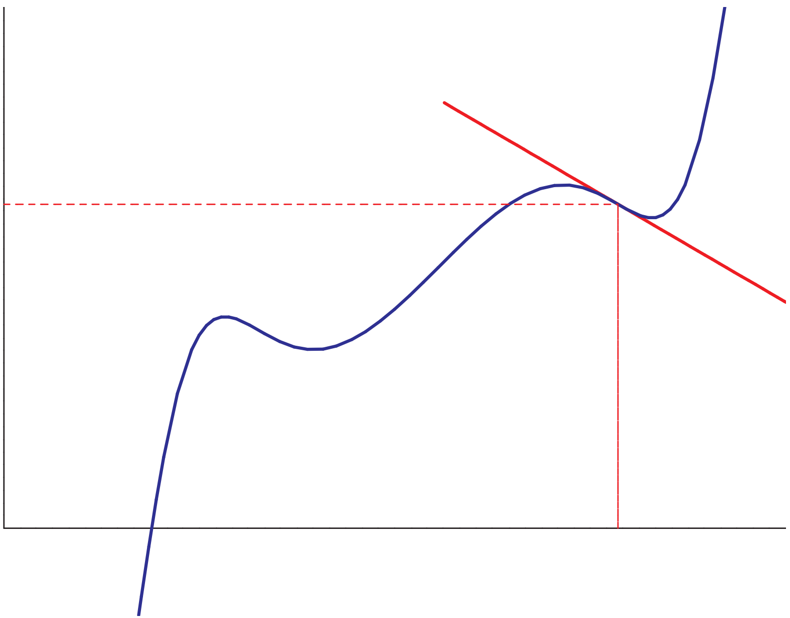

Definition

Suppose that a function f has a finite derivative at $a \in \mathbb{R}$ and let T_a denote the tangent to the graph of f at $[a, f(a)]$. We say that the point $[x, f(x)]$ lies below the tangent T_a if

$$f(x) < f(a) + f'(a) \cdot (x - a).$$

We say that the point $[x, f(x)]$ lies above the tangent T_a if the opposite inequality holds.

Definition

Suppose that a function f has a finite derivative at $a \in \mathbb{R}$ and let T_a denote the tangent to the graph of f at $[a, f(a)]$. We say that a is an inflection point of f if there is $\Delta > 0$ such that

(i) $\forall x \in (a - \Delta, a)\colon [x, f(x)]$ lies below the tangent T_a,

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Theorem 51 (necessary condition for inflection)

Let $a \in \mathbb{R}$ be an inflection point of a function f. Then $f''(a)$ either does not exist or equals zero.

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Theorem 52 (sufficient condition for inflection)

Suppose that a function f has a continuous first derivative on an interval (a, b) and $z \in (a, b)$. Suppose further that

- $\forall x \in (a, z)\colon f''(x) > 0$,
- $\forall x \in (z, b)\colon f''(x) < 0$.

Then z is an inflection point of f.

Definition

The line which is a graph of an affine function $x \mapsto kx + q$, $k, q \in \mathbb{R}$, is called an asymptote of the function f at $+\infty$ (resp. v $-\infty$) if

$$\lim_{x \to +\infty} (f(x) - kx - q) = 0, \quad \text{(resp. } \lim_{x \to -\infty} (f(x) - kx - q) = 0\text{).}$$

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$$\lim_{x\to+\infty}(f(x)-kx-q) = 0, \quad (\text{resp. } \lim_{x\to-\infty}(f(x)-kx-q) = 0).$$

Proposition 53

A function f has an asymptote at $+\infty$ given by the affine function $x \mapsto kx + q$ if and only if

$$\lim_{x\to+\infty}\frac{f(x)}{x} = k \in \mathbb{R} \quad \textit{and} \quad \lim_{x\to+\infty}(f(x) - kx) = q \in \mathbb{R}.$$

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7. Draw the graph of the function.