Řešené příklady

1. **Integrály typu** $\int_{-\infty}^{\infty} R(x)\mathrm{dx}$

c) $\int_{-\infty}^{\infty} \frac{x\,\mathrm{dx}}{(x^2+4x+13)^2}$.

Řešení:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{x\,\mathrm{dx}}{(x^2+4x+13)^2} &= \int_{-\infty}^{\infty} \frac{x\,\mathrm{dx}}{\big((x-\alpha_-)(x-\alpha_+)\big)^2} \\
\Big| \alpha_+ &= -2+3\mathbf{i},\ \alpha_- = -2-3\mathbf{i} \Big| \\
&= 2\pi\mathbf{i}\ \mathrm{res}_{\alpha_+} \frac{z}{((z-\alpha_-)(z-\alpha_+))^2} \\
&= 2\pi\mathbf{i}\left(\frac{z}{(z-\alpha_-)^2}\right)'\Bigg|_{z=\alpha_+} \\
&= 2\pi\mathbf{i}\left(\frac{1}{(\alpha_+-\alpha_-)^2} - \frac{2\alpha_+}{(\alpha_+-\alpha_-)^3}\right) \\
&= 2\pi\mathbf{i}\left(\frac{1}{(6\mathbf{i})^2} - \frac{2(-2+3\mathbf{i})}{(6\mathbf{i})^3}\right) \\
&= 2\pi\mathbf{i}\left(\frac{1}{-36} + \frac{-2+3\mathbf{i}}{3.36\mathbf{i}}\right) \\
&= -\frac{\pi}{3.9} = -\frac{\pi}{27}.
\end{aligned}
$$

d) $\int_{-\infty}^{\infty} \frac{\mathrm{dx}}{(x^2+a^2)(x^2+b^2)},\ a,b>0.$

Řešení:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{\mathrm{dx}}{(x^2+a^2)(x^2+b^2)} &= 2\pi\mathbf{i}\left(\mathrm{res}_{a\mathbf{i}} \frac{z}{(z^2+a^2)(z^2+b^2)} + \mathrm{res}_{b\mathbf{i}} \frac{z}{(z^2+a^2)(z^2+b^2)}\right) \\
&= \pi\mathbf{i}\left(\frac{1}{(\mathbf{i}a)(-a^2+b^2)} + \frac{1}{(-b^2+a^2)(\mathbf{i}b)}\right) \\
&= \frac{\pi(a-b)}{ab(a^2-b^2)} \\
&= \frac{\pi}{ab(a+b)}.
\end{aligned}
$$

e) $\int_0^\infty \frac{x^2 \mathrm{dx}}{(x^2+a^2)^3}, \ a > 0.$

Řešení:

$$\begin{aligned}
\int_0^\infty \frac{x^2 \,\mathrm{dx}}{(x^2+a^2)^3} &= \frac{1}{2}\int_{-\infty}^\infty \frac{x^2 \,\mathrm{dx}}{(x^2+a^2)^3} \\
&= \pi\mathbf{i}\,\mathrm{res}_{a\mathbf{i}} \frac{z^2}{(z^2+a^2)^3} \\
&= \frac{\pi\mathbf{i}}{2}\left(\frac{z^2}{(z+\mathbf{i}a)^3}\right)''\Bigg|_{z=\mathbf{i}a} \\
&= \frac{\pi\mathbf{i}}{2}\left(\frac{2z}{(z+\mathbf{i}a)^3} - \frac{3z^2}{(z+\mathbf{i}a)^4}\right)'\Bigg|_{z=\mathbf{i}a} \\
&= \frac{\pi\mathbf{i}}{2}\left(\frac{2}{(z+\mathbf{i}a)^3} - \frac{6z}{(z+\mathbf{i}a)^4} - \frac{6z}{(z+\mathbf{i}a)^4} + \frac{12z^2}{(z+\mathbf{i}a)^5}\right)\Bigg|_{z=\mathbf{i}a} \\
&= \frac{\pi\mathbf{i}}{2}\left(\frac{2}{-8a^3\mathbf{i}} - 2\frac{6\mathbf{i}a}{16a^4} + \frac{-12a^2}{2^5a^5\mathbf{i}}\right) \\
&= \frac{\pi\mathbf{i}}{2}\left(\frac{\mathbf{i}}{4a^3} - \frac{3\mathbf{i}}{4a^3} + \frac{3\mathbf{i}}{8a^3}\right) \\
&= \frac{\pi}{16\,a^3}.
\end{aligned}$$

2. **Integrály typu** $\int_{-\infty}^\infty \cos x\, R(x)\,\mathrm{dx}, \int_{-\infty}^\infty \sin x\, R(x)\,\mathrm{dx}$

a) $\int_{-\infty}^\infty \frac{\cos x \,\mathrm{dx}}{x^2+a^2}, a > 0.$

Řešení:

$$\begin{aligned}
\int_{-\infty}^\infty \frac{\cos x \,\mathrm{dx}}{x^2+a^2} &= \Re \int_{\mathbb{R}} \frac{e^{iz}\mathrm{dz}}{z^2+a^2} \\
&= \Re\left(2\pi\mathbf{i}\,\mathrm{res}_{a\mathbf{i}}\, \frac{e^{\mathbf{i}z}}{z^2+a^2}\right) \\
&= \Re\left(2\pi\mathbf{i}\, \frac{e^{-a}}{2\mathbf{i}a}\right) \\
&= \frac{\pi e^{-a}}{a}.
\end{aligned}$$

b) $\int_0^\infty \frac{\sin(\pi x)}{x^3-x}\,\mathrm{dx}$.

Řešení:

$$\begin{aligned}
\int_0^\infty \frac{\sin(\pi x)}{x^3-x}\,\mathrm{dx} &= \frac{1}{2}\Im\int_{\mathbb{R}} \frac{\exp(\mathbf{i}\pi z)}{z^3-z}\,\mathrm{dz} \\
&= \frac{1}{2}\Im\big(\pi\mathbf{i}\Big(\mathrm{res}_{z=0}\frac{\exp(\mathbf{i}\pi z)}{z^3-z} + \mathrm{res}_{z=1}\frac{\exp(\mathbf{i}\pi z)}{z^3-z} + \mathrm{res}_{z=-1}\frac{\exp(\mathbf{i}\pi z)}{z^3-z}\Big)\big) \\
&= \frac{1}{2}\Im\big(\pi\mathbf{i}(-1+\frac{e^{\mathbf{i}\pi}}{2}+\frac{e^{-\mathbf{i}\pi}}{2})\big) \\
&= \frac{1}{2}\Im(-2\pi\mathbf{i}) \\
&= -\pi.
\end{aligned}$$

c) $\int_{-\infty}^\infty \frac{(x-8)\cos(\pi x)}{4x^2-1}\,\mathrm{dx}$.

Řešení:

$$\begin{aligned}
\int_{-\infty}^\infty \frac{(x-8)\cos(\pi x)}{4x^2-1}\,\mathrm{dx} &= -8\int_{-\infty}^\infty \frac{\cos(\pi x)}{4x^2-1}\,\mathrm{dx} \\
&= -2\Re\int_{\mathbb{R}} \frac{\exp(\mathbf{i}\pi z)}{z^2-\frac{1}{4}}\,\mathrm{dz} \\
&= -2\Re\big(\pi\mathbf{i}\Big(\mathrm{res}_{\frac{1}{2}}\frac{e^{\mathbf{i}\pi z}}{z^2-\frac{1}{4}} + \mathrm{res}_{-\frac{1}{2}}\frac{e^{\mathbf{i}\pi z}}{z^2-\frac{1}{4}}\Big)\big) \\
&= -2\Re\Big(\pi\mathbf{i}(e^{\frac{\mathbf{i}}{2}\pi}-e^{-\frac{\mathbf{i}}{2}\pi})\Big) \\
&= 4\Re\big(\pi\sin(\frac{\pi}{2})\big) \\
&= 4\pi.
\end{aligned}$$

V postupu b) a c) jsme použili Lemma o obcházení jednoduchých pólů.

3.a) $\int_0^{2\pi} \frac{\mathrm{dx}}{a+\cos x}, \; |a| > 1.$

Řešení:

$$\begin{aligned}
\int_0^{2\pi} \frac{\mathrm{dx}}{a+\cos x} &= \int_{|z|=1} \frac{1}{a+\frac{z+z^{-1}}{2}} \frac{\mathrm{dz}}{\mathbf{i}z} \\
&= 2\int_{|z|=1} \frac{1}{z^2+2az+1} \frac{\mathrm{dz}}{\mathbf{i}} \\
&= -2\mathbf{i}\int_{|z|=1} \frac{\mathrm{dz}}{z^2+2az+1} \\
&= -2\mathbf{i}\int_{|z|=1} \frac{\mathrm{dz}}{(z-\alpha_1)(z-\alpha_2)} \\
\Big| \alpha_1 &= -a+\sqrt{a^2-1}, \; \alpha_2 = -a-\sqrt{a^2-1} \Big| \\
&= \begin{cases} \frac{4\pi}{\alpha_1-\alpha_2} = \frac{2\pi}{\sqrt{a^2-1}}, & a > 1, \\ \frac{4\pi}{\alpha_2-\alpha_1} = \frac{-2\pi}{\sqrt{a^2-1}}, & a < -1, \end{cases} \\
\int_0^{2\pi} \frac{\mathrm{dx}}{a+\cos x} &= \frac{2\pi\,\mathrm{sgn}(a)}{\sqrt{a^2-1}}.
\end{aligned}$$

3.b) $\int_0^{2\pi} \frac{\mathrm{dx}}{(a+b\cos x)^2}$, $a > b > 0$.

Řešení:

$$
\begin{aligned}
\int_0^{2\pi} \frac{\mathrm{dx}}{(a+b\cos x)^2} &= \int_{|z|=1} \frac{1}{(a+b\frac{z+z^{-1}}{2})^2} \frac{\mathrm{dz}}{\mathbf{i}z} \\
&= \frac{4}{\mathbf{i}b^2} \int_{|z|=1} \frac{z\,\mathrm{dz}}{(z^2+2cz+1)^2} \quad \Big| \; c = \frac{a}{b} \\
&= \frac{4}{\mathbf{i}b^2} \int_{|z|=1} \frac{z\,\mathrm{dz}}{(z^2+2cz+1)^2} \\
\Big| \alpha_1 = -c + \sqrt{c^2-1},\ \alpha_2 &= -c - \sqrt{c^2-1},\ |c| > 1 \Rightarrow |\alpha_1| < 1,\ |\alpha_2| > 1 \Big| \\
&= \frac{8\pi}{b^2} \left(\frac{z}{(z-\alpha_2)^2} \right)' \Bigg|_{z=\alpha_1} \\
&= \frac{8\pi}{b^2} \left(\frac{1}{(\alpha_1-\alpha_2)^2} - \frac{2\alpha_1}{(\alpha_1-\alpha_2)^3} \right) \\
&= \frac{8\pi}{b^2} \left(\frac{1}{4|c^2-1|} - \frac{-2c+2\sqrt{c^2-1}}{8(\sqrt{c^2-1})^3} \right) \\
&= \frac{2\pi}{b^2} \frac{c}{(\sqrt{c^2-1})^3} \\
&= \frac{2\pi a}{(\sqrt{a^2-b^2})^3}
\end{aligned}
$$

3.c) $\int_0^\pi \frac{\cos(nx)\mathrm{dx}}{1-2a\cos x+a^2}$, $a \in \mathbb{R}$, $a \neq \pm 1$, $n \in \mathbb{N}$.

Řešení: 1. postup

$$\begin{aligned}\int_{-\pi}^{0} \frac{\cos(nx)\mathrm{dx}}{1-2a\cos x+a^2} &= -\int_{\pi}^{0} \frac{\cos(-ny)\mathrm{dy}}{1-2a\cos(-y)+a^2} \quad \Big| \; x=-y \\ &= \int_0^\pi \frac{\cos(ny)\mathrm{dy}}{1-2a\cos(y)+a^2}\end{aligned}$$

$$\begin{aligned}\int_0^\pi \frac{\cos(nx)\mathrm{dx}}{1-2a\cos x+a^2} &= \frac{1}{2}\int_0^{2\pi} \frac{\cos(nx)\mathrm{dx}}{1-2a\cos x+a^2} \\ &= \frac{1}{4}\int_{|z|=1} \frac{z^n+z^{-n}}{1-a(z+z^{-1})+a^2}\,\frac{\mathrm{dz}}{\mathbf{i}z} \\ &= \frac{1}{4\mathbf{i}}\int_{|z|=1} \frac{(z^n+z^{-n})\,\mathrm{dz}}{z-az^2-a+a^2z} \\ &= -\frac{1}{4\mathbf{i}a}\int_{|z|=1} \frac{(z^{2n}+1)\,\mathrm{dz}}{z^n(z^2-az-a^{-1}z+1)} \quad \Big| \; b=\frac{1}{2}(a+a^{-1}) \\ &= -\frac{1}{4\mathbf{i}a}\int_{|z|=1} \frac{(z^{2n}+1)\,\mathrm{dz}}{z^n(z^2-2bz+1)} \\ \Big| \; \alpha_- &= b-\sqrt{b^2-1}, \quad \alpha_+ = b+\sqrt{b^2-1} \; \Big| \\ &= \begin{cases} -\frac{\pi}{2a}\left(\mathrm{res}_{\alpha_-} \frac{z^{2n}+1}{z^n(z^2-2bz+1)} + \mathrm{res}_0 \frac{z^{2n}+1}{z^n(z^2-2bz+1)}\right), & a>0, \\ -\frac{\pi}{2a}\left(\mathrm{res}_{\alpha_+} \frac{z^{2n}+1}{z^n(z^2-2bz+1)} + \mathrm{res}_0 \frac{z^{2n}+1}{z^n(z^2-2bz+1)}\right), & a<0. \end{cases}\end{aligned}$$

$$
\begin{aligned}
res_0 \frac{z^{2n}+1}{z^n(z^2-2bz+1)} &= res_0 \frac{1}{z^n(z^2-2bz+1)} \\
&= \lim_{z\to 0} \frac{1}{(n-1)!}\left(\frac{1}{(z-\alpha_-)(z-\alpha_+)}\right)^{(n-1)} \\
&= \lim_{z\to 0} \frac{1}{(n-1)!}\left(\frac{(\alpha_- - \alpha_+)^{-1}}{(z-\alpha_-)} + \frac{(\alpha_+ - \alpha_-)^{-1}}{(z-\alpha_+)}\right)^{(n-1)} \\
&= \lim_{z\to 0} \frac{1}{(n-1)!}\left(\frac{(\alpha_+ - \alpha_-)^{-1}}{\alpha_-(1-\frac{z}{\alpha_-})} + \frac{(\alpha_- - \alpha_+)^{-1}}{\alpha_+(-\frac{z}{\alpha_+})}\right)^{(n-1)} \\
&= \lim_{z\to 0} \frac{1}{(n-1)!}\left(\frac{(\alpha_+ - \alpha_-)^{-1}}{\alpha_-}\sum_{k=0}^{+\infty}\big(\frac{z}{\alpha_-}\big)^k + \frac{(\alpha_- - \alpha_+)^{-1}}{\alpha_+}\sum_{k=0}^{+\infty}\big(\frac{z}{\alpha_+}\big)^k\right)^{(n-1)} \\
&= \frac{1}{(\alpha_+ - \alpha_-)\alpha_-^n} + \frac{1}{(\alpha_- - \alpha_+)\alpha_+^n} \\
&= \frac{\alpha_-^n - \alpha_+^n}{\alpha_- - \alpha_+} \\
res_{\alpha_-} \frac{z^{2n}+1}{z^n(z^2-2bz+1)} &= \frac{\alpha_-^{2n}+1}{\alpha_-^n(\alpha_- - \alpha_+)} = \frac{\alpha_-^n + \alpha_+^n}{\alpha_- - \alpha_+} \\
res_{\alpha_+} \frac{z^{2n}+1}{z^n(z^2-2bz+1)} &= \frac{\alpha_+^{2n}+1}{\alpha_+^n(\alpha_+ - \alpha_-)} = \frac{\alpha_-^n + \alpha_+^n}{\alpha_+ - \alpha_-}
\end{aligned}
$$

Uvažujme čtyři případy, a to $a > 1$ nebo $1 > a > 0$ nebo $0 > a > -1$ nebo $-1 > a$.

- $1 < a$

$$
\begin{aligned}
-\frac{\pi}{2a}\left(\frac{\alpha_-^n - \alpha_+^n}{\alpha_- - \alpha_+} + \frac{\alpha_-^n + \alpha_+^n}{\alpha_- - \alpha_+}\right) &= -\frac{\pi}{a}\frac{\alpha_-^n}{(\alpha_- - \alpha_+)} = -\frac{\pi}{a}\frac{a^{-n}}{(a^{-1}-a)} = \frac{\pi}{a^n(a^2-1)} \\
\alpha_- &= b - \sqrt{b^2-1} = \frac{1}{2}\big((a+a^{-1}) - \sqrt{(a-a^{-1})^2}\big) = a^{-1} \\
\alpha_- - \alpha_+ &= -2\sqrt{b^2-1} = -\sqrt{(a-a^{-1})^2} = a^{-1} - a
\end{aligned}
$$

- $0 < a < 1$

$$
\begin{aligned}
-\frac{\pi}{2a}\left(\frac{\alpha_-^n - \alpha_+^n}{\alpha_- - \alpha_+} + \frac{\alpha_-^n + \alpha_+^n}{\alpha_- - \alpha_+}\right) &= -\frac{\pi}{a}\frac{\alpha_-^n}{(\alpha_- - \alpha_+)} = -\frac{\pi}{a}\frac{a^{n}}{(a-a^{-1})} = \frac{\pi a^n}{1-a^2} \\
\alpha_- &= b - \sqrt{b^2-1} = \frac{1}{2}\big((a+a^{-1}) - \sqrt{(a-a^{-1})^2}\big) = a \\
\alpha_- - \alpha_+ &= -2\sqrt{b^2-1} = -\sqrt{(a-a^{-1})^2} = a - a^{-1}
\end{aligned}
$$

- $-1 < a < 0$

$$\begin{aligned}
-\frac{\pi}{2a}\left(\frac{\alpha_-^n - \alpha_+^n}{\alpha_- - \alpha_+} + \frac{\alpha_-^n + \alpha_+^n}{\alpha_+ - \alpha_-}\right) &= \frac{\pi}{a}\frac{\alpha_+^n}{(\alpha_- - \alpha_+)} = \frac{\pi}{a}\frac{a^n}{(a^{-1} - a)} = \frac{\pi a^n}{1 - a^2} \\
\alpha_+ &= b + \sqrt{b^2 - 1} = \frac{1}{2}\big((a + a^{-1}) + \sqrt{(a - a^{-1})^2}\big) = a \\
\alpha_- - \alpha_+ &= -2\sqrt{b^2 - 1} = -\sqrt{(a - a^{-1})^2} = -a + a^{-1}
\end{aligned}$$

- $a < -1$

$$\begin{aligned}
-\frac{\pi}{2a}\left(\frac{\alpha_-^n - \alpha_+^n}{\alpha_- - \alpha_+} + \frac{\alpha_-^n + \alpha_+^n}{\alpha_+ - \alpha_-}\right) &= \frac{\pi}{a}\frac{\alpha_+^n}{(\alpha_- - \alpha_+)} = \frac{\pi}{a}\frac{a^{-n}}{a - a^{-1}} = \frac{\pi}{a^n(a^2 - 1)} \\
\alpha_+ &= b + \sqrt{b^2 - 1} = \frac{1}{2}\big((a + a^{-1}) + \sqrt{(a - a^{-1})^2}\big) = a^{-1} \\
\alpha_- - \alpha_+ &= -2\sqrt{b^2 - 1} = -\sqrt{(a - a^{-1})^2} = a - a^{-1}
\end{aligned}$$

Celkově tedy máme

$$\int_0^{\pi} \frac{\cos(nx)\mathrm{dx}}{1 - 2a\cos x + a^2} = \begin{cases} \frac{\pi}{a^n(a^2-1)}, & 1 < a, \\ \frac{\pi a^n}{1-a^2}, & 0 < a < 1, \\ \frac{\pi a^n}{1-a^2}, & -1 < a < 0, \\ \frac{\pi}{a^n(a^2-1)}, & a < -1. \end{cases}$$

2. postup:

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos(nx)\mathrm{dx}}{1 - 2a\cos x + a^2} &= \Re \int_0^{2\pi} \frac{e^{inx}\,\mathrm{dx}}{1 - 2a\cos x + a^2} \\
&= \int_{|z|=1} \frac{z^n}{1 - a(z + z^{-1}) + a^2}\frac{\mathrm{dz}}{\mathbf{i}z} \\
&= -\mathbf{i}\int_{|z|=1} \frac{z^n\,\mathrm{dz}}{z - az^2 - a + a^2 z} \\
&= \frac{\mathbf{i}}{a}\int_{|z|=1} \frac{z^n\,\mathrm{dz}}{z^2 - (a + a^{-1})z + 1} \\
&= \frac{\mathbf{i}}{a}\int_{|z|=1} \frac{z^n\,\mathrm{dz}}{(z - a)(z - a^{-1})} \\
&= \begin{cases} -\frac{2\pi}{a}\left(\frac{a^{-n}}{a^{-1}-a}\right) = \frac{2\pi}{a^n(a^2-1)}, & 1 < a, \\ -\frac{2\pi}{a}\left(\frac{a^n}{a-a^{-1}}\right) = \frac{2\pi a^n}{1-a^2}, & 0 < a < 1, \\ -\frac{2\pi}{a}\left(\frac{a^n}{a-a^{-1}}\right) = \frac{2\pi a^n}{1-a^2}, & -1 < a < 0, \\ -\frac{2\pi}{a}\left(\frac{a^{-n}}{a^{-1}-a}\right) = \frac{2\pi}{a^n(a^2-1)}, & a < -1. \end{cases}
\end{aligned}$$

3.d) $\int_0^\pi \tan(x+\mathbf{i}a)\,\mathrm{dx},\ a\neq 0,\ a\in\mathbb{R}.$

Řešení:

$$\begin{aligned}
\int_0^\pi \tan(x+\mathbf{i}a)\,\mathrm{dx} &= \frac{1}{2}\int_0^{2\pi} \tan(x+\mathbf{i}a)\,\mathrm{dx} \\
&= \frac{1}{2}\int_0^{2\pi} \frac{\sin(x+\mathbf{i}a)}{\cos(x+\mathbf{i}a)}\,\mathrm{dx} \\
&= \frac{1}{2\mathbf{i}}\int_0^{2\pi} \frac{\exp(\mathbf{i}x-a)-\exp(-\mathbf{i}x+a)}{\exp(\mathbf{i}x-a)+\exp(-\mathbf{i}x+a)}\,\mathrm{dx} \\
&= \frac{1}{2\mathbf{i}}\int_{|z|=1} \frac{z\exp(-a)-z^{-1}\exp(a)}{z\exp(-a)+z^{-1}\exp(a)}\,\frac{\mathrm{dz}}{\mathbf{i}z} \\
&= -\frac{1}{2}\int_{|z|=1} \frac{z^2-\exp(2a)}{z^2+\exp(2a)}\,\frac{\mathrm{dz}}{z} \\
&= -\frac{1}{2}\int_{|z|=1} \frac{(z-\exp(a))(z+\exp(a))}{(z+\mathbf{i}\exp(a))(z-\mathbf{i}\exp(a))}\,\frac{\mathrm{dz}}{z} \\
&= \begin{cases} \pi\mathbf{i}, & a>0 \\ -\pi\mathbf{i}\left(-1+\frac{\exp(2a)(\mathbf{i}-1)(\mathbf{i}+1)}{2\mathbf{i}\exp(a)\mathbf{i}\exp(a)}+\frac{\exp(2a)(-\mathbf{i}-1)(-\mathbf{i}+1)}{-2\mathbf{i}\exp(a)(-\mathbf{i})\exp(a)}\right)=-\pi\mathbf{i}, & a<0 \end{cases} \\
\int_0^\pi \tan(x+\mathbf{i}a)\mathrm{dx} &= sgn(a)\pi\mathbf{i}
\end{aligned}$$