Representations of algebraic lattices

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Grätzer-Schmidt’s theorem

Definition

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Theorem (G. Grätzer, E.T. Schmidt 1963)

*Every algebraic lattice \( L \) is represented as the congruence lattice \( \text{Con} A \) of some unary algebra \( A \).*
Lampe’s theorem

**Definition**

An algebraic lattice $L$ is a **pinched lattice** provided that there is a set $I$ of compact elements in $L$ such that $\bigvee I = 1$ and each element of $L$ is comparable to every element of $I$. 
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Theorem (W.A. Lampe 1982)

*Each pinched lattice can be represented as the congruence lattice of a grupoid.*
**Lampe’s theorem**

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<td>Every algebraic lattice with compact maximal element is isomorphic to a congruence lattice $\text{Con } M$ of a monoid $M$.</td>
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Theorem (R. Freese, W.A. Lampe, W. Taylor 1979)

Let $V$ be an infinitely-dimensional vector space over an uncountable field $F$. If $\text{Con} V$ is isomorphic to the congruence lattice, $\text{Con} A$, of an algebra $A$, then $A$ has at least $|F|$ operations.
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Proof.

Let $A$ be an algebra with $\not\exists$ non-nulary operations. If for every compact congruence $\Psi$ of $A$ there exist congruences $\Theta, \Phi$ such that $\Psi \geq \Phi \lor \Theta$ and $\Psi \land \Phi = 0 = \Psi \land \Theta$, then a block of every compact congruence of $A$ has at most $\aleph + \aleph_0$ elements.
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Proof.

1. Let $A$ be an algebra with $\kappa$ non-nulary operations. If for every compact congruence $\Psi$ of $A$ there exist congruences $\Theta, \Phi$ such that $\Psi \geq \Phi \lor \Theta$ and $\Psi \land \Phi = 0 = \Psi \land \Theta$, then a block of every compact congruence of $A$ has at most $\kappa + \aleph_0$ elements.

2. Let $A$ be an algebra, $\Psi$ a compact congruence of $A$ and suppose that there is a set $X$ of compact congruences of $A$ such that for all distinct $\Phi, \Theta \in X$, $\Phi \land \Theta = 0$ and $\Phi \lor \Theta \geq \Psi$. Then every nontrivial block of $\Psi$ has at least $|X|$ elements.
A join-semilattice \( S \) is **distributive** if for all \( a, b, c \in S \) with \( a \vee b \geq c \) there are \( a' \leq a, b' \leq b \) with \( a' \vee b' = c \).
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An algebraic lattice $L$ is distributive iff the join-semilattice $L_C$ of compact elements of $L$ is a distributive semilattice.
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A distributive algebraic lattice with at most $\aleph_0$ compact elements is isomorphic to the ideal lattice of a locally matricial algebra (a direct limit of $F$-algebras of type $\mathbb{M}_{n_1}(F) \times \cdots \times \mathbb{M}_{n_k}(F)$, where $F$ is a field).
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Theorem (F. Wehrung 2005)

A distributive algebraic lattice with at most $\aleph_1$ compact elements is isomorphic to the ideal lattice of a von Neumann regular ring.
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Every dimension group of size at most \( \aleph_1 \) is represented as \( K_0(R) \) of some locally matricial algebra.
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Every dimension group of size at most $\aleph_1$ is represented as $K_0(R)$ of some locally matricial algebra.

**Theorem (F. Wehrung 1998)**

There is a vector space with interpolation and order unit, of size $\aleph_2$, which is not isomorphic to $K_0(R)$ of any von Neumann regular ring.
Theorem (R 2004)

There is an algebraic distributive lattice with $\aleph_2$ compact elements not isomorphic to the lattice of convex subgroups (\(=\) ideals) of any dimension group.
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There is a distributive join-semilattice with zero, $S$, of size $\aleph_1$, non isomorphic to the maximal semilattice quotient of any strongly separative Riesz-monoid. In particular, the ideal lattice, $L$, of $S$ is not isomorphic to the lattice of convex subgroups of any dimension group.
Ideal lattices of dimension groups

**Theorem (R 2004)**

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**Corollary**

The lattice L is not isomorphic to the ideal lattice of any unit-regular ring, in particular, the ideal lattice of any locally matricial algebra.
Theorem (F. Wehrung 1998)

There is a partially ordered vector-space with interpolation, $E$, not isomorphic to $K_0(R)$ of any von-Neumann regular ring. The size of $E$ is $\aleph_2$. 

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The maximal semilattice quotient $S$ of the positive cone of $E$ (from the previous theorem) is not isomorphic to the join-semilattice finitely generated ideals of any von-Neumann regular ring.
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Theorem (R, F. Wehrung, J. Tůma)

There is an algebraic distributive lattice \( D \) with \( \aleph_2 \) compact elements which is not isomorphic to the congruence lattice of any algebra with almost permutable congruences. In particular, \( D \) is isomorphic to neither the submodule lattice of a module nor the lattice of normal subgroups of a group.
Theorem (N. Funayama, T. Nakayama 1942)

The congruence lattice of a lattice is distributive.
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Problem (R.P. Dilworth 194?)

Is every distributive algebraic lattice isomorphic to a congruence lattice of a lattice?
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Every finite distributive lattice is isomorphic to the congruence lattice of a finite lattice.

Theorem (A.P. Huhn 1989)

Every algebraic distributive lattice with at most $\aleph_1$ compact elements is isomorphic to the congruence lattice of a lattice.
Theorem (M. Ploščica, J. Tůma, F. Wehrung 1998)

The congruence lattice of a free lattice with $\aleph_2$ free generators in any non-distributive lattice variety is not isomorphic to the congruence lattice of a lattice, with permutable congruences.
Solution to the congruence lattice problem

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There is an algebraic distributive lattice with $\aleph_{\omega+1}$ compact elements not isomorphic to the congruence lattice of a lattice.
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Open problems

Problem (F. Wehrung)
Is every algebraic distributive lattice isomorphic to the congruence lattice of an algebra generating a congruence distributive variety?

Problem (G. Grätzer, E.T. Schmidt)
Is every algebraic distributive lattice isomorphic to the congruence lattice of an algebra with only finitely many non-nulary operations?