FREE TREES AND THE OPTIMAL BOUND IN WEHRUNG'S THEOREM

PAVEL RŮŽIČKA

ABSTRACT. We prove that there is a distributive $(\vee, 0, 1)$-semilattice $\mathcal{G}$ of size $\aleph_2$ such that there is no weakly distributive $(\vee, 0)$-homomorphism from $\mathrm{Con}_c\, A$ to $\mathcal{G}$ with 1 in its range, for any algebra A with either a congruence-compatible structure of a $(\vee, 1)$-semilattice or a congruence-compatible structure of a lattice. In particular, $\mathcal{G}$ is not isomorphic to the $(\vee, 0)$-semilattice of compact congruences of any lattice. This improves Wehrung's solution of Dilworth's Congruence Lattice Problem, by giving the best cardinality bound possible. The main ingredient of our proof is the modification of Kuratowski's Free Set Theorem, which involves what we call *free trees*.

1. Introduction

Congruence lattices of universal algebras correspond to algebraic lattices. By the theorem of N. Funayama and T. Nakayama [2], the congruence lattice of a lattice is, in addition, distributive (see also [3, II. 3. Theorem 11]). On the other hand, R. P. Dilworth proved that every finite distributive lattice is isomorphic to the congruence lattice of a finite lattice (first published in [5]) and he conjectured that every distributive algebraic lattice is isomorphic to the congruence lattice of a lattice (see again [5]). This conjecture, referred to as the *Congruence Lattice Problem*, despite many attempts (see surveys [3, Appendix C] and [12]), remained open for over sixty years until, recently, F. Wehrung disproved it in [17].

Wehrung's solution involves a combination of new ideas, see, in particular, Lemmas 4.4, 5.1, and 6.2 in [17], and methods developed in earlier papers, which originated in [14] and were pursued further in [9, 10, 11, 13, 15]. In these papers, counterexamples to various problems related to the Congruence Lattice Problem were obtained. The optimal cardinality bound for all these counterexamples is $\aleph_2$, however Wehrung's argument requires an algebraic distributive lattice with at least $\aleph_{\omega+1}$ compact elements. In the present paper, we improve Wehrung's result by proving that there is a counterexample of size $\aleph_2$. As in the related cases, $\aleph_2$ turns out to be the optimal cardinality bound for a negative solution of the Congruence Lattice Problem. Our proof closely follows Wehrung's ideas. The main difference consists in an enhancement of Kuratowski's Free Set Theorem by a new combinatorial principle which involves finite trees.

Date: 24/9/2007.

2000 *Mathematics Subject Classification.* 06B15, 06B10, 06A12, 08A30, 08B10.

Key words and phrases. Lattice; algebraic; semilattice; distributive; congruence; weakly distributive; free tree.

This work is part of the research project MSM 0021620839, financed by MSMT; it is also partly supported by INTAS project 03-51-4110 and grant GAUK 448/2004/B-MAT.

Wehrung's construction in [17] uses a "*free*" distributive extension of a $(\vee, 0)$-semilattice; a functor that assigns to every $(\vee, 0)$-semilattice a distributive $(\vee, 0)$-semilattice, constructed previously by M. Ploščica and J. Tůma in [9]. The main features of this construction for the refutation of the Congruence Lattice Problem are extracted in the so-called Evaporation Lemma [17, Lemma 4.4]. We generalize this idea by defining a *diluting functor* whose properties are sufficient to prove the Evaporation Lemma, and we prove that the free distributive extension of a $(\vee, 0)$-semilattice is, indeed, a diluting functor.

Further, we modify Kuratowski's Free Set Theorem, the combinatorial essence of the above mentioned counterexamples. Given a set Ω and a map $\Phi\colon [\Omega]^{<\omega} \to [\Omega]^{<\omega}$, we define a *free k-tree* (*with respect to* Φ), for every positive integer k, which is a k-ary tree with some combinatorial properties derived from the Kuratowski's Free Set Theorem. We prove that a free k-tree exists whenever the cardinality of the set Ω is at least $\aleph_{k-1}$, and we apply the existence of a free 3-tree in every set of cardinality at least $\aleph_2$ to attain the optimal cardinality bound in the Wehrung's result.

2. Basic Concepts

A $(\vee, 0)$-semilattice S is *distributive* if for every $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in S$ satisfying $\boldsymbol{c} \leq \boldsymbol{a} \vee \boldsymbol{b}$, there are $\boldsymbol{a}' \leq \boldsymbol{a}$ and $\boldsymbol{b}' \leq \boldsymbol{b}$ such that $\boldsymbol{a}' \vee \boldsymbol{b}' = \boldsymbol{c}$. A homomorphism $\mu\colon S \to T$ of join-semilattices is called *weakly distributive* at $\boldsymbol{x} \in S$, if for all $\boldsymbol{y}_0, \boldsymbol{y}_1 \in T$ such that $\mu(\boldsymbol{x}) \leq \boldsymbol{y}_0 \vee \boldsymbol{y}_1$, there are $\boldsymbol{x}_0, \boldsymbol{x}_1 \in S$ such that $\boldsymbol{x} \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1$ and $\mu(\boldsymbol{x}_i) \leq \boldsymbol{y}_i$, for all $i < 2$ (see [17]). The homomorphism μ is *weakly distributive* if it is weakly distributive at every element of S.

Let A be an algebra. We say that an n-ary operation f on A is *congruence-compatible* (see [8, 17]) if for every congruence θ of A, $(x_i, y_i) \in \theta$, for $i = 0, \dots, n-1$, implies that $(f(x_0, \dots, x_{n-1}), f(y_0, \dots, y_{n-1})) \in \theta$. In particular, a semilattice operation $\vee$, resp. $\wedge$, on A is congruence-compatible providing that $(x, y) \in \theta$ implies $(x \vee z, y \vee z) \in \theta$, resp. $(x \wedge z, y \wedge z) \in \theta$, for every $x, y, z \in A$ and every $\theta \in \operatorname{Con} A$.

Given an algebra A and elements $x, y \in A$, we denote by $\Theta_A(x, y)$ the smallest congruence (i.e., intersection of all the congruences) of A identifying x and y. We denote by $\operatorname{Con} A$, resp. $\operatorname{Con}_c A$ the lattice of all congruences of A, resp. the join-semilattice of all compact congruences of A. We say that A has permutable congruences if $\boldsymbol{a} \vee \boldsymbol{b} = \boldsymbol{a} \circ \boldsymbol{b} = \boldsymbol{b} \circ \boldsymbol{a}$, for all $\boldsymbol{a}, \boldsymbol{b} \in \operatorname{Con} A$.

We will use the standard set theoretic notation and terminology. We identify each ordinal number with the set of its predecessors, in particular, $n = \{0, \dots, n-1\}$, for each positive integer n. We denote by ω the first infinite ordinal, and by ω_n the first ordinal of size $\aleph_n$, for every positive integer n. For a set X, we denote by $[X]^{<\omega}$ the set of all finite subsets of X and by $[X]^n$ the set of all its n-element subset, for every natural number n. We denote by $|X|$ the cardinality of a set X. As in [17], we put $\varepsilon(n) = n \mod 2$, for every integer n.

3. Diluting functors

Denote by $\mathbf{S}$ the category of $(\vee, 0)$-semilattices (with $(\vee, 0)$-homomorphisms).

Definition 1. An *expanding functor* on $\mathbf{S}$ is a pair $(\mathcal{F}, \iota)$, where $\mathcal{F}$ is an endofunctor on $\mathbf{S}$ and ι is a natural transformation from the identity to $\mathcal{F}$ such that $\iota_S\colon S \to \mathcal{F}(S)$ is an embedding, for every $(\vee, 0)$-semilattice S. We shall denote the expanding functor above by $\mathcal{F}$ once the natural transformation ι is understood, and we shall identify $\iota_S(\boldsymbol{x})$ with $\boldsymbol{x}$, for all $\boldsymbol{x} \in S$.

An expanding functor $\mathcal{F}$ on $\mathbf{S}$ is a *diluting functor*, if for all $(\vee, 0)$-semilattices S and T and every $(\vee, 0)$-homomorphism $f : S \to T$, the following property is satisfied: for every $\boldsymbol{v} \in \mathcal{F}(S)$, and $\boldsymbol{u}_0, \boldsymbol{u}_1 \in \mathcal{F}(T)$, if $\mathcal{F}(f)(\boldsymbol{v}) \le \boldsymbol{u}_0 \vee \boldsymbol{u}_1$, then there are $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \mathcal{F}(S)$ and $\boldsymbol{y} \in S$ such that

$$f(\boldsymbol{y}) \le \boldsymbol{u}_0 \vee \boldsymbol{u}_1, \quad \mathcal{F}(f)(\boldsymbol{x}_i) \le \boldsymbol{u}_i, \text{ for all } i < 2, \quad \text{and} \quad \boldsymbol{v} \le \boldsymbol{x}_0 \vee \boldsymbol{x}_1 \vee \boldsymbol{y}.$$

Given a $(\vee, 0)$-semilattice S and subsets U, V of S, we shall use the notation

$$U \vee V = \{\boldsymbol{u} \vee \boldsymbol{v} \mid \boldsymbol{u} \in U \text{ and } \boldsymbol{v} \in V\}.$$

Lemma 3.1. *Let S be a $(\vee, 0)$-semilattice and let S_i, $i < 2$, be $(\vee, 0)$-subsemilattices of S such that $S = S_0 \vee S_1$, and there are retractions $r_i : S \to S_i$, for $i < 2$. Put $s_i = \mathcal{F}(r_i)$, for every $i < 2$. Let $\boldsymbol{u}_i \in \mathcal{F}(S_i)$, $i < 2$, be such that $s_i(\boldsymbol{u}_{1-i}) = 0$, for all $i < 2$. Then for every $\boldsymbol{y} \in S$ such that $\boldsymbol{y} \le \boldsymbol{u}_0 \vee \boldsymbol{u}_1$, there are $\boldsymbol{y}_i \in S_i$, $i < 2$, such that $\boldsymbol{y} \le \boldsymbol{y}_0 \vee \boldsymbol{y}_1$ and $\boldsymbol{y}_i \le \boldsymbol{u}_i$, for all $i < 2$.*

Proof. Put $\boldsymbol{y}_i = r_i(\boldsymbol{y})$, for every $i < 2$. Since $S = S_0 \vee S_1$, there are $\boldsymbol{y}'_i \in S_i$, for $i < 2$, such that $\boldsymbol{y} = \boldsymbol{y}'_0 \vee \boldsymbol{y}'_1$. Since the maps r_i, $i < 2$, are retractions, $\boldsymbol{y}'_i \le r_i(\boldsymbol{y}) = \boldsymbol{y}_i$, for all $i < 2$, whence $\boldsymbol{y} \le \boldsymbol{y}_0 \vee \boldsymbol{y}_1$.

It remains to prove that $\boldsymbol{y}_i \le \boldsymbol{u}_i$, for all $i < 2$. Fix $i < 2$. Since $s_i \restriction S = r_i$ and $s_i : \mathcal{F}(S) \to \mathcal{F}(S_i)$ is a retraction, $s_i(\boldsymbol{u}_i) = \boldsymbol{u}_i$. Since, by the assumptions, $s_i(\boldsymbol{u}_{1-i}) = 0$, we conclude that

$$\boldsymbol{y}_i = s_i(\boldsymbol{y}) \le s_i(\boldsymbol{u}_0 \vee \boldsymbol{u}_1) = s_i(\boldsymbol{u}_0) \vee s_i(\boldsymbol{u}_1) = s_i(\boldsymbol{u}_i) = \boldsymbol{u}_i. \qquad \square$$

Define $\mathcal{F}^0$ to be the identity functor and, inductively, $\mathcal{F}^{n+1} = \mathcal{F} \circ \mathcal{F}^n$, for every natural number n. By our assumption, the inclusion map defines a natural transformation from the identity functor on $\mathbf{S}$ to $\mathcal{F}$, therefore we can define $\mathcal{F}^\infty(S) = \bigcup_{n\in\omega} \mathcal{F}^n(S)$, resp. $\mathcal{F}^\infty(f) = \bigcup_{n\in\omega} \mathcal{F}^n(f)$, for every $(\vee, 0)$-semilattice S, resp. every $(\vee, 0)$-homomorphism $f : S \to T$ and, again, the inclusion map defines a natural transformation from the identity functor on $\mathbf{S}$ to $\mathcal{F}^\infty$. In particular, if $\mathcal{F}$ is an expanding functor on $\mathbf{S}$, then $\mathcal{F}^\infty$ is expanding as well.

Lemma 3.2. *Let $\mathcal{F}$ be a diluting functor on $\mathbf{S}$. Then the functor $\mathcal{F}^\infty$ is diluting as well.*

Proof. Let S and T be $(\vee, 0)$-semilattices, and let $f : S \to T$ be a $(\vee, 0)$-homomorphism. Let $\boldsymbol{v} \in \mathcal{F}^\infty(S)$ and let $\boldsymbol{u}_0, \boldsymbol{u}_1 \in \mathcal{F}^\infty(T)$ be such that $\mathcal{F}^\infty(f)(\boldsymbol{v}) \le \boldsymbol{u}_0 \vee \boldsymbol{u}_1$. We are looking for $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \mathcal{F}^\infty(S)$ and $\boldsymbol{y} \in S$ such that

$$f(\boldsymbol{y}) \le \boldsymbol{u}_0 \vee \boldsymbol{u}_1, \quad \mathcal{F}^\infty(f)(\boldsymbol{x}_i) \le \boldsymbol{u}_i, \text{ for all } i < 2, \quad \text{and} \quad \boldsymbol{v} \le \boldsymbol{x}_0 \vee \boldsymbol{x}_1 \vee \boldsymbol{y}.$$

We shall argue by induction on the least natural number n such that $\boldsymbol{v} \in \mathcal{F}^n(S)$. If $n = 0$, we put $\boldsymbol{x}_0 = \boldsymbol{x}_1 = 0$, $\boldsymbol{y} = \boldsymbol{v}$, and we are done. Suppose that $\boldsymbol{v} \in \mathcal{F}^{n+1}(S)$, for some natural number n, and that the property is proved at stage n. Let $k \ge n$ be a natural number such that $\boldsymbol{u}_0, \boldsymbol{u}_1 \in \mathcal{F}^{k+1}(T)$. Denote by g the composition of the $(\vee, 0)$-homomorphism $\mathcal{F}^n(f)$ and the inclusion map from $\mathcal{F}^n(T)$ to $\mathcal{F}^k(T)$. By applying the assumption that $\mathcal{F}$ is a diluting functor to the $(\vee, 0)$-homomorphism $g : \mathcal{F}^m(S) \to \mathcal{F}^k(T)$, we obtain elements $\boldsymbol{x}'_0, \boldsymbol{x}'_1 \in \mathcal{F}^{m+1}(S)$ and $\boldsymbol{y}' \in \mathcal{F}^m(S)$ such that

$$g(\boldsymbol{y}') \le \boldsymbol{u}_0 \vee \boldsymbol{u}_1, \quad \mathcal{F}(g)(\boldsymbol{x}'_i) \le \boldsymbol{u}_i, \text{ for all } i < 2, \quad \text{and} \quad \boldsymbol{v} \le \boldsymbol{x}'_0 \vee \boldsymbol{x}'_1 \vee \boldsymbol{y}'.$$

Since $g(\boldsymbol{y}') \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1$ implies $\mathcal{F}^\infty(f)(\boldsymbol{y}') \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1$, there are, by the induction hypothesis, elements $\boldsymbol{x}_0'', \boldsymbol{x}_1'' \in \mathcal{F}^\infty(S)$ and $\boldsymbol{y} \in S$ such that

$$f(\boldsymbol{y}) \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1, \quad \mathcal{F}^\infty(f)(\boldsymbol{x}_i'') \leq \boldsymbol{u}_i, \text{ for all } i < 2, \quad \text{ and } \boldsymbol{y}' \leq \boldsymbol{x}_0'' \vee \boldsymbol{x}_1'' \vee \boldsymbol{y}.$$

Now it is easy to conclude that $\boldsymbol{x}_i = \boldsymbol{x}_i' \vee \boldsymbol{x}_i''$, for $i < 2$, and $\boldsymbol{y}$ are the desired elements. □

As in [17], denote by $\mathcal{L}$ the functor from the category of sets to $\mathbf{S}$, which assigns to a set Ω the $(\vee, 0, 1)$-semilattice $\mathcal{L}(\Omega)$ defined by generators 1, and $\boldsymbol{a}_0^\xi$, $\boldsymbol{a}_1^\xi$, for $\xi \in \Omega$, subjected to the relations

$$\boldsymbol{a}_0^\xi \vee \boldsymbol{a}_1^\xi = 1, \quad \text{for all } \xi \in \Omega, \tag{3.1}$$

and to a map $f\colon X \to Y$ the unique $(\vee, 0, 1)$-homomorphism $\mathcal{L}(f)\colon \mathcal{L}(X) \to \mathcal{L}(Y)$ such that $\mathcal{L}(f)(\boldsymbol{a}_i^\xi) = \boldsymbol{a}_i^{f(\xi)}$, for all $\xi \in X$ and all $i < 2$.

Given a finite subset A of Ω and a map $\varphi\colon A \to 2$, we put $\boldsymbol{a}_\varphi^A = \bigvee_{\alpha \in A} \boldsymbol{a}_{\varphi(\alpha)}^\alpha$. By the coming Corollary 4.2, the following lemma is a generalization of Wehrung's original "Evaporation Lemma" [17, Lemma 4.4].

Lemma 3.3. *Let $\mathcal{F}$ be a diluting functor on $\mathbf{S}$. Define $\mathcal{G} = \mathcal{F} \circ \mathcal{L}$. Let Ω be a set, let A_0, A_1 be finite disjoint subsets of Ω, and let $\delta \in \Omega \smallsetminus (A_0 \cup A_1)$. Let $\boldsymbol{v} \in \mathcal{G}(\Omega \smallsetminus \{\delta\})$, let $\varphi_i\colon A_i \to 2$, and let $\boldsymbol{u}_i \in \mathcal{G}(\Omega \smallsetminus A_{1-i})$, for $i < 2$. Then*

$$\boldsymbol{v} \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1 \quad \textit{and} \quad \boldsymbol{u}_i \leq \boldsymbol{a}_{\varphi_i}^{A_i}, \boldsymbol{a}_i^\delta, \textit{ for all } i < 2,$$

implies that $\boldsymbol{v} = 0$.

Proof. Denote by f the inclusion map from $\Omega \smallsetminus \{\delta\}$ to Ω, and observe that $\mathcal{L}(f)$ corresponds to the inclusion $\mathcal{L}(\Omega \smallsetminus \{\delta\}) \subseteq \mathcal{L}(\Omega)$. Since $\mathcal{F}$ is diluting, there are elements $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \mathcal{G}(\Omega \smallsetminus \{\delta\})$ and $\boldsymbol{y} \in \mathcal{L}(\Omega \smallsetminus \{\delta\})$ such that

$$\boldsymbol{y} \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1, \quad \mathcal{G}(f)(\boldsymbol{x}_i) \leq \boldsymbol{u}_i, \text{ for all } i < 2, \quad \text{and} \quad \boldsymbol{v} \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1 \vee \boldsymbol{y}.$$

Fix $i < 2$. There is a unique retraction $p_i\colon \mathcal{L}(\Omega) \to \mathcal{L}(\Omega \smallsetminus \{\delta\})$ satisfying $p_i(\boldsymbol{a}_i^\delta) = 0$ and $p_i(\alpha_{1-i}^\delta) = 1$. Observe that $q_i = \mathcal{F}(p_i)\colon \mathcal{G}(\Omega) \to \mathcal{G}(\Omega \smallsetminus \{\delta\})$ is a retraction with respect to $\mathcal{G}(f)$. Since $\boldsymbol{x}_i \in \mathcal{G}(\Omega \smallsetminus \{\delta\})$, $q_i(\mathcal{G}(f)(\boldsymbol{x}_i)) = \boldsymbol{x}_i$, while, by our assumptions, $q_i(\boldsymbol{a}_i^\delta) = 0$. Since $\mathcal{G}(f)(\boldsymbol{x}_i) \leq \boldsymbol{u}_i \leq \boldsymbol{a}_i^\delta$, we conclude that $\boldsymbol{x}_i = 0$.

Let $r_i\colon \mathcal{L}(\Omega) \to \mathcal{L}(\Omega \smallsetminus A_{1-i})$ be a unique retraction such that $r_i(\boldsymbol{a}_{\varphi_{1-i}}^{A_{1-i}}) = 0$, and put $s_i = \mathcal{F}(r_i)$. From $\boldsymbol{u}_{1-i} \leq \boldsymbol{a}_{\varphi_{1-i}}^{A_{1-i}}$ it follows that $s_i(\boldsymbol{u}_{1-i}) = 0$. By Lemma 3.1, there are $\boldsymbol{y}_j \in \mathcal{L}(\Omega \smallsetminus A_{1-j})$ with $\boldsymbol{y}_j \leq \boldsymbol{u}_j$, for all $j < 2$, such that $\boldsymbol{y} \leq \boldsymbol{y}_0 \vee \boldsymbol{y}_1$. Since $\boldsymbol{y}_j \leq \boldsymbol{u}_j \leq \boldsymbol{a}_{\varphi_j}^{A_j}, \boldsymbol{a}_j^\delta$ and $\delta \notin A_j$, we conclude that $\boldsymbol{y}_j = 0$, for all $j < 2$. □

4. Free Distributive Extension is Diluting

We summarize the main properties of the construction of the extension $\mathcal{R}(S)$ of a $(\vee, 0)$-semilattice S (see [9, Section 2]) referring to the outline in [17, Sections 3,4]. We shall prove that the functor $\mathcal{R}$ is diluting. For a $(\vee, 0)$-semilattice S, we shall put $\mathcal{C}(S) = \{(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in S^3 \mid \boldsymbol{c} \leq \boldsymbol{a} \vee \boldsymbol{b}\}$. We say that a finite subset $\boldsymbol{v}$ of $\mathcal{C}(S)$ is *reduced*, if the following properties are satisfied:

(1) the set $\boldsymbol{v}$ contains exactly triple of the form $(\boldsymbol{a}, \boldsymbol{a}, \boldsymbol{a})$; we define $\pi(\boldsymbol{v}) = \boldsymbol{a}$ and $\boldsymbol{v}^* = \boldsymbol{v} \smallsetminus \{(\boldsymbol{a}, \boldsymbol{a}, \boldsymbol{a})\}$.
(2) $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{v}$ and $(\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c}) \in \boldsymbol{v}$ implies that $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{c}$, for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in S$.
(3) if $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{v}^*$, then $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \not\leq \pi(\boldsymbol{v})$, for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in S$.

Observe that if $\boldsymbol{v}$ is a reduced subset of $\mathcal{C}(S)$ and $\boldsymbol{u} \subseteq \boldsymbol{v}^*$, then $\boldsymbol{u} \cup \{(0,0,0)\}$ is a reduced subset as well.

We denote by $\mathcal{R}(S)$ the set of all reduced subsets of $\mathcal{C}(S)$. By [9, Lemma 2.1] (see also [17, Corollary 3.2]), $\mathcal{R}(S)$ is a $(\vee, 0)$-semilattice with respect to the partial ordering $\leq$ defined by

$$\boldsymbol{v} \leq \boldsymbol{u} \text{ iff } \text{ for all } (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{v} \smallsetminus \boldsymbol{u} \text{ either } \boldsymbol{a} \leq \pi(\boldsymbol{u}) \text{ or } \boldsymbol{c} \leq \pi(\boldsymbol{u}) \tag{4.1}$$

and the assignment $\boldsymbol{v} \mapsto \{(\boldsymbol{v},\boldsymbol{v},\boldsymbol{v})\}$ is a $(\vee,0)$-embedding from S into $\mathcal{R}(S)$.

As in [17], we use the symbol $\bowtie_S$ to denote the elements of $\mathcal{R}(S)$ defined as

$$\bowtie_S(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = \begin{cases} \boldsymbol{c}, & \text{if either } \boldsymbol{a} = \boldsymbol{b} \text{ or } \boldsymbol{b} = 0 \text{ or } \boldsymbol{c} = 0, \\ 0, & \text{if } \boldsymbol{a} = 0, \\ \{(0,0,0), (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\}, & \text{otherwise,} \end{cases}$$

for all $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \mathcal{C}(S)$. Recall that by formula (3.3) in [17],

$$\boldsymbol{x} = \bigvee \bigl(\bowtie_S(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \mid (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{x}\bigr), \quad \text{for all } \boldsymbol{x} \in \mathcal{R}(S). \tag{4.2}$$

By [17, Proposition 3.5], every $(\vee,0)$-homomorphism $f\colon S \to T$ extends to a unique $(\vee,0)$-homomorphism $\mathcal{R}(f)\colon S \to T$ such that

$$\mathcal{R}(f)(\bowtie_S(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})) = \bowtie_T(f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c})), \quad \text{for all } (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \mathcal{C}(S), \tag{4.3}$$

and the assignment $S \mapsto \mathcal{R}(S)$, $f \mapsto \mathcal{R}(f)$ is an functor on the category $\mathbf{S}$. It follows that if $f\colon S \to T$ is a $(\vee,0)$-homomorphism, $\boldsymbol{v} \in S$, and $\boldsymbol{u} \in T$, then

$$\mathcal{R}(f)(\boldsymbol{v}) \leq \boldsymbol{u} \text{ iff } \bowtie_T(f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c})) \leq \boldsymbol{u} \text{ for all } (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{v}. \tag{4.4}$$

Lemma 4.1. *The functor $\mathcal{R}$ is diluting.*

Proof. Let S and T be $(\vee,0)$-semilattices and let $f\colon S \to T$ be a $(\vee,0)$-homomorphism. We have to verify that for every $\boldsymbol{v} \in \mathcal{R}(S)$ and $\boldsymbol{u}_0, \boldsymbol{u}_1 \in \mathcal{R}(T)$ such that $\mathcal{R}(f)(\boldsymbol{v}) \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1$, there are elements $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \mathcal{R}(S)$ and $\boldsymbol{y} \in S$ such that

$$f(\boldsymbol{y}) \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1, \quad \mathcal{R}(f)(\boldsymbol{x}_i) \leq \boldsymbol{u}_i, \text{ for all } i < 2, \quad \text{and} \quad \boldsymbol{v} \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1 \vee \boldsymbol{y}.$$

For all $i < 2$ define

$$\boldsymbol{x}_i = \{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{v} \mid (f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c})) \in \boldsymbol{u}_i^*\} \cup \{(0,0,0)\},$$

and observe that $\boldsymbol{x}_0$, $\boldsymbol{x}_1$, as subsets of $\boldsymbol{v}^* \cup \{(0,0,0)\}$, are reduced, that is, $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \mathcal{R}(S)$. It follows from (4.3) that $\mathcal{R}(f)(\boldsymbol{x}_i) \leq \boldsymbol{u}_i$, for all $i < 2$. An easy application of [17, Lemma 3.1] yields that $(\boldsymbol{u}_0 \vee \boldsymbol{u}_1)^* \subseteq \boldsymbol{u}_0^* \cup \boldsymbol{u}_1^*$, and so $\bowtie_S(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1$, for every $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{v}$ such that $(f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c})) \in (\boldsymbol{u}_0 \vee \boldsymbol{u}_1)^*$.

For all $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{v}$ define

$$\varrho((\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})) = \begin{cases} \boldsymbol{a} & \text{if } f(\boldsymbol{a}) \leq \pi(\boldsymbol{u}_0 \vee \boldsymbol{u}_1), \\ \boldsymbol{c} & \text{otherwise,} \end{cases}$$

and put

$$\boldsymbol{y} = \bigvee \bigl(\varrho((\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})) \mid (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{v} \text{ and } (f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c})) \notin (\boldsymbol{u}_0 \vee \boldsymbol{u}_1)^*\bigr).$$

Clearly, $\boldsymbol{y} \in S$, and, by (4.1), $\bowtie_S(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \leq \boldsymbol{y}$, for all $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{v}$ such that $(f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c})) \notin (\boldsymbol{u}_0 \vee \boldsymbol{u}_1)^*$. Overall, we have proved that $\bowtie_S(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1 \vee \boldsymbol{y}$, for all $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{v}$, and so, by (4.2), $\boldsymbol{v} \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1 \vee \boldsymbol{y}$.

Finally, since $\mathcal{R}(f)(\boldsymbol{v}) \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1$, it follows from (4.1) that $f(\varrho((\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}))) \leq \pi(\boldsymbol{u}_0 \vee \boldsymbol{u}_1)$, for every $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{v}$ such that $(f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c})) \notin (\boldsymbol{u}_0 \vee \boldsymbol{u}_1)^*$, whence $f(\boldsymbol{y}) \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1$. □

Observe that $\mathcal{R}(S)$ is distributive "relatively to" the $(\vee, 0)$-semilattice S, that is, for every $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in S$ with $\boldsymbol{c} \leq \boldsymbol{a} \vee \boldsymbol{b}$, there are $\boldsymbol{a}' \leq \boldsymbol{a}$, $\boldsymbol{b}' \leq \boldsymbol{b}$ in $\mathcal{R}(S)$ such that $\boldsymbol{c} = \boldsymbol{a}' \vee \boldsymbol{b}'$. It follows that the $(\vee, 0)$-semilattice $\mathcal{R}^\infty(S)$ is distributive. By Lemma 3.2, we conclude that

Corollary 4.2. *The functor $\mathcal{R}^\infty$ is diluting. Moreover, $\mathcal{R}^\infty(S)$ is a distributive $(\vee, 0)$-semilattice, for every $(\vee, 0)$-semilattice S.*

Observe that the functor $\mathcal{R}^\infty$ is denoted by $\mathcal{D}$ in [17].

5. Free Trees

Let k be a positive integer, let Ω be a set. For a map $\Psi\colon [X]^{k-1} \to [X]^{<\omega}$, we say that a k-element subset B is *free* (*with respect to* Ψ) if $b \notin \Psi(B \smallsetminus \{b\})$, for all $b \in B$.

Kuratowski's Free Set Theorem [7]**.** *Let k be a positive integer, let Ω be a set, and let $\Psi\colon [\Omega]^{k-1} \to [\Omega]^{<\omega}$ be any map. If $|\Omega| \geq \aleph_{k-1}$, then there is a k-element free subset of Ω.*

Notation. Let k and n be natural numbers with $k > 0$. Given a natural number $m \leq n$ and a map $g\colon \{m, \dots, n-1\} \to k$, we shall put

$$T_{n,k}(g) = \{f\colon n \to k \mid f \text{ extends } g\}$$

Given a natural number $m < n$, a map $g\colon \{m+1, \dots, n-1\} \to k$, and $i < k$, we shall use the notation

$$T_{n,k}(g,i) = \{f \in T_{n,k}(g) \mid f(m) = i\},$$
$$T_{n,k}(g,\neg i) = \{f \in T_{n,k}(g) \mid f(m) \neq i\}.$$

Definition 2. Let Ω be a set and let $\Phi\colon [\Omega]^{<\omega} \to [\Omega]^{<\omega}$ be a map. Let k and n be natural numbers with $k > 0$. We say that a family $\mathcal{T} = (\alpha(f) \mid f\colon n \to k)$ of distinct elements of Ω is a *free k-tree of height n* (*with respect to* Φ) if

$$\{\alpha(f) \mid f \in T_{n,k}(g,i)\} \cap \Phi(\{\alpha(f) \mid f \in T_{n,k}(g,\neg i)\}) = \emptyset, \tag{5.1}$$

for all $m < n$, all maps $g\colon \{m+1, \dots, n-1\} \to k$, and all $i < k$. We will call the set $\operatorname{rng} \mathcal{T} = \{\alpha(f) \mid f\colon n \to k\}$ the *range* of $\mathcal{T}$.

Lemma 5.1. *Let Ω be a set and let $\Phi\colon [\Omega]^{<\omega} \to [\Omega]^{<\omega}$ be a map. Let k be a positive integer. Every subset X of Ω of cardinality at least $\aleph_{k-1}$ contains the range of a free k-tree of height n, for every natural number n.*

Proof. We shall argue by induction on n. If $n = 0$, pick $\mathcal{T} = \{\alpha_\emptyset\}$, where $\alpha_\emptyset$ is an arbitrary element of X. Let n be natural number and suppose that the statement holds for n. We shall prove that X contains a free k-tree, $\mathcal{T}$, of height $n+1$. Cut up the set X as a union of pairwise disjoint subsets X_ξ, for $\xi < \omega_{k-1}$, of cardinality at least $\aleph_{k-1}$. By the induction hypothesis, each X_ξ contains the range of a free tree $\mathcal{T}_\xi = (\alpha_\xi(f) \mid f\colon n \to k)$ of height n. Define a map $\Psi\colon [\omega_{k-1}]^{k-1} \to [\omega_{k-1}]^{<\omega}$ by

$$\Psi(F) = \Big\{\nu < \omega_{k-1} \mid \operatorname{rng} \mathcal{T}_\nu \cap \Phi\Big(\bigcup_{\xi \in F} \operatorname{rng} \mathcal{T}_\xi\Big) \neq \emptyset\Big\}, \tag{5.2}$$

for all $F \in [\omega_{k-1}]^{k-1}$ (since the sets $\operatorname{rng} \mathcal{T}_\xi$ are pairwise disjoint and finite, $\Psi(F)$ is finite, for every $F \in [\omega_{k-1}]^{k-1}$). By Kuratowski's Free Set Theorem, there is a k-element free subset, $B = \{\xi_0, \dots, \xi_{k-1}\}$, of X with respect to Ψ. Put $\alpha(f) = \alpha_{\xi_{f(n)}}(f \restriction n)$, for all maps $f\colon (n{+}1) \to k$. We claim that $\mathcal{T} = (\alpha(f) \mid f\colon (n{+}1) \to k)$ is a free k-tree with respect to Φ. Let $m < n{+}1$ and fix a map $g\colon \{m{+}1, \dots, n\} \to k$. If $m = n$, the only possibility is $g = \emptyset$. Then

$$\{\alpha(f) \mid f \in T_{n+1,k}(g,i)\} = \operatorname{rng} \mathcal{T}_{\xi_i},$$
$$\{\alpha(f) \mid f \in T_{n+1,k}(g,\neg i)\} = \bigcup_{j<k,\ j \neq i} \operatorname{rng} \mathcal{T}_{\xi_j},$$

for all $i < k$. Since B is a free set with respect to Ψ,

$$\operatorname{rng} \mathcal{T}_{\xi_i} \cap \Phi\Big(\bigcup_{j<k,\ j\neq i} \operatorname{rng} \mathcal{T}_{\xi_j}\Big) = \emptyset,$$

by (5.2). Let $m < n$ and $i < k$. Put $g' = g \restriction \{m+1, \dots, n-1\}$. Then

$$\{\alpha(f) \mid f \in T_{n+1,k}(g,i)\} = \{\alpha_{\xi_{g(n)}}(f) \mid f \in T_{n,k}(g',i)\},$$
$$\{\alpha(f) \mid f \in T_{n+1,k}(g,\neg i)\} = \{\alpha_{\xi_{g(n)}}(f) \mid f \in T_{n,k}(g',\neg i)\}.$$

Since $\mathcal{T}_{\alpha_{\xi_{g(n)}}}$ is a free k-tree with respect to Φ,

$$\{\alpha_{\xi_{g(n)}}(f) \mid f \in T_{n,k}(g',i)\} \cap \Phi(\{\alpha_{\xi_{g(n)}}(f) \mid f \in T_{n,k}(g',\neg i)\}) = \emptyset,$$

by (5.1). □

6. The optimal bound in Wehrung's Theorem

Let $\mathcal{F}$ be an expanding functor on $\mathbf{S}$ satisfying the following properties: For every $(\vee,0)$-semilattice S and every family $(S_i \mid i \in I)$ of $(\vee,0)$-subsemilattices of S:

$$\bigcap_{i\in I} \mathcal{F}(S_i) = \mathcal{F}\Big(\bigcap_{i\in I} S_i\Big). \tag{6.1}$$

For a nonempty upwards directed poset P and every family $(S_p \mid p \in P)$ of $(\vee,0)$-semilattices such that S_p is a $(\vee,0)$-subsemilattice of S_q, whenever $p \le q$ in P:

$$\bigcup_{p\in P} \mathcal{F}(S_p) = \mathcal{F}\Big(\bigcup_{p\in P} S_p\Big). \tag{6.2}$$

Put $\mathcal{G} = \mathcal{F} \circ \mathcal{L}$. Then for every set Ω and every family $(A_i \mid i \in I)$ of subsets of Ω:

$$\bigcap_{i\in I} \mathcal{G}(A_i) = \mathcal{G}\Big(\bigcap_{i\in I} A_i\Big),$$

and for a nonempty upwards directed poset P and every family $(A_p \mid p \in P)$ of sets such that $A_p \subseteq A_q$, whenever $p \le q$ in P:

$$\bigcup_{p\in P} \mathcal{G}(A_p) = \mathcal{G}\Big(\bigcup_{p\in P} A_p\Big).$$

It follows that, given a set Ω and an element $\boldsymbol{a} \in \mathcal{G}(\Omega)$, there is a smallest finite subset F of Ω such that $\boldsymbol{a} \in \mathcal{G}(F)$. We shall call the subset F the *support* of $\boldsymbol{a}$. We denote the support of $\boldsymbol{a}$ by $\operatorname{Supp}(\boldsymbol{a})$ (see [17]). Now we rephrase [17, Theorem 6.1].

Theorem 6.1. *Let Ω be a set of cardinality at least $\aleph_2$, let $\mathcal{F}$ be a diluting functor satisfying properties* (6.1) *and* (6.2)*. Let A be an algebra with either a congruence-compatible structure of a $(\vee, 1)$-semilattice or a congruence compatible structure of a lattice. Then there does not exist a weakly distributive $(\vee, 0)$-homomorphism from $\mathrm{Con_c}\, A$ to $\mathcal{G}(\Omega)$ containing* 1 *in its range.*

Proof. Assume for contradiction that there is a weakly distributive $(\vee, 0)$-homomorphism $\mu\colon \mathrm{Con_c}\, A \to \mathcal{G}(\Omega)$ with 1 in its range. Since 1 is in the range of μ, there is a finite subset T of Ω such that

$$\mu\Big(\bigvee\{\Theta_A(s,t) \mid s,t \in T\}\Big) = 1.$$

Put $s = 1$ if A has a congruence-compatible structure of a $(\vee, 0)$-semilattice and $s = \bigvee T$ if A has a congruence compatible structure of a lattice. Then

$$\mu\big(\bigvee_{t\in T} \Theta_A(s,t)\Big) = \bigvee_{t\in T} \mu(\Theta_A(s,t)) = 1.$$

Let $\xi \in \Omega$. Since the homomorphism μ is weakly distributive, there are $\theta_0^\xi, \theta_1^\xi \in \mathrm{Con_c}\, A$ such that $\bigvee_{t\in T} \Theta_A(s,t) \leq \theta_0^\xi \vee \theta_1^\xi$ and $\mu(\theta_i^\xi) \leq \boldsymbol{a}_i^\xi$, for $i = 0, 1$. In particular, $\Theta_A(s,t) \leq \theta_0^\xi \vee \theta_1^\xi$, for every $t \in T$. Fix $t \in T$. Since $\Theta_A(s,t) \leq \theta_0^\xi \vee \theta_1^\xi$, there are a positive integer n_ξ, and elements $t = z_0^\xi, z_1^\xi, \dots, z_{n_\xi}^\xi = s$ in A, such that

$$\mu\Theta_A(z_i^\xi, z_{i+1}^\xi) \leq \boldsymbol{a}_{\varepsilon(i)}^\xi, \text{ for all } i < n_\xi, \tag{6.3}$$

for all $\xi \in \Omega$. (Recall that $\varepsilon(i) = i \mod 2$.)

If A has a congruence-compatible structure of a $(\vee, 1)$-semilattice, we replace each z_i^ξ with $\bigvee_{j=0}^i z_j^\xi$. If A has a congruence-compatible structure of a lattice, we replace each z_i^ξ with $s \wedge \Big(\bigvee_{j=0}^i z_j^\xi\Big)$. In both cases we obtain an increasing chain $t = z_0^\xi \leq z_1^\xi \leq \dots \leq z_{n_\xi}^\xi = s$ in A such that (6.3) remains satisfied.

As in [17, Section 6], we denote by $\mathrm{Con_c^U}\, A$ the $(\vee, 0)$-subsemilattice of $\mathrm{Con_c}\, A$ generated by all principal congruences $\Theta_A(u,v)$, where $u, v \in U$, for every subset U of A. Further, we denote by $S(F)$ the $(\vee, 0)$-subsemilattice of A generated by $\{z_i^\xi \mid \xi \in F \text{ and } 0 \leq i \leq n_\xi\}$, and we put

$$\Phi(F) = \bigcup\Big\{\mathrm{Supp}(\mu\theta) \mid \theta \in \mathrm{Con_c^{S(F)}}\, A\Big\}, \tag{6.4}$$

for every subset F of Ω. Observe that if the subset F is finite, then both $S(F)$ and $\Phi(F)$ are finite.

Since the size of Ω is at least $\aleph_2$, there are a positive integer n and a subset X of Ω of cardinality at least $\aleph_2$ such that $n_\xi = n$, for all $\xi \in X$. The following crucial claim is analogous to [17, Lemma 6.2], giving another illustration of the "erosion method".

Claim 1. *Let $\mathcal{T} = (\alpha(f) \mid f\colon n \to 3)$ be a free* 3*-tree with respect to Φ with* $\operatorname{rng} \mathcal{T} \subseteq X$*. Then*

$$\Theta_A\Big(s, \bigvee_{f\in T_{n,2}(g)} z_{n-m}^{\alpha(f)}\Big) = 0, \tag{6.5}$$

for every non-negative integer $m \leq n$ and every map $g\colon \{m, \dots, n-1\} \to 2$.

Proof of Claim. We shall argue by induction on m. If $m = 0$, then the equality (6.5) is trivially satisfied, for every map $g\colon \{m, \dots, n-1\} \to 2$. Let $m < n$, let $g\colon \{m+1, \dots, n-1\} \to 2$ be a map, and suppose that (6.5) is satisfied at stage n. Put

$$x_i = \bigvee_{f \in T_{n,2}(g,i)} z_{n-m-1}^{\alpha(f)}, \text{ for all } i < 2.$$

Fix $i < 2$. Clearly,

$$\mu\Theta_A(s, x_i) \leq \Big(\bigvee_{f \in T_{n,2}(g,i)} \mu\Theta_A(z_{n-m-1}^{\alpha(f)}, z_{n-m}^{\alpha(f)}) \Big) \vee \mu\Theta_A\Big(s, \bigvee_{f \in T_{n,2}(g,i)} z_{n-m}^{\alpha(f)}\Big).$$

Put

$$\boldsymbol{v} = \mu\Theta_A\Big(s, \bigvee_{f \in T_{n,2}(g)} z_{n-m-1}^{\alpha(f)}\Big) = \mu\Theta_A(s, x_0 \vee x_1).$$

Put $A_i = \{\alpha(f) \mid f \in T_{n,2}(g,i)\}$, and let $\varphi_i\colon A_i \to 2$ be the constant map with the value $\varepsilon(n-m-1)$. By the induction hypothesis

$$\mu\Theta_A\Big(s, \bigvee_{f \in T_{n,2}(g,i)} z_{n-m}^{\alpha(f)}\Big) = 0,$$

and, by (6.3), $\mu\Theta_A(z_{n-m-1}^{\alpha(f)}, z_{n-m}^{\alpha(f)}) \leq \boldsymbol{a}_{\varepsilon(n-m-1)}^{\alpha(f)}$, for all $f \in T_{n,2}(g,i)$. Thus

$$\mu\Theta_A(s, x_i) \leq \bigvee_{f \in T_{n,2}(g,i)} \boldsymbol{a}_{\varepsilon(n-m-1)}^{\alpha(f)} = \boldsymbol{a}_{\varphi_i}^{A_i}.$$

Let δ be any element of X. By the Erosion Lemma [17, Lemma 5.1], there are $\boldsymbol{u}_j \in \mathrm{Con}_c^{S(A_j \cup \{\delta\})} A$ such that $\boldsymbol{v} \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1$ satisfying $\boldsymbol{u}_j \leq \boldsymbol{a}_{\varepsilon(j)}^{\delta}, \mu\Theta_A(s, x_j)$ (and so $\boldsymbol{u}_j \leq \boldsymbol{a}_{\varepsilon(j)}^{\delta}, \boldsymbol{a}_{\varphi_j}^{A_j}$), for all $j < 2$.

Now let $\delta = \alpha(f)$, for some $f \in T_{n,3}(g,2)$. By (6.4), $\mathrm{Supp}(\boldsymbol{v}) \subseteq \Phi(\{\alpha(f) \mid f \in T_{n,2}(g)\}) = \Phi(A_0 \cup A_1)$ and $\mathrm{Supp}(\boldsymbol{u}_j) \subseteq \Phi(A_j \cup \{\delta\})$, for all $j < 2$. Since $\mathcal{T}$ is a free 3-tree with respect to Φ, $\delta \notin \Phi(A_0 \cup A_1)$ and $A_{1-j} \cap \Phi(A_j \cup \{\delta\}) = \emptyset$, for all $j < 2$. It follows, that $\boldsymbol{v} \in \mathcal{G}(\Omega \setminus \{\delta\})$ and $\boldsymbol{u}_j \in \mathcal{G}(\Omega \setminus A_{1-j})$, for all $j < 2$. Since the functor $\mathcal{F}$ is diluting, applying Lemma 3.3, we conclude that $\boldsymbol{v} = 0$ as desired . □ Claim 1.

Now we finish the proof of Theorem 6.1. By Lemma 5.1, the set X contains the range of a free 3-tree $\mathcal{T} = (\alpha(f) \mid f : n \to 3)$ of height n. By Claim 1,

$$\Theta_A\Big(s, \bigvee_{f \in T_{n,2}(g)} z_{n-m}^{\alpha(f)}\Big) = 0,$$

for every non-negative integer $m \leq n$ and every map $g\colon \{m, \dots, n-1\} \to 2$. For $m = n$ and $g = \emptyset$ we have that

$$\bigvee_{f \in T_{n,2}(g)} z_{n-m}^{\alpha(f)} = \bigvee_{f\colon n \to 2} z_0^{\alpha(f)} = t,$$

whence $\Theta_A(s,t) = 0$ for every $t \in T$. Consequently $\bigvee_{t \in T} \Theta_A(s,t) = 0$, which leads to a contradiction. □

By Corollary 4.2, the functor $\mathcal{R}^\infty$ is diluting and by [17, Lemma 3.6], it satisfies both (6.1) and (6.2). Let us denote by $\mathcal{G}$ the composition $\mathcal{R}^\infty \circ \mathcal{L}$ (It is the same $\mathcal{G}$ as the one considered in [17].) Since $\mathcal{R}^\infty(S)$ is distributive for every $(\vee, 0)$-semilattice S, we obtain the following corollary.

Corollary 6.2. *Let Ω be a set of cardinality at most $\aleph_2$. Then there is no lattice L such that $\mathcal{G}(\Omega)$ is isomorphic to $\mathrm{Con_c}\, L$.*

A. P. Huhn [6] (see also [3, Theorem 13 in Appendix C]) proved that every distributive $(\vee, 0)$-semilattice of size at most $\aleph_1$ is isomorphic to $\mathrm{Con_c}\, L$, for some lattice L. Moreover, the lattice L can be taken sectionally complemented and modular [16, Corollary 5.3] or relatively complemented, locally finite, and with zero [4]. In particular, in all these cases, the lattice L has permutable congruences [1].

Acknowledgment

I thank Miroslav Ploščica, Philippe Toffin, Jiří Tůma, and Friedrich Wehrung for their support during my visit of the University of Caen. Special thanks are due to Friedrich Wehrung who carefully read the manuscript of the paper. His suggestions considerably improved its final version.

References

[1] R. P. Dilworth *The structure of relatively complemented lattices*, Ann. of Math. (2), Ser. **51** (1950), 348–359.
[2] N. Funayama and T. Nakayama, *On the distributivity of a lattice of lattice-congruences*, Proc. Imp. Acad. Tokyo **18** (1942), 553–554.
[3] G. Grätzer, “General Lattice Theory. (Second edition.)”, Birkhäuser Verlag, Basel, 1998. xix + 663 pp.
[4] G. Grätzer, H. Lakser, and F. Wehrung, *Congruence amalgamation of lattices*, Acta Sci. Math. (Szeged) **66** (2000), 339–358.
[5] G. Grätzer and E. T. Schmidt, *On congruence lattices of lattices*, Acta Math. Acad. Sci. Hungar. **13** (1962), 179–185.
[6] A. P. Huhn, *On the representation of distributive algebraic lattices I-III*, Acta Sci. Math. (Szeged) **45** (1983), 139–246; **53** (1989) 3–10, and 11–18.
[7] C. Kuratowski, *Sur une caractérisation des alephs*, Fund. Math. **38**, (1951), 475–50.
[8] M. Ploščica, *Congruence lattices of lattices with m-permutable congruences*, preprint (2007).
[9] M. Ploščica and J. Tůma, *Uniform refinements in distributive semilattices*, Contributions to General Algebra **10**, Proceedings of the Klagenfurt Conference, May 29 - June 10 1997, Verlag Johannes Heyn, Klagenfurt, 1998.
[10] M. Ploščica, J. Tůma, and F. Wehrung, *Congruence lattices of free lattices in non-distributive varieties*, Colloquium Mathematicum **76**, no. 2 (1998), 269–278.
[11] P. Růžička, J. Tůma, and F. Wehrung, *Distributive congruence lattices of congruence-permutable algebras*, J. Algebra 311, no. 1 (2007), 96–116.
[12] J. Tůma and F. Wehrung, *A survey on recent results on congruence lattices of lattices*, Algebra Universalis **45**, no. 4 (2002), 439–471.
[13] J. Tůma and F. Wehrung, *Simultaneous representations of semilattices by lattices with permutable congruences*, Internat. J. Algebra Comput. **11**, no. 2 (2001), 217–246.
[14] F. Wehrung, *Non-measurability properties of interpolation vector spaces*, Israel J. Math. **103** (1998), 177–206.
[15] F. Wehrung, *A uniform refinement property for congruence lattices*, Proc. Amer. Math. Soc. **127** (1999), 363–370.
[16] F. Wehrung, *Representation of algebraic distributive lattices with $\aleph_1$ compact elements as ideal lattices of regular rings*, Publ. Mat. (Barcelona) **44** (2000), 419–435.
[17] F. Wehrung, *A solution to Dilworth’s congruence lattice problem*, Adv. Math 216, no. 2 (2007), 610–625.

Charles University, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail address: `ruzicka@karlin.mff.cuni.cz`