Řešení DÚ
7. sada

2.c)

$$\int \frac{\sin x \cos x}{1+\sin^3 x} = \int \frac{y}{1+y^3}\,\mathbf{dy} \quad \Big|\; y=\sin x,\ \mathbf{dy}=\cos x\,\mathbf{dx}$$

$$= \int \frac{y}{(y+1)(y^2-y+1)}\,\mathbf{dy}$$

$$= \int \frac{y}{(y+1)(y^2-y+1)}\,\mathbf{dy} = (*)$$

$$\frac{y}{(y+1)(y^2-y+1)} = \frac{a}{y+1} + \frac{by+c}{y^2-y+1}$$

$$y = a(y^2-y+1) + (by+c)(y+1) = (a+b)y^2 + (-a+b+c)y + (a+c)$$

$$a = -\frac{1}{3},\ b=c=\frac{1}{3}$$

$$(*) = -\frac{1}{3}\int \frac{1}{y+1}\,\mathbf{dy} + \frac{1}{3}\int \frac{y+1}{y^2-y+1}\,\mathbf{dy}$$

$$= -\frac{1}{3}\ln|y+1| + \frac{1}{2.3}\int \frac{2y-1+3}{y^2-y+1}\,\mathbf{dy}$$

$$= -\frac{1}{3}\ln|y+1| + \frac{1}{6}\ln|y^2-y+1| + \frac{1}{2}\int \frac{1}{y^2-y+1}\,\mathbf{dy} = (\square)$$

$$\int \frac{1}{y^2-y+1}\,\mathbf{dy} = \int \frac{1}{(y-\frac{1}{2})^2+\frac{3}{4}}\,\mathbf{dy}$$

$$= \frac{4}{3}\int \frac{1}{(\frac{2}{\sqrt{3}}(y-\frac{1}{2}))^2+1}\,\mathbf{dy} \quad \Big|\; u=\frac{2}{\sqrt{3}}(y-\frac{1}{2}),\ \mathbf{du}=\frac{2}{\sqrt{3}}\mathbf{dy}$$

$$= \frac{2}{\sqrt{3}}\int \frac{1}{u^2+1}\,\mathbf{du}$$

$$= \frac{2}{\sqrt{3}}\arctan(\frac{2}{\sqrt{3}}(y-\frac{1}{2})) + c$$

$$(\square) = -\frac{1}{3}\ln|y+1| + \frac{1}{6}\ln|y^2-y+1| + \frac{1}{\sqrt{3}}\arctan(\frac{2}{\sqrt{3}}(y-\frac{1}{2})) + c/2$$

$$= -\frac{1}{3}\ln|\sin x+1| + \frac{1}{6}\ln|\sin^2 x-\sin x+1| + \frac{1}{\sqrt{3}}\arctan(\frac{2}{\sqrt{3}}(\sin x-\frac{1}{2})) + c/2,\ c\in\mathbb{R}$$

$$\sin^3 x \neq -1 \Leftrightarrow \sin x \neq -1 \Leftrightarrow x \neq -\frac{\pi}{2}+2k\pi,\ k\in\mathbb{Z}$$

$$x \in (-\frac{\pi}{2}+2k\pi, \frac{3\pi}{2}+2k\pi),\ k\in\mathbb{Z}$$

3.e)

$$\int \sqrt{a^2+x^2}\,\mathbf{dx} = \int \frac{(a^2+t^2)^2}{4t^3}\,\mathbf{dt} = \int \frac{a^4+2a^2t^2+t^4}{4t^3}\,\mathbf{dt} = (\square)$$

$$\sqrt{a^2+x^2} = t-x,\ x=\frac{t^2-a^2}{2t},\ \sqrt{a^2+x^2}=\frac{a^2+t^2}{2t},\ \mathbf{dx}=\frac{t^2+a^2}{2t^2}\mathbf{dt}$$

$$(\square) = -\frac{a^4}{8t^2} + \frac{a^2\ln|t|}{2} + \frac{t^2}{8} + c$$

$$= -\frac{a^4}{8(2x^2+a^2+2x\sqrt{a^2+x^2})} + \frac{a^2}{2}\ln(x+\sqrt{a^2+x^2}) + \frac{1}{8}(2x^2+a^2+2x\sqrt{a^2+x^2}) + c$$

$$= -\frac{(2x^2+a^2-2x\sqrt{a^2+x^2})}{8} + \frac{a^2}{2}\ln(x+\sqrt{a^2+x^2}) + \frac{1}{8}(2x^2+a^2+2x\sqrt{a^2+x^2}) + c$$

$$= \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln(x+\sqrt{a^2+x^2}) + c,\quad x,c\in\mathbb{R}$$