CHARACTERIZATION OF ABELIAN GROUPS WITH A MINIMAL GENERATING SET

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Abstract. We characterize abelian groups with a minimal generating set: Let τA denote the maximal torsion subgroup of A. An infinitely generated abelian group A of cardinality $\varkappa$ has a minimal generating set iff at least one of the following conditions is satisfied:

(1) $\dim(A/pA) = \dim(A/qA) = \varkappa$ for at least two different primes p, q;
(2) $\dim(\tau A/p\tau A) = \varkappa$ for some prime number p;
(3) $\sum\{\dim(A/(pA + B)) \mid \dim(A/(pA + B)) < \varkappa\} = \varkappa$ for every finitely generated subgroup B of A.

Moreover, if the group A is uncountable, property (3) can be simplified to
(3') $\sum\{\dim(A/pA) \mid \dim(A/pA) < \varkappa\} = \varkappa$,
and if the cardinality of the group A has uncountable cofinality, then A has a minimal generating set iff any of properties (1) and (2) is satisfied.

1. Introduction

For the notion of $\mathcal{S}$-independence we refer to [3, pages 26 and 46]. In particular, a subset X of a universal algebra $\mathfrak{A}$ is called $\mathcal{S}$-independent provided that x is not in the subalgebra generated by $X \setminus \{x\}$ for all $x \in X$. A subset X of $\mathfrak{A}$ which is both $\mathcal{S}$-independent and generating is called a minimal generating set of $\mathfrak{A}$. One should notice right away that minimal generating sets correspond to generating sets minimal with respect to inclusion.

In general, unless the algebra $\mathfrak{A}$ is finitely generated, the existence of a minimal generating set is not guaranteed. The question of the existence of a minimal generating set for various concrete algebraic structures, e.g. groups, rings, fields, was studied in [1] and [4]. A deeper insight into this problem restricted to abelian groups is in [6]. There the question of the existence of a minimal generating set is decided for torsion and partially for torsion free abelian groups, and some nontrivial examples are shown.

More should be said about [6]. The first author of this paper noticed that [6, Theorem 3.1], [6, Lemma 5.3] (and consequently [6, Theorem 5.5]) are not correct. In [6, Theorem 3.1] we have to restrict to torsion abelian groups while in [6, Lemma 5.3] some additional assumptions need to be added. In particular, there should not be a single prime p such that all but finitely many torsion-free groups of rank 1 in the direct sum decomposition are divisible by all primes $q \neq p$. In both cases we wrongly applied [6, Proposition 1.5] as we assumed that if D is a divisible abelian

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group and A is an abelian group with a minimal generating set, say X, such that $\operatorname{gen}(A) \geq \operatorname{gen}(D)$, then the direct sum $A \oplus D$ has a minimal generating set which lifts X over D (following our terminology below). This holds only under some additional assumptions (see Corollary 3.12), e.g., if the group A is torsion.

In this paper we correct both the results and we extend [6] to reach at the end the complete description of abelian groups with a minimal generating set. Let us outline the structure of the paper. This introduction is followed by Basic concept where we sum up necessary definitions and terminology. In Section 3 we reprove and generalize [6, Proposition 1.5] and [6, Lemma 2.1] which are main tools in [6] and prove some other general statements which will be applied in the sequel. Most of the time we restrict ourself to abelian groups, even though we believe that some of these results permit further generalizations for modules over a ring. In Section 4 we redo the characterization of torsion abelian groups with a minimal generating set. The last two sections deal with torsion-free and general abelian groups, respectively.

2. Basic concept

By $\operatorname{card}(X)$ we denote the cardinality of a set X. By $\operatorname{cf}(\varkappa)$ we denote the cofinality of a cardinal $\varkappa$. Given a collection $\{\varkappa_\iota \mid \iota < I\}$ of cardinals, we denote by $\sum_{\iota \in I} \varkappa_\iota$ the cardinality of its disjoint union. Given an ordinal σ and maps $f, g\colon \sigma \to \sigma$ we use $g \leq f$ to denote that $g(\alpha) \leq f(\alpha)$ for every $\alpha < \sigma$.

We denote by ω the first infinite ordinal, that is $\omega = \{0, 1, 2, \dots\}$. Finite ordinals will also be called natural numbers, in particular, 0 is a natural number. We will unify each ordinal with the set of its predecessors, e.g., $2 = \{0, 1\}$. By $\mathbb{P}$ we denote the set of all prime numbers and we put $\hat{p} = \mathbb{P} \setminus \{p\}$ for all $p \in \mathbb{P}$.

Let X be a set and let $f\colon X \to Y$ be a map. Given $V \subseteq Y$, we denote by $[V]f^{-1}$ the f-preimage of V, i.e., $[V]f^{-1} = \{x \in X \mid f(x) \in V\}$. Similarly, given $U \subseteq X$, we denote by $f[U]$ the f-image of U, i.e., $f[U] = \{f(u) \mid u \in U\}$.

Let A be an abelian group. We denote by τA the maximal torsion subgroup of A and by ϕA the torsion free quotient $A/\tau A$. Given a prime number p, we denote by $\tau_p A$ the p-primary component of τA and we set $\tau_{\hat{p}} A = \bigoplus_{q \in \hat{p}} \tau_q A$. Similarly, given a domain R and an R-module A we denote by τA the maximal torsion submodule of A and by ϕA the torsion free quotient $A/\tau A$.

All rings are supposed to be commutative. Given a ring R, an R-module A, and a subset $X \subseteq A$, we denote by $\operatorname{Span}(X)$ the submodule generated by the set X. Further, we denote by $\pi_X\colon A \to A/\operatorname{Span}(X)$ the canonical projection (sending $a \mapsto a + \operatorname{Span}(X)$ for all $a \in A$). We will use the notation $B \leq A$ to say that B is a submodule of A. By $\operatorname{gen}(A)$ we denote the minimal cardinality of a generating set of A.

Let R a ring an let I be an ideal of R. An R-module A is said to be *I-divisible* provided that $IA = A$. In particular, given $a \in \mathbb{Z}$, an abelian group A is said to be *a-divisible* provided that $aA = A$.

3. General principles

Most of the results in [6] claiming the existence of a minimal generating set in a certain class of abelian groups are based on two statements, namely [6, Proposition 1.5] and [6, Lemma 2.1]. In this section we closely examine and generalize them (although we restrict ourselves to abelian groups). In particular we show that both these results are based on the same simple property of linear maps (Corollary 3.2 below). On top of that we prove some of their consequences which will be applied in the rest of the paper.

We start with a nearly trivial observation. We leave its proof as an exercise.

Lemma 3.1. *Let R be a ring, let A, B be R-modules, let C, D be submodules of the module A, and let γ, $\delta \in \hom_R(A, B)$. Let $D \leq \ker\delta$ and $C \leq \ker\gamma$. Then $(\gamma+\delta)[C+D] = \gamma[D] + \delta[C]$.*

Corollary 3.2. *Let R be a ring, let A, B be R-modules, and let γ, $\delta \in \hom_R(A, B)$. If $\ker\gamma + \ker\delta = A$, then $(\gamma+\delta)[A] = \gamma[A] + \delta[A]$.*

Proof. By the assumption $\ker\gamma + \ker\delta = A$, hence $\gamma[A] = \gamma[\ker\delta]$ and $\delta[A] = \delta[\ker\gamma]$. Applying Lemma 3.1 we get that $(\gamma+\delta)[A] = (\gamma+\delta)[\ker\gamma + \ker\delta] = \gamma[\ker\delta] + \delta[\ker\gamma] = \gamma[A] + \delta[A]$. □

Definition. Let R be a ring, let A, B be R-modules, and let $\alpha\colon A \to B$ be an R-linear map. Let X, resp. Y be a subset of A, resp. B. We say that X is *$\mathcal{S}$-independent over Y via α* provided that $x \notin \operatorname{Span}(X \setminus \{x\}) + [\operatorname{Span} Y]\alpha^{-1}$ for all $x \in X$. We say shortly that X is *$\mathcal{S}$-independent over Y* in case α is the identity map. Similarly, we say that *X is $\mathcal{S}$-independent (via α)* provided that $Y = \emptyset$.

Let Z be a subset of $B/\operatorname{Span}(Y)$. We say *X lifts Z over Y via α* provided that $\pi_Y \circ \alpha[X] = Z$ and the restriction $\pi_Y \circ \alpha \restriction X$ is one-to-one. We say shortly that *X lifts Z over Y* provided that α is the identity map and we say that *X lifts Z (over α)* if $Y = \emptyset$.

Let $X' \subseteq A$. We say that X and X' are *$\mathcal{S}$-equivalent via α over Y* provided that both the sets are $\mathcal{S}$-independent via α over Y and $(\pi_Y \circ \alpha)[X] = (\pi_Y \circ \alpha)[X']$ (i.e., they lift the same set via α over Y). We say simply that X and X' are *$\mathcal{S}$-equivalent over Y* provided that the map α is the identity map.

Lemma 3.3. *Let R be a ring, let A, B be R-modules and let $\alpha\colon A \to B$ be an R-linear map. Let $\{X_i \mid i \in I\}$ be a collection of subsets of A. If X_j is $\mathcal{S}$-independent over $\alpha[\bigcup_{i\neq j} X_i]$ for all $j \in I$, then $\bigcup_{i\in I} X_i$ is $\mathcal{S}$-independent via α.*

Proof. Let $j \in I$ and let $x \in X_j$. According to our assumption, we have that

$$\begin{aligned} x &\notin \operatorname{Span}(X_j \setminus \{x\}) + [\operatorname{Span}(\alpha[\bigcup_{i\neq j} X_i])]\alpha^{-1} \\ &= \operatorname{Span}(X_j \setminus \{x\}) + \operatorname{Span}(\bigcup_{i\neq j} X_i) + \ker\alpha \\ &\supseteq \operatorname{Span}(\bigcup_{i\in I} X_i \setminus \{x\}) + \ker\alpha. \end{aligned}$$

This means that $\bigcup_{i\in I} X_i$ is $\mathcal{S}$-independent via α. □

We are going to derive a couple of corollaries of this simple lemma, not in full generality but in formulations allowing direct applications.

Corollary 3.4. *Let A be an abelian group, let p_0, p_1 be a couple of different primes and let X_i, for $i \in 2$, be a subset of $p_{1-i}A$ which lifts a basis of A/p_iA over p_iA. Then $X_0 \cup X_1$ is an $\mathcal{S}$-independent subset of A.*

Proof. Since X_i lifts a basis of A/p_iA over p_iA and $X_i \subseteq p_{1-i}A$ we see that the set X_i is $\mathcal{S}$-independent over X_{1-i} for all $i \in 2$. By Lemma 3.3 we conclude that the set $X_0 \cup X_1$ is $\mathcal{S}$-independent. $\square$

Corollary 3.5. *Let A be an abelian group. Let $\{p_i \mid i \in \omega\}$ be a set of primes and $\{X_i \mid i \in \omega\}$ a set of subsets of A such that for all $j \in \omega$, $X_j \subseteq q_jA$, where $q_j = p_0 \cdots p_{j-1}$ and it lifts a basis of $A/(B_j + p_jA)$ over $B_j + p_jA$, where $B_j = \operatorname{Span}(\bigcup_{i<j} X_i)$. Then the set $X = \bigcup_{i\in\omega} X_i$ is $\mathcal{S}$-independent.*

Proof. Fix $j \in \omega$. Observe $X_k \subseteq q_kA \subseteq p_jA$ for all $k > j$. It follows that $\bigcup_{i\neq j} X_i$ is contained in $B_j + p_jA$ and so X_j is $\mathcal{S}$-independent over $\bigcup_{i\neq j} X_i$. Applying Lemma 3.3, we conclude that the set $X = \bigcup_{i\in\omega} X_i$ is $\mathcal{S}$-independent. $\square$

From now on, with a few exceptions, we restrict ourselves to abelian groups aiming to complete the characterization of abelian groups with a minimal generating sets. However, we believe that most of the results obtained on the way can be generalized for wider classes of modules over commutative or even non-commutative rings.

Lemma 3.6. *Let A be an abelian group, let Y be a subset of A and let p_0, p_1 be a couple of different primes such that* $\operatorname{card}(Y) \leq \dim(A/p_iA)$ *for all $i \in 2$. Then there are subsets $X_i \subseteq p_{1-i}A$, $i \in 2$, such that X_i lifts a basis of A/p_iA over p_iA for all $i \in 2$ and $Y \subseteq \operatorname{Span}(X_0 \cup X_1)$.*

Proof. We can without loss of generality assume that $\dim(A/p_0A) \leq \dim(A/p_1A)$.

Claim 1. *There are subsets $Z_0 \subseteq Z_1$ of A such that Z_i lifts a basis of A/p_iA over p_iA for all $i \in 2$ and $Z_1 \setminus Z_0 \subseteq p_0A$.*

Proof of Claim 1. Put $\lambda_i = \dim(A/p_iA)$, $i \in 2$, and let $\{y_{\iota,i} \mid \iota < \lambda_i\}$ lift a basis of A/p_iA for all $i = 0, 1$. For all $\iota < \lambda_1$ put

$$z_\iota = \begin{cases} p_1 y_{\iota,0} + p_0 y_{\iota,1} & \text{if } \iota < \lambda_0, \\ p_0 y_{\iota,1} & \text{if } \lambda_0 \leq \iota < \lambda_1. \end{cases}$$

It is straightforward to verify that the sets $Z_i = \{z_\iota \mid \iota < \lambda_i\}$, $i \in 2$, satisfy the desired properties. $\square_{\text{Claim 1}}$

Since Z_i lifts a basis of A/p_iA, we have that $A = \operatorname{Span}(Z_i) + p_iA$ for all $i \in 2$. Since $Z_0 \subseteq Z_1$, we conclude that $A = \operatorname{Span}(Z_1) + p_0p_1A$. Thus there is a map $f\colon Y \to p_0p_1A$ such that $y \in \operatorname{Span}(Z_1) + f(y)$ for all $y \in Y$. Since $\operatorname{card}(Y) \leq \dim(A/p_0A) = \operatorname{card}(Z_0)$, there is a projection $g\colon Z_0 \to Y$. Put $h = f \circ g$ and observe that $h\colon Z_0 \to p_0p_1A$ is a map such that $Y \subseteq \operatorname{Span}(Z_1) + h[Z_0]$.

Let F denote a free abelian group with a basis $Z_0 \times 2$. Note that, since p_0, p_1 are different primes, $a_0p_0 + a_1p_1 = 1$ for some integers a_0, a_1. Let $\gamma'\colon Z_0 \times 2 \to A$ be a map defined by the correspondence $(z, i) \mapsto a_{1-i}p_{1-i}z$. Let $\gamma\colon F \to A$ be a unique extension of γ' to a group homomorphism. Observe that $\gamma[F] = \operatorname{Span}(Z_0)$ and $\ker\gamma \supseteq \{a_0p_0(z,0) - a_1p_1(z,1) \mid z \in Z_0\}$.

Let $\delta'\colon Z_0 \times 2 \to p_0p_1A$ be a map defined by the correspondence $(z, i) \mapsto (-1)^i h(z))$. Let $\delta\colon F \to A$ be its unique extension to a group homomorphism. Observe that $\delta[F] = \operatorname{Span}(h[Z_0])$ and $\ker\delta \subseteq \{(z,0) + (z,1) \mid z \in Z_0\}$.

Claim 2. *The equality $F = \ker\gamma + \ker\delta$ holds.*

Proof of Claim 2. Let $z \in Z_0$. Since $a_0p_0 + a_1p_1 = 1$, we have that $(z,0) = (a_0p_0 + a_1p_1)(z,0) = (a_0p_0(z,0) - a_1p_1(z,1)) + a_1p_1((z,0)+(z,1)) \in \ker\gamma + \ker\delta$. Now $(z,1) = ((z,1)+(z,0))-(z,0) \in \ker\delta + (\ker\gamma + \ker\delta) = \ker\gamma + \ker\delta$. $\square_{\text{Claim 2}}$

Put $\beta = \gamma + \delta$. By Corollary 3.2 and Claim 2 we have that $\beta[F] = \operatorname{Span}(Z_0) + \operatorname{Span}(h[Z_0])$. Put $X_0 = \beta[Z_0 \times \{0\}]$ and $X_1 = \beta[Z_0 \times \{1\}] \cup (Z_1 \setminus Z_0)$. Then $\operatorname{Span}(X_0 \cup X_1) = \operatorname{Span}(\beta[Z_0 \times 2]) + \operatorname{Span}(Z_1 \setminus Z_0) = \beta[\operatorname{Span}(Z_0 \times 2)] + \operatorname{Span}(Z_1 \setminus Z_0) = \beta[F] + \operatorname{Span}(Z_1 \setminus Z_0) = (\operatorname{Span}(Z_0) + \operatorname{Span}(h[Z_0])) + \operatorname{Span}(Z_1 \setminus Z_0) = (\operatorname{Span}(Z_0) + \operatorname{Span}(Z_1 \setminus Z_0)) + \operatorname{Span}(h[Z_0]) = \operatorname{Span}(Z_1) + \operatorname{Span}(h[Z_0]) \supseteq Y$.

It is clear from the definition of the maps γ, δ that $\gamma[Z_0 \times \{i\}] \subseteq p_{1-i}A$, for all $i \in 2$, and $\delta[F] \subseteq p_0p_1A$. Thus $\beta[Z_0 \times \{i\}] \subseteq p_{1-i}A$ for all $i \in 2$. It follows that $X_0 \subseteq p_1A$. Since by Claim 1 we have that $Z_1 \setminus Z_0 \subseteq p_0A$, we infer that also $X_1 \subseteq p_0A$.

Let $z \in Z_0$ and $i \in 2$. Then $\beta((z,i)) = \gamma((z,i)) + \delta((z,i)) = a_{1-i}p_{1-i}z + (-1)^i h(z) = (a_ip_i + a_{1-i}p_{1-i})z - a_ip_iz + (-1)^i h(z) = z + (a_ip_iz + (-1)^i h(z)) \in z + p_iA$. Thus $X_0 = \beta[Z_0 \times \{0\}]$ is $\mathcal{S}$-equivalent to Z_0 over p_0A and $X_1 = \beta[Z_0 \times \{1\}] \cup (Z_1 \setminus Z_0)$ is $\mathcal{S}$-equivalent to Z_1 over p_1A. In particular, X_i lifts a basis of A/p_iA over p_iA for all $i \in 2$. $\square$

Notation. Let A be an abelian group. We will use the following notation:

$$\operatorname{Spec}(A,<) = \{p \in \mathbb{P} \mid 0 < \dim(A/pA) < \operatorname{gen}(A)\},$$
$$\operatorname{Spec}(A,=) = \{q \in \mathbb{P} \mid \dim(A/qA) = \operatorname{gen}(A)\}.$$

Proposition 3.7 ([6, Lemma 2.1])**.** *Let A be an abelian group. If*

$$\operatorname{card}(\operatorname{Spec}(A,=)) \geq 2,$$

then A has a minimal generating set.

Proof. Let p_i, $i \in 2$, be a couple of different primes from $\operatorname{Spec}(A,=)$ and let Y be a generating set of A of a minimal cardinality. Then $\operatorname{card}(Y) \leq \dim(A/p_iA)$ for all $i \in 2$ and, applying Lemma 3.6, we get $X_i \subseteq A$ such that $X_i \subseteq p_{1-i}A$, X_i lifts a basis of A/p_iA over p_iA, for all $i \in 2$, and $Y \subseteq \operatorname{Span}(X_0 \cup X_1)$. Since $A = \operatorname{Span} Y$, we conclude that $A = \operatorname{Span}(X_0 \cup X_1)$. It follows from Corollary 3.4 that the set $X_0 \cup X_1$ is $\mathcal{S}$-independent. $\square$

Proposition 3.8. *Let A be a countable abelian group. If* $\operatorname{Spec}(A/B,<) \neq \emptyset$ *for every finitely generated subgroup B of A, then A has a minimal generating set.*

Proof. Let $Y = \{y_i \mid i \in \omega\}$ be a generating set of A (note that since A is countable, we can as well put $Y = A$).

Claim 1. *There is a sequence $p_0, p_1, p_2, \ldots$ of primes and a sequence $X_0, X_1, X_2, \ldots$ of finite sets such that, putting $q_0 = 1$, $q_j = \prod_{i=0}^{j-1} p_i$, and $B_j = \operatorname{Span}(\bigcup_{i=0}^{j-1} X_i)$, the following properties are satisfied for all $j \in \omega$: $X_j \subseteq q_jA$, X_j lifts a basis of $A/(p_jA + B_j)$, and $y_j \in B_{2(j+1)}$.*

Proof of Claim 1. First observe that the assumptions of Proposition 3.8 imply that $\operatorname{Spec}(A/B,<)$ is infinite for every finitely generated subgroup B of A. Indeed, since the abelian group A is countable, there is a finitely generated subgroup B_p of A with $p(A/(B+B_p)) = A/(B+B_p)$, for every prime $p \in \operatorname{Spec}(A/B,<)$. Thus if $\operatorname{Spec}(A/B,<)$ was finite, $B' = B + \sum_{p \in \operatorname{Spec}(A/B,<)} B_p$ would have been a finitely generated subgroup of A with $\operatorname{Spec}(A/B',<) = \emptyset$. (Note that $\operatorname{Spec}(A,=) =$

$\operatorname{Spec}(A/B'', =)$ for every finitely generated subgroup B'' of A.) This contradicts our assumptions.

Let p_0, p_1 be two different primes from $\operatorname{Spec}(A, <)$. By Lemma 3.6 there are subsets X_i, $i \in 2$, such that $X_i \subseteq p_{1-i}A$, X_i lifts a basis of A/p_iA over p_iA, for all $i \in 2$ and $y_0 \in \operatorname{Span}(X_0 \cup X_1)$. Since $p_i \in \operatorname{Spec}(A, <)$, $\dim(A/p_iA) < \aleph_0$, for all $i \in \omega$, and so both the sets X_0, X_1 are finite. Since $q_0 = 1$, we have that $X_0 \subseteq q_0A$. Since $q_1 = p_0$ and $X_1 \subseteq p_0A$, we have that $X_1 \subseteq q_1A$. Since $B_0 = \operatorname{Span}(\emptyset) = 0$, X_0 lifts a basis of $A/(p_0A + B_0)$ over $p_0A + B_0 = p_0A$. Since $X_0 \subseteq p_1A$, $B_1 = \operatorname{Span}(X_0) \subseteq p_1A$, and so $p_1A + B_1 = p_1A$. Thus X_1 lifts a basis of $A/(p_1A + B_1)$ over $p_1A + B_1$. Since $B_2 = \operatorname{Span}(X_0 \cup X_1)$, $y_0 \in B_2$.

Let $j \in \omega$ and suppose that we have already picked prime numbers $p_0, \dots, p_{2j-1}$ and constructed finite sets $X_0, \dots, X_{2j-1}$ satisfying the desired properties. Put $A' = A/B_{2j}$ and $y'_j = y_j + B_{2j} \in A'$. Observe that $A = p_iA + B_{2j}$ for every $i \in 2j$. It follows that $A = q_{2j}A + B_{2j}$, whence $A' = q_{2j}A'$. Since the constructed sets are finite, the group B_{2j} is finitely generated, hence, by our assumption, $\operatorname{Spec}(A', <) \neq \emptyset$. As noted above, this set is in fact infinite. Pick a couple p_{2j}, p_{2j+1} of different primes from $\operatorname{Spec}(A', <)$. By Lemma 3.6, there are sets X'_i, $i \in 2$, such that $X'_i \subseteq p_{2j+1-i}A'$, X'_i lifts a basis of $A'/p_{2j+i}A'$ over $p_{2j+i}A'$, for all $i \in 2$, and $y'_j \in \operatorname{Span}(X'_0 \cup X'_1)$. Since $A = q_{2j}A + B_{2j}$, there is $X_{2j+i} \subseteq p_{2j+1-i}q_{2j}A$ lifting the set X'_i over B_{2j}, for all $i \in 2$. Clearly $X_{2j} \subseteq q_{2j}A$ and, since X'_0 lifts a basis of $A'/p_{2j}A'$ over $p_{2j}A'$, X_{2j} lifts a basis of $A/(p_{2j}A + B_{2j})$ over $p_{2j}A + B_{2j}$. Since $q_{2j+1} = p_{2j}q_{2j}$, $X_{2j+1} \subseteq q_{2j+1}A$ and since X'_1 lifts a basis of $A'/p_{2j+1}A'$ over $p_{2j+1}A'$, we get that X_{2j+1} lifts a basis of $A/(p_{2j+1}A + B_{2j})$ over $p_{2j+1}A + B_{2j}$. Since $X_{2j} \subseteq p_{2j+1}A$, we have that $B_{2j+1} = \operatorname{Span}(X_{2j}) + B_{2j} \subseteq p_{2j+1}A + B_{2j}$. We conclude that X_{2j+1} lifts a basis of $A/(p_{2j+1}A + B_{2j+1})$ over $p_{2j+1}A + B_{2j+1}$. Finally, since $y'_j \in \operatorname{Span}(X'_0 \cup X'_1)$, $y_j \in \operatorname{Span}(X_{2j} \cup X_{2j+1}) + B_{2j} = B_{2(j+1)}$. $\square_{\text{Claim 1}}$

Put $X = \bigcup_{j \in \omega} X_j$ and observe that the set X is $\mathcal{S}$-independent by Corollary 3.5. Since $y_j \in B_{2j} \subseteq \operatorname{Span}(X)$ for all $j \in \omega$, $Y \subseteq \operatorname{Span}(X)$, and so X generates A. It follows that X is a minimal generating set of A. $\square$

Proposition 3.10 below guarantee the existence of a minimal generating set in an uncountable abelian group under hypothesis similar to those of Proposition 3.8. In order to demonstrate closer similarity of both the statements we reformulate Proposition 3.8 as follows:

Corollary 3.9. *Let A be a countable abelian group. If for every finitely generated subgroup B of A*

$$\sum_{p \in \operatorname{Spec}(A,<)} \dim(A/(pA + B)) = \operatorname{gen}(A),$$

then A has a minimal generating set.

Now the promised proposition:

Proposition 3.10. *Let A be an uncountable abelian group. Suppose that*

$$\sum_{p \in \operatorname{Spec}(A,<)} \dim(A/pA) = \operatorname{gen}(A). \tag{3.1}$$

Then A has a minimal generating set.

Proof. Put $\varkappa = \mathrm{gen}(A)$. Observe that the equality (3.1) can be satisfied only if $\mathrm{cf}(\varkappa) = \aleph_0$. In this case there is a sequence $p_0, p_1, p_2, \dots$ of primes from $\mathrm{Spec}(A, <)$ such that, having denoted $\lambda_i = \dim(A/p_i A)$, we get an increasing sequence $\aleph_0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ of cardinals whose supremum is $\varkappa$. Similarly as in Proposition 3.8 set $q_0 = 1$ and $q_{j+1} = q_j p_j$ for all $j \in \omega$. Let $Y = \{y_\iota \mid \iota < \varkappa\}$ be a generating set of A. For each $i \in \omega$ put $Y_i = \{y_\iota \mid \iota < \lambda_i\}$.

From now on the proof closely follows the proof of Proposition 3.8.

Claim 1. *There is a collection $\{X_j \mid j \in \omega\}$ of subsets of A such that $X_j \subseteq q_j A$, for all $j \in \omega$ and, putting $B_j = \mathrm{Span}(\bigcup_{i=0}^{j-1} X_i)$, the set X_j lifts a basis of $A/(p_j A + B_j)$ over $p_j A + B_j$ and $Y_j \subseteq B_{2(j+1)}$, for all $j \in \omega$.*

Proof of Claim 1. Repeatedly applying Lemma 3.6, we will construct inductively the sets X_j, $j \in \omega$, adding a couple of them in each step. By Lemma 3.6, there are subsets $X_0 \subseteq p_1 A$ and $X_1 \subseteq p_0 A$ such that X_i lifts a basis of $A/p_i A$ over $p_i A$ for all $i \in 2$ and $Y_0 \subseteq B_2 = \mathrm{Span}(X_0 \cup X_1)$. Notice that $X_1 \subseteq p_0 A = q_1 A$ and $B_1 = \mathrm{Span}(X_0) \subseteq p_1 A$, hence $p_1 A + B_1 = p_1 A$. Thus the sets X_0, X_1 satisfy the required properties.

Let $1 \le j \in \omega$ and suppose that we have constructed sets $X_0, \dots, X_{2j-1}$ so that the required properties are satisfied. Put $Y' = \{y_\iota + B_{2j} \mid \iota < \lambda_{2j}\}$ and $A' = A/B_{2j}$. Since $A = p_i A + B_{2j}$ for each $i \in 2j$, we have $A = q_{2j} A + B_{2j}$ and thus $A' = q_{2j} A'$. Observe that $\dim(A'/p_k A') = \lambda_k$, for all $2j \le k \in \omega$. Indeed, $\dim(A'/p_k A') \le \dim(A/p_k A) \le \dim(A'/p_k A') + \mathrm{gen}(B_{2j})$, and, since $\mathrm{gen}(B_{2j}) \le \sum_{i=0}^{2j-1} \mathrm{card}(X_i) \le \sum_{i=0}^{2j-1} \lambda_i < \lambda_{2j}$, we get that $\lambda_k = \dim(A/p_k A) = \dim(A'/p_k A')$.

By Lemma 3.6 there are $X_i' \subseteq p_{2j+1-i} A'$ such that X_i' lifts a basis of $A'/p_{2j+i} A'$ over $p_{2j+i} A'$, for all $i \in 2$, and $Y' \subseteq \mathrm{Span}(X_0' \cup X_1')$. Since $A' = q_{2j} A'$, there are $X_{2j+i} \subseteq p_{2j+1-i} q_{2j} A$ lifting X_i' over B_{2j}, for $i \in 2$. Clearly $X_{2j} \subseteq q_{2j} A$ and $X_{2j+1} \subseteq p_{2j} q_{2j} A = q_{2j+1} A$. Since X_0' lifts a basis of $A'/p_{2j} A'$ over $p_{2j} A'$, X_{2j} lifts a basis of $A/(p_{2j} A + B_{2j})$ over $p_{2j} A + B_{2j}$. Since X_1' lifts a basis of $A'/p_{2j+1} A'$ over $p_{2j+1} A'$, X_{2j+1} lifts a basis of $A/(p_{2j+1} A + B_{2j})$ over $p_{2j+1} A + B_{2j}$. Since $X_{2j} \subseteq p_{2j+1} A$, it follows that X_{2j+1} lifts a basis of $A/(p_{2j+1} A + B_{2j+1})$ over $p_{2j+1} A + B_{2j+1}$. Finally, since $Y' \subseteq \mathrm{Span}(X_0' \cup X_1')$, we conclude that $Y_j \subseteq \mathrm{Span}(X_{2j} \cup X_{2j+1}) + B_{2j} = B_{2(j+1)}$. $\square_{\text{Claim 1}}$

Set $X = \bigcup_{i \in \omega} X_i$. It follows from Claim 1 that $Y_j \subseteq \mathrm{Span}(X)$ for all $j \in \omega$, hence $Y \subseteq \mathrm{Span}(X)$, whence $\mathrm{Span}(X) = A$. The set X is $\mathcal{S}$-independent by Corollary 3.5. It follows that X is a minimal generating set of A. $\square$

Lemma 3.11 ([6, Proposition 1.5])**.** *Let A be an abelian group with a minimal generating set and let B be a group such that* $\mathrm{Ext}(A, B) = 0$. *If* $\mathrm{gen}(A) \ge \mathrm{gen}(B)$, *then $A \oplus B$ has a minimal generating set.*

Proof. It is clear if A is finitely generated. Suppose otherwise and let X be a minimal generating set of A. Let F be a free abelian group with a basis X and let $\gamma\colon F \to A$ be a unique projection extending the identity map on X. Put $K = \ker \gamma$. Since a subgroup of a free abelian group is free [5, Theorem 10.18], K is free. We will consider two cases. First suppose that $\mathrm{gen}(K) < \mathrm{gen}(A) = \mathrm{card}(X)$ and pick any prime p. Then $\dim(A/pA) = \dim(F/pF + K) = \mathrm{card}(X) = \mathrm{gen}(A)$, indeed, $\mathrm{gen}(K) < \mathrm{card}(X)$. It follows that A satisfies the assumptions of Proposition 3.7, and so it has a minimal generating set. It remains to assume that $\mathrm{gen}(K) = \mathrm{gen}(A)$. Since $\mathrm{gen}(A) \ge \mathrm{gen}(B)$ and K is free, there is a projection $\delta'\colon K \to B$. Since

$\operatorname{Ext}(A,B)=0$, the projection δ' extends to a homomorphism $\delta\colon F\to B$. Since $\delta[K]=\delta[F]$, $K+\ker(\delta)=F$. Since $K=\ker(\gamma)$, we conclude that $F=\ker(\gamma)+\ker(\delta)$. By Corollary 3.2 $(\gamma+\delta)[F]=\gamma[F]+\delta[F]=A\oplus B$. Since $F=\operatorname{Span}(X)$, we get that $\operatorname{Span}((\gamma+\delta)[X])=A\oplus B$. Since X (as a basis of F) is $\mathcal{S}$-independent via γ, it is $\mathcal{S}$-independent via $\gamma+\delta$ over B (note that $[B](\gamma+\delta)^{-1}=K$), and so it is $\mathcal{S}$-independent via $\gamma+\delta$. It follows that $(\gamma+\delta)[X]$ is a minimal generating set of $A\oplus B$.

□

Note that in the second case the minimal generating set lifts X over B. Some obvious properties of A would guarantee that $\operatorname{gen}(K)=\operatorname{gen}(A)$.

Corollary 3.12. *Let A be an abelian group with an infinite minimal generating set X and let B be an such that $\operatorname{gen}(B)\leq\operatorname{card}(X)$ and $\operatorname{Ext}(A,B)=0$. Suppose that either $\operatorname{card}(\tau A)=\operatorname{card}(X)$ or A has no free direct summand of rank equal $\operatorname{card}(X)$. Then $A\oplus B$ has a minimal generating set which lifts X over B.*

4. Torsion groups

Starting with torsion abelian groups, we correct [6, Theorem 3.1], formulated in [6] for all uncountable abelian groups. The theorem holds under the additional assumption that the group in question is torsion but it is not true in general as follows from Corollary 5.3. Then we rephrase the characterization of countable torsion abelian groups with a minimal generating set (see [6, Lemma 4.3]) and combining the countable and uncountable cases we reformulate properties characterizing torsion abelian groups with a minimal generating set [6, Theorem 4.4].

Lemma 4.1. *Let A be an infinitely generated abelian group. If*

$$\sum_{p\in\mathbb{P}}\dim(A/pA)<\operatorname{card}(A),$$

then A has not a minimal generating set.

Proof. Note that since the group A is infinitely generated, $\operatorname{card}(A)=\operatorname{gen}(A)$. For every prime number p there is a subset X_p of A such that $\operatorname{card}(X_p)=\dim(A/pA)$ and $A=\operatorname{Span}(X_p)+pA$. Put $B=\operatorname{Span}(\bigcup_{p\in\mathbb{P}}X_p)$. Since $\operatorname{gen}(B)\leq\sum_{p\in\mathbb{P}}\dim A/pA<\operatorname{gen}(A)$ and $\operatorname{gen}(A)$ is infinite, $\operatorname{gen}(B)<\operatorname{gen}(A)$. It follows that $\operatorname{gen}(B)<\operatorname{gen}(A/B)$ and since $p(A/B)=A/B$ for every prime number p, the abelian group A/B is divisible. Then A has not a minimal generating set by [6, Lemma 1.3]. □

Lemma 4.2. *Let $\lambda\leq\varkappa$ be infinite cardinals and let X be a subset of $\varkappa$ of cardinality $\varkappa$. Then there is a map $h\colon X\to\varkappa$ such that $h(\xi)\leq\xi$ and $\operatorname{card}([\{\xi\}]h^{-1})=\lambda$ for all $\xi\in X$.*

Proof. Let $\{X_\alpha\mid\alpha<\lambda\}$ be a partition of the set X into λ pairwise disjoint subsets of cardinality $\varkappa$. Let $X_\alpha=\{x_{\alpha,\xi}\mid\xi<\varkappa\}$, where $x_{\alpha,\beta}<x_{\alpha,\gamma}$ whenever $\beta<\gamma<\varkappa$, for all $\alpha<\lambda$. By induction we easily prove that $\xi\leq x_{\alpha,\xi}$ for all $\alpha<\lambda$ and $\xi<\varkappa$. Defining $h(x_{\alpha,\xi})=\xi$ for all $\alpha<\lambda$ and all $\xi<\varkappa$ we get the map h with the desired properties. □

Lemma 4.3. *Let $\lambda\leq\varkappa$ be infinite cardinals, let Y be a set of cardinality $\varkappa$, and let $f\colon Y\to\varkappa$ be a map such that $\operatorname{card}([\{\xi\}]f^{-1})\leq\lambda$ for every $\xi<\varkappa$. Then there is a map $g\colon Y\to\varkappa$ such that $g\leq f$ and $\operatorname{card}([\{\xi\}]g^{-1})=\lambda$ for all $\xi<\varkappa$.*

Proof. Put $X = f[Y]$, let $h\colon X \to \varkappa$ be a map from Lemma 4.2, and set $g = h \circ f$. The desired properties of the map g follows readily from the properties of h. $\square$

Lemma 4.4. *Let A be an uncountable abelian group. If $\dim(\tau A/p\tau A) = \operatorname{gen}(A)$ for some prime number p, then A has a minimal generating set.*

Proof. Let $\varkappa$ denote the cardinality of A. Let U^τ be a subset of τA lifting a basis of $\tau A/p\tau A$ over $p\tau A$. Since τA is pure subgroup of A (e.g., see [2, p. 77 C)]), $p\tau A = \tau A \cap pA$. It follows that U^τ lifts a linearly independent subset of A/pA over pA. Let $U^\tau \subseteq U$, where U lifts a basis of A/pA over pA. Let $U = \{u_\iota \mid \iota \in \varkappa\}$ be some ordering of the set U. Put $I^\tau = \{\iota \in \varkappa \mid u_\iota \in U^\tau\}$ and observe that $\operatorname{card}(I^\tau) = \varkappa$. Find pairwise disjoint subsets I_q, $q \in \hat{p}$, of $\varkappa$ such that $\operatorname{card}(I_q) = \dim(A/qA)$ for all $q \in \hat{p}$ and $\operatorname{card}(I^*) = \varkappa$, where we set $I^* = I^\tau \setminus \bigcup_{q\in\hat{p}} I_q$. For each $q \in \hat{p}$ pick a subset $Y_q = \{a_\iota \mid \iota \in I_q\}$ of A such that $A = \operatorname{Span}(Y_q) + qA$. Put $v_\iota = pa_\iota + qu_\iota$ for all $\iota \in I_q$ and all $q \in \hat{p}$, and $v_\iota = u_\iota$ for all $\iota \in \varkappa \setminus I_q$. Set $V = \{v_\iota \mid \iota \in \varkappa\}$ and $B = \operatorname{Span}(V)$. Observe that V lifts a basis of A/pA over pA and $A = B + qA$ for all $q \in \hat{p}$. It follows that $D = A/B$ is divisible.

Decompose $D = \oplus_{\alpha<\varkappa} D_\alpha$ into a direct sum of at most countable divisible groups. For each $\alpha < \varkappa$ pick a countable subgroup E_α of A such that $D_\alpha = (E_\alpha + B)/B$ and a countable subset J_α of $\varkappa$ such that $E_\alpha \cap B \subseteq \operatorname{Span}(V_\alpha)$, where $V_\alpha = \{v_\iota \mid \iota \in J_\alpha\}$. Let $f\colon \varkappa \to \varkappa$ be a map defined by

$$f(\xi) = \begin{cases} \min\{\alpha \mid \xi \in J_\alpha\} & \text{if } \xi \in J_\alpha \text{ for some } \alpha \in \varkappa; \\ \xi & \text{otherwise.} \end{cases}$$

Put $f^* = f \restriction I^*$. Observe that $[\{\alpha\}]f^{-1} \subseteq J_\alpha \cup \{\alpha\}$, in particular, $[\{\alpha\}]f^{-1}$ is at most countable for every $\alpha \in \varkappa$. Consequently, $[\{\alpha\}](f^*)^{-1}$ is at most countable, for every $\alpha \in \varkappa$. By Lemma 4.3, there is a map $g^*\colon I^* \to \varkappa$ such that $g^* \leq f^*$ and $\operatorname{card}([\{\alpha\}](g^*)^{-1}) = \aleph_0$ for all $\alpha \in \varkappa$. Let $g\colon \varkappa \to \varkappa$ be a map such that $g \restriction I^* = g^*$ and $g \restriction (\varkappa \setminus I^*) = f \restriction (\varkappa \setminus I^*)$. Observe that $\xi \in J_\alpha$ implies that $f(\xi) \leq \alpha$, for all ξ and α from $\varkappa$. It follows that $J_\alpha \subseteq [\alpha + 1]g^{-1}$ for all $\alpha \in \varkappa$.

Put $G_\alpha = \{v_\iota \mid \iota \in [\{\alpha\}]g^{-1}\}$, resp. $G_{<\alpha} = \{v_\iota \mid \iota \in [\alpha]g^{-1}\}$, and set $B_\alpha = \operatorname{Span}(G_\alpha)$, resp. $B_{<\alpha} = \operatorname{Span}(G_{<\alpha})$, for all $\alpha \in \varkappa$. Observe that, since $\operatorname{card}([\{\alpha\}](g^*)^{-1}) = \aleph_0$, we have that $\operatorname{card}(G_\alpha \cap \tau A) = \aleph_0$, for all $\alpha \in \varkappa$. Now put

$$C_\alpha = B_{<\alpha+1}/B_{<\alpha} \simeq B_\alpha/(B_\alpha \cap B_{<\alpha}),$$

for all $\alpha \in \varkappa$. Since $J_\alpha \subseteq [\alpha + 1]g^{-1}$, we have that $V_\alpha \subseteq G_{<\alpha+1}$, hence $E_\alpha \cap B \subseteq B_{<\alpha+1}$, for all $\alpha \in \varkappa$. It follows that

$$D_\alpha = (E_\alpha + B)/B \simeq E_\alpha/(E_\alpha \cap B) = E_\alpha/(E_\alpha \cap B_{<\alpha+1}) \simeq (E_\alpha + B_{<\alpha+1})/B_{<\alpha+1},$$

for all $\alpha < \varkappa$. Consequently, $D_\alpha \oplus C_{\alpha+1} \simeq (E_\alpha + B_{<\alpha+2})/B_{<\alpha+1}$, for all $\alpha \in \varkappa$. The set V lifts a basis of A/pA over pA, in particular, it is $\mathcal{S}$-independent. It follows that $G_{\alpha+1}$ lifts a minimal generating set of $C_{\alpha+1}$ over $B_{<\alpha+1}$, for all $\alpha \in \varkappa$. Denote this set by $H_{\alpha+1}$ (note that $H_{\alpha+1} = \pi_{B_{<\alpha+1}}[G_{\alpha+1}]$ for all $\alpha \in \varkappa$). Since $\operatorname{card}([\{\alpha\}](g^*)^{-1}) = \aleph_0$, we have that $\operatorname{card}(G_{\alpha+1} \cap \tau A) = \aleph_0$, we infer that $\operatorname{card}(H_{\alpha+1} \cap \tau(A/B_{<\alpha+1})) = \aleph_0$, for all $\alpha \in \varkappa$. It follows that, for all $\alpha \in \varkappa$, the direct sum $D_\alpha \oplus C_{\alpha+1}$ has a minimal generating set, say $W_{\alpha+1}$, which lifts $H_{\alpha+1}$ over D_α by Corollary 3.12. Since D_α is divisible, the set $W_{\alpha+1}$ is formed by suitable elements $v_\iota + pe_\iota + B_{<\alpha+1}$, where $\iota \in [\{\alpha + 1\}]g^{-1}$ and $e_\iota \in E_\alpha$ for all

$\iota \in [\{\alpha+1\}]g^{-1}$. Given $\alpha \in \varkappa$ and $\iota \in [\{\alpha\}]g^{-1}$, we define

$$x_\iota = \begin{cases} v_\iota & \text{provided that ordinal } \alpha \text{ has no predecessor,} \\ v_\iota + pe_\iota & \text{otherwise,} \end{cases} \tag{4.1}$$

and put $X = \{x_\iota \mid \iota \in \varkappa\}$.

Claim 1. *The set X forms a minimal generating set of A.*

Proof of Claim 1. It is straightforward to see that the set X is $\mathcal{S}$-independent, indeed, $x_\iota + pA = v_\iota + pA$ for every $\iota \in \varkappa$, whence X lifts a basis of A/pA over pA.

It remains to verify that $A = \operatorname{Span}(X)$. Put $X_\alpha = \{x_\iota \mid \iota \in [\{\alpha\}]g^{-1}\}$, resp. $X_{<\alpha} = \{x_\iota \mid \iota \in [\alpha]g^{-1}\}$, and set $A_\alpha = \operatorname{Span}(X_\alpha)$, resp. $A_{<\alpha} = \operatorname{Span}(X_{<\alpha})$, for all $\alpha \in \varkappa$. By transfinite induction on $\beta \in \varkappa$ we prove simultaneously that $B_{<\beta} \subseteq A_{<\beta}$ and $E_\alpha \subseteq A_{<\beta}$, whenever $\beta = \alpha + 2$.

First, observe that $B_{<0} = A_{<0} = \operatorname{Span}(\emptyset) = \{0\}$. Now let $0 < \beta \in \varkappa$, and suppose that the assertion holds for all $\alpha \in \beta$. If β is a limit ordinal, then $B_{<\beta} = \bigcup_{\alpha<\beta} B_{<\alpha} \subseteq \bigcup_{\alpha<\beta} A_{<\alpha} = A_{<\beta}$, by the induction hypothesis. Let $\beta = \alpha+1$, where the ordinal α has no predecessor. Then $B_{<\alpha} \subseteq A_{<\alpha}$ by the induction hypothesis. Since α has no predecessor, it follows from (4.1) that $G_\alpha = V_\alpha$, hence $A_\alpha = B_\alpha$, whence $B_{<\alpha+1} \subseteq A_{<\alpha+1}$. Finally suppose that $\beta = \alpha + 2$ for some $\alpha \in \varkappa$. Then $X_{\alpha+1}$ lifts $W_{\alpha+1}$ over $B_{<\alpha+1}$ by (4.1) and $B_{<\alpha+1} \subseteq A_{<\alpha+1}$ by the induction hypothesis. Since $W_{\alpha+1}$ forms a minimal generating set of $D_\alpha \oplus C_{\alpha+1}$, we conclude that $E_\alpha + B_{<\alpha+2} \subseteq A_{<\alpha+2}$. $\square_{\text{Claim 1}}$

This verification concludes the proof of the statement. $\square$

Proposition 4.5. *Let A be a torsion abelian group. If* $\operatorname{Spec}(A,=) \neq \emptyset$*, then A has a minimal generating set.*

Proof. If A is uncountable, them the proposition follows from Lemma 4.4. Suppose that the group A is countable and decompose it into $A = R \oplus D$, where R is reduced and D is divisible. Then $\dim(R/pR) = \dim(A/pA) = \aleph_0$, for some prime number p. In particular, R is infinite. Since R is torsion, it cannot be finitely generated, hence it has an infinite minimal generating set by [6, Lemma 4.3]. Applying Lemma 3.11, we conclude that A has a minimal generating set. $\square$

Lemma 4.6. *Let A be a countable torsion abelian group. If* $\operatorname{Spec}(A,<)$ *is infinite, then A has a minimal generating set.*

Proof. Let B a finitely generated subgroup of A. Since B is torsion, it is finite, hence $pB = B$ for all but finitely many primes. Thus there is a finite subset $F \subseteq \mathbb{P}$ such that $\dim(A/(pA+B)) = \dim(A/pA)$ for all $p \in \mathbb{P} \setminus F$. It follows that $\operatorname{Spec}(A,<) \setminus F \subseteq \operatorname{Spec}(A/B,<)$, thus $\operatorname{Spec}(A/B,<)$ is infinite, in particular, it is non-empty. By Proposition 3.8, A has a minimal generating set. $\square$

Lemma 4.7. *Let A be an abelian p-group and let B be its basic subgroup. There is a minimal generating set X of B which lifts a basis of A/pA over pA.*

Proof. The group B is a direct sum of cyclic p-groups, say $B = \bigoplus_{x\in X} C_x$, where X denotes a set of generators of the cyclic summands in the decomposition and C_x is a finite cyclic group generated by x for all $x \in X$. Obviously, the set X lifts a basis of B/pB over pB. Since B is a basic subgroup of A, the factor group A/B is divisible, whence $A = B + pA$, and B is pure subgroup of A, whence $pB = B \cap pA$. It follows that there is a natural isomorphism

$$B/pB = B/(pA \cap B) \simeq (B + pA)/pA = A/pA,$$

given by the correspondence $x + pB \mapsto x + pA$, $x \in X$. We conclude that X lifts a basis of A/pA over pA. □

Theorem 4.8. *Let A be a torsion abelian group of an infinite cardinality $\varkappa$ and let B be its basic subgroup. Then the following are equivalent.*

(1) *A has a minimal generating set.*
(2) $\mathrm{card}(B) = \varkappa$.
(3) $\sum_{p\in\mathbb{P}} \dim(A/pA) = \varkappa$.

Proof. $(1 \Rightarrow 2)$ Since B is a basic subgroup of A, the factor group A/B is divisible and the implication follows from [6, Lemma 1.3]. $(2 \Rightarrow 3)$ By Lemma 4.7, $\tau_p B$ has a minimal generating set X_p which lifts a basis of $\tau_p A/p\tau_p A$ over $p\tau_p A$, for every $p \in \mathbb{P}$. Since $p\tau_{\hat{p}} A = \tau_{\hat{p}} A$, we get that $\dim(A/pA) = \dim(\tau_p A/p\tau_p A) = \mathrm{card}(X_p)$, for all $p \in \mathbb{P}$. Since B is an infinite torsion group,

$$\mathrm{card}(B) = \mathrm{gen}(B) = \mathrm{card}(\bigcup_{p\in\mathbb{P}} X_p) = \sum_{p\in\mathbb{P}} \mathrm{card}(X_p) = \sum_{p\in\mathbb{P}} \dim(A/pA).$$

$(3 \Rightarrow 1)$ If A is countable, then (3) implies that either $\mathrm{Spec}(A, =) \neq \emptyset$ or $\mathrm{Spec}(A, <)$ is infinite and we infer that A has a minimal generating set by Proposition 4.5 or by Lemma 4.6, respectively. If A is uncountable, then (3) implies that either $\mathrm{Spec}(A, =) \neq \emptyset$ or $\sum_{p\in\mathrm{Spec}(A,<)} \dim(A/pA) = \varkappa$. Then A has a minimal generating set by Proposition 4.5 or by Proposition 3.10, respectively. □

Theorem 4.8 can be simplified in case the cofinality of the cardinality of the group A is uncountable. Indeed, in this case $\sum_{p\in\mathrm{Spec}(A,<)} \dim(A/pA) < \mathrm{gen}\, A$, and we can simplify the statement of the previous theorem as follows:

Corollary 4.9. *Let A be a torsion abelian group such that* $\mathrm{cf}(\mathrm{card}(A)) > \aleph_0$. *Then A has a minimal generating set iff* $\dim(A/pA) = \varkappa$ *for some prime number p (i.e.,* $\mathrm{Spec}(A, =) \neq \emptyset$*).*

5. Torsion free abelian groups

In [6] we did not succeed to characterize torsion free abelian groups with a minimal generating set. Here we complete this characterization. What in [6] was missing is Lemma 5.2. Roughly saying, it states that for a torsion free abelian group A to have a minimal generating set one prime in $\mathrm{Spec}(A, =)$ is not enough. We prove its more general version applicable also for mixed groups.

Definition. Let A be an abelian group and let $X \subseteq A$. We say that a X is $\mathbb{Z}$-*linearly independent* if the only vanishing linear combination of elements of X with integer coefficients is a trivial combination.

The next lemma is well-known, we leave the proof to the reader.

Lemma 5.1. *Let A be a torsion free abelian group and let p be a prime number. Then every $X \subseteq A$ which lifts a linearly independent subset of A/pA over pA is $\mathbb{Z}$-linearly independent.*

Lemma 5.2. *Let A be an abelian group. Suppose that there is a prime number p and a subset U of A with $\operatorname{card}(U) < \operatorname{gen}(A)$ such that $A = qA + \operatorname{Span}(U)$ for all $q \in \hat{p}$ and $\tau A \subseteq pA + \operatorname{Span}(U)$. Then A has not a minimal generating set.*

Proof. We start with proving that the abelian group A is not finitely generated. Suppose otherwise. Since a finitely generated abelian group is a direct sum of cyclic groups [5, Crollary 10.22], we observe that $\bigcap_{p\in\mathbb{P}} pA = 0$. It follows from our assumptions that $\tau A \subseteq pA + \operatorname{Span}(U)$ for all $p \in \mathbb{P}$, hence $\tau A \subseteq \operatorname{Span}(U)$. Since the group $\operatorname{Span}(U)$ is finitely generated, we get that $\operatorname{Span}(U) \simeq \tau A \oplus \phi \operatorname{Span}(U)$. The equality $\operatorname{rank} \phi \operatorname{Span}(U) = \operatorname{rank} A$ would imply that $\operatorname{Span}(U) \simeq A$ which is not the case, since $\operatorname{gen}(\operatorname{Span}(U)) \leq \operatorname{card} U < \operatorname{gen}(A)$. Applying [5, Exercise 10.15], we get that the group $\phi \operatorname{Span}(U)$ has not finite index in ϕA. It follows that the factor group $A/\operatorname{Span}(U)$ is not finite. Since it is finitely generated, it has a free direct summand. But then $A/\operatorname{Span}(U)$ is not divisible by any prime number, which contradicts our assumptions.

For the rest of the proof assume that A is not finitely generated. Towards a contradiction, suppose that A has a minimal generating set X. Pick a finite $X_u \subseteq X$ such that $u \in \operatorname{Span}(X_u)$, for every $u \in U$, and put $Y = \bigcup_{u\in U} X_u$. Observe that either U is finite and then Y is finite as well or $\aleph_0 \leq \operatorname{card}(U) = \operatorname{card}(Y)$. Put $Z = X \setminus Y$. Since the group A is not finitely generated and $\operatorname{card}(U) < \operatorname{gen}(A) = \operatorname{card}(X)$, we infer that $\operatorname{card}(Y) < \operatorname{card}(X)$, and so $\operatorname{card}(Z) = \operatorname{card}(X)$. Further deduce from the properties of the set U that $A = qA + \operatorname{Span}(Y)$ for all $q \in \hat{p}$ and $\tau A \subseteq pA + \operatorname{Span}(Y)$.

Claim 1. *The set Z lifts a linearly independent subset of $A/(pA + \operatorname{Span}(Y))$ over $pA + \operatorname{Span}(Y)$.*

Proof of Claim 1. Let $\mathcal{Z}$ denote the collection of all subsets of Z which are $\mathcal{S}$-independent over $pA+\operatorname{Span}(Y)$, i.e., those subsets, which lift a linearly independent subset of $A/(pA+\operatorname{Span}(Y))$ over $pA+\operatorname{Span}(Y)$. The set $\mathcal{Z}$ has a maximal element, say Z', by Zorn's lemma. Suppose that $Z' \neq Z$ and put $X' = Z' \cup Y$. Then $A/\operatorname{Span}(X')$ is a nontrivial divisible group with a minimal generating set (lifted by nonempty $X \setminus X'$), which cannot be the case. $\square_{\text{Claim 1}}$

Since $\tau A \subseteq pA + \operatorname{Span}(Y)$, the set Z is $\mathcal{S}$-independent over τA. Put $Z_{\mathcal{F}} = \pi_{\tau A}(Z)$. The set Z, and so the set $Z_{\mathcal{F}}$ as well, lifts a linearly independent subset of $A/(pA + \operatorname{Span}(Y))$ over $pA + \operatorname{Span}(Y)$ by Claim 1. Since $\tau A \subseteq pA + \operatorname{Span}(Y)$, we get that $Z_{\mathcal{F}}$ lifts a linearly independent subset of $A/(pA+\tau A) \simeq (A/\tau A)/p(A/\tau A)$. By Lemma 5.1 we get that $Z_{\mathcal{F}}$ is a $\mathbb{Z}$-linearly independent subset of $A/\tau A$. That is, $\operatorname{Span}(Z_{\mathcal{F}}) = \operatorname{Span}(Z) + \tau A/\tau A$ is a free subgroup of $A/\tau A$. Put $Y_{\mathcal{F}} = \pi_{\tau A}(Y)$ and note that $\operatorname{Span}(Y_{\mathcal{F}}) = (\tau A + \operatorname{Span} Y)/\tau A$ and $\operatorname{Span}(Z_{\mathcal{F}}) + \operatorname{Span}(Y_{\mathcal{F}}) = A/\tau A$. Since $\operatorname{card}(Y_{\mathcal{F}}) \leq \operatorname{card}(Y) < \operatorname{card}(Z) = \operatorname{card}(Z_{\mathcal{F}})$ and the group $\operatorname{Span}(Z_{\mathcal{F}})$ is free of rank $\operatorname{card}(Z_{\mathcal{F}})$, we infer that the group

$$\begin{aligned} A/(\tau A + \operatorname{Span}(Y)) &\simeq (A/\tau A)/\operatorname{Span}(Y_{\mathcal{F}}) = (\operatorname{Span}(Z_{\mathcal{F}} + \operatorname{Span}(Y_{\mathcal{F}}))/\operatorname{Span}(Y_{\mathcal{F}}) \\ &\simeq \operatorname{Span}(Z_{\mathcal{F}})/(\operatorname{Span}(Z_{\mathcal{F}}) \cap \operatorname{Span}(Y_{\mathcal{F}})) \end{aligned}$$

has a nontrivial free direct summand. But this is impossible, since the group $A/(\tau A + \operatorname{Span}(Y))$ is divisible by every $q \in \hat{p}$. $\square$

Notice that Lemma 5.2 generalizes [6, Lemma 1.3]. For a torsion free abelian group we have its following corollary:

Corollary 5.3. *Let A be a torsion free abelian group. Suppose that there is a prime number p and a subset $Y \subseteq A$ of cardinality less than* $\mathrm{gen}(A)$ *such that $A = qA + \mathrm{Span}(Y)$ for every prime $q \neq p$. Then A has not a minimal generating set.*

Theorem 5.4. *A torsion free abelian group A has a minimal generating set iff either* $\mathrm{card}(\mathrm{Spec}(A, =)) \geq 2$ *or*

$$\sum_{p \in \mathrm{Spec}(A,<)} \dim(A/(pA + B)) = \mathrm{gen}(A), \tag{5.1}$$

for every finitely generated subgroup B of A.

Proof. ($\Leftarrow$) If $\mathrm{card}(\mathrm{Spec}(A, =)) \geq 2$, then the group A has a minimal generating set due to Proposition 3.7, while if (5.1) is satisfied, then A is not finitely generated and it has a minimal generating set by Proposition 3.10, resp. Proposition 3.8 in case that $\mathrm{gen}(A) > \aleph_0$, resp. $\mathrm{gen}(A) = \aleph_0$. Note that if $\mathrm{gen}(A) > \aleph_0$, then the equation (5.1) can be simplified to (3.1). ($\Rightarrow$) A finitely generated torsion free abelian group A is free, in which case $\mathrm{Spec}(A, =) = \mathbb{P}$. Thus we can assume that A is not finitely generated. Suppose that $\mathrm{card}(\mathrm{Spec}(A, =) \leq 1$. Then $\dim(A/pA) < \mathrm{gen}(A)$ for all but a single prime, say p. Suppose that there is a subgroup B of A generated by a finite set Y_0 such that $\sum_{p \in \mathrm{Spec}(A,<)} \dim(A/(pA + B)) < \mathrm{gen}(A)$. It follows that there is a subset Y of A containing Y_0 such that $\mathrm{card}(Y) < \mathrm{gen}(A)$ and $\mathrm{Span}(Y) + qA = A$ for every prime $q \neq p$. We conclude that A has not a minimal generating set by Corollary 5.3. □

In case the cardinality of A is of an uncountable cofinality, we can remove (5.1) from the previous statement.

Corollary 5.5. *Let A be a torsion free abelian group. If the cardinality of A has uncountable cofinality, then the group A has a minimal generating set iff* $\mathrm{card}(\mathrm{Spec}(A, =)) \geq 2$.

6. Mixed Groups - General Case

One would expect that combining the characterization in torsion and torsion-free case would suffice to characterize all abelian groups with a minimal generating set. It is quite true in the uncountable case while for countable abelian groups we need one more result, namely Lemma 6.4. Thus we will treat the uncountable and countable case separately and combine both the cases to gain the final characterization in Theorem 6.6.

6.1. **Uncountable abelian groups.**

Theorem 6.1. *Let A be an abelian group of an uncountable cardinality $\varkappa$. Then A has a minimal generating set iff at least one of the following conditions is satisfied:*

1. $\mathrm{card}(\mathrm{Spec}(A, =)) \geq 2$*;*
2. $\dim(\tau A/p\tau A) = \varkappa$ *for some prime number p;*
3. $\sum_{p \in \mathrm{Spec}(A,<)} \dim(A/pA) = \varkappa$.

Proof. ($\Leftarrow$) If $\mathrm{Spec}(A, =)$ has at least two elements, then A has a minimal generating set by Proposition 3.7, if $\dim(\tau A/p\tau A) = \varkappa$, for some prime number p, then A has a minimal generating set by Lemma 4.4, and, finally, if $\sum_{p \in \mathrm{Spec}(A,<)} \dim(A/pA) = \varkappa$, then A has a minimal generating set by Proposition 3.10. ($\Rightarrow$) Suppose that

$\sum_{p\in\operatorname{Spec}(A,<)} \dim(A/pA) < \varkappa$. If $\operatorname{Spec}(A,=) = \emptyset$, then $\sum_{p\in\mathbb{P}} \dim(A/pA) < \varkappa$ and the group A has not a minimal generating set by Lemma 4.1. Let $\operatorname{Spec}(A,=) = \{p\}$ for a single prime p. Then there is a subset $Y \subseteq A$ with $\operatorname{card}(Y) < \varkappa$ such that $A = qA + \operatorname{Span}(Y)$ for all $q \in \hat{p}$. If, moreover, $\dim(\tau A/p\tau A) < \varkappa$, then $\tau A \subseteq pA + \operatorname{Span}(Z)$ for some $Z \subseteq A$ with $\operatorname{card}(Z) < \varkappa$. Putting $U = Y \cup Z$, we conclude that A has not a minimal generating set by Lemma 5.2. $\square$

Corollary 6.2. *Let A be an abelian group and suppose that cardinality of A is of an uncountable cofinality. Then A has a minimal generating set if either* $\operatorname{card}(\operatorname{Spec}(A,=)) \geq 2$ *or* $\dim(\tau A/p\tau A) = \operatorname{card}(A)$ *for some prime number p.*

Combining Theorem 6.1 with Theorem 4.8 and Theorem 5.4 we get readily another of its corollaries.

Corollary 6.3. *Let A be an uncountable abelian group. Then A has a minimal generating set iff either τA has a minimal generating set and* $\operatorname{card}(\tau A) = \operatorname{card}(A)$ *or $A/\tau A$ has a minimal generating set and* $\operatorname{card}(A/\tau A) = \operatorname{card}(A)$.

6.2. **Countable abelian groups and the final statement.**

Lemma 6.4. *Let A be a countable abelian group such that* $\dim(\tau A/p\tau A) = \aleph_0$ *for some prime p. Then A has a minimal generating set.*

Proof. If there is a prime $q \neq p$ such that $\dim(A/qA) = \aleph_0$, then $\operatorname{Spec}(A,=) \geq 2$ and A has a minimal generating set by Proposition 3.7. Thus we can suppose that $\operatorname{Spec}(A,=) = \{p\}$. If $\operatorname{Spec}(A/B,<) \neq \emptyset$ for every finitely generated subgroup B of A, then A has a minimal generating set by Proposition 3.8. So assume that there is a finitely generated subgroup B of A such that $A = qA + B$ for all $q \in \hat{p}$. Since the subgroup B is finitely generated, $\dim(\tau(A/B)/p\tau(A/B)) = \aleph_0$ and A has a minimal generating set iff A/B has a minimal generating set by [6, Lemma 5.1]. Observe that the factor group A/B is divisible by every $q \in \hat{p}$. Thus replacing A with A/B, we can without loss of generality assume that $qA = A$ for all $q \in \hat{p}$.

Put $T = \tau A$, $\Phi = \phi A$, and $A' = A/pT$. Observe that $\tau A' = T/pT$ is a bounded subgroup of A', hence it is its direct summand by [5, Corollary 10.42] . Therefore $A' = \tau A' \oplus \phi A' \simeq (T/pT) \oplus \Phi$. Let $Z' \subseteq \Phi$ lift a basis of $\Phi/p\Phi$. Put $\Phi' = \Phi/\operatorname{Span}(Z')$. Recall that $\dim(T/pT) = \aleph_0$, and so we can pick linearly independent $Y'' \subseteq T/pT$ such that $\dim(\operatorname{Span}(Y'')) = \operatorname{codim}(\operatorname{Span}(Y'')) = \aleph_0$. Put $A'' = \operatorname{Span}(Y'') \oplus \Phi'$. Observe that the group Φ' is divisible, hence A'' has a minimal generating set Y' which lifts Y'' over Φ' by Corollary 3.12. Let Z be a subset of A which lifts Z' over pT, and let Y be a subset of A which lifts Y' over $\operatorname{Span}(Z) + pT$. Denote by C the subgroup of A generated by $Y \cup Z$. Observe that $A = C + T$, hence the group $T' = A/C \simeq T/(C \cap T)$ is torsion. Since $\dim(T'/pT') = \operatorname{codim}(\operatorname{Span}(Y'')) = \aleph_0$, the group T' has a minimal generating set X' by Proposition 4.5. Note that X' lifts a basis of T'/pT' over pT', indeed, $T' = \operatorname{Span}(X') + pT'$ and if $T' = \operatorname{Span}(X'') + pT'$ for some $X'' \subsetneq X'$, we would get a nontrivial divisible group $T'/\operatorname{Span}(X'')$ with a minimal generating set (corresponding to the canonical image of $X' \setminus X'' \neq \emptyset$).

Let X lift X' over C. Note that Y lifts a linearly independent subset of $A/(pA+\operatorname{Span}(Z))$ over $pA+\operatorname{Span}(Z)$ and X lifts a linearly independent subset of $A/(pA+C) = A/(pA+\operatorname{Span}(Y\cup Z))$ over $pA+\operatorname{Span}(Y\cup Z)$. It follows that $X\cup Y\cup Z$ lifts a linearly independent subset of A/pA, in particular, the union $X\cup Y\cup Z$ forms an $\mathcal{S}$-independent subset of A. Since $A = \operatorname{Span}(X)+C$ and $C = \operatorname{Span}(Y\cup Z)$, we have that $A = \mathrm{Span}(X \cup Y \cup Z)$. We conclude that $X \cup Y \cup Z$ is a minimal generating set of A. $\square$

Theorem 6.5. *Let A be an infinitely generated countable abelian group. Then A has a minimal generating set iff at least one of the following conditions is satisfied:*

(1) $\mathrm{card}(\mathrm{Spec}(A,=)) \geq 2$*;*
(2) $\dim(\tau A/p\tau A) = \aleph_0$ *for some prime number p;*
(3) $\mathrm{Spec}(A/B,<) \neq \emptyset$ *for every finitely generated subgroup B of A.*

Proof. ($\Leftarrow$) If $\mathrm{Spec}(A,=) \geq 2$, then A has a minimal generating set by Proposition 3.7, if $\dim(\tau A/p\tau A) = \aleph_0$ for some prime number p, then has a minimal generating set by Lemma 6.4, and if $\mathrm{Spec}(A/B,<) \neq \emptyset$ for every finitely generated subgroup B of A, then the existence of a minimal generating set of A follows from Proposition 3.8. ($\Rightarrow$) Suppose that $\mathrm{Spec}(A/B,<) = \emptyset$ for some finitely generated subgroup B of A. Note that $\mathrm{Spec}(A,=) = \mathrm{Spec}(A/B,=)$ and $\dim(\tau A/p\tau A) = \aleph_0$ iff $\dim(\tau A/B/p\tau A) = \aleph_0$ for each prime number p. By [6, Lemma 5.1], the group A has minimal generating set iff the factor-group A/B has a minimal generating set. Thus, replacing A by A/B, we can without loss of generality assume that $\mathrm{Spec}(A,<) = \emptyset$. If $\mathrm{Spec}(A,=) = \emptyset$, then A is divisible and so it has not a minimal generating set. If $\mathrm{Spec}(A,=) = \{p\}$ for a single prime p and $\dim(\tau A/p\tau A) < \aleph_0$, then A has not a minimal generating set by Lemma 5.2. $\square$

Note that applying Corollary 3.9, we can replace property (3) in Theorem 6.5 by requiring that $\sum_{p \in \mathrm{Spec}(A,<)} \dim(A/(pA+B)) = \mathrm{gen}(A)$ for every finitely generated subgroup B of A. Combining Theorem 6.1 and Theorem 6.5 treating uncountable and countable case, respectively, we get the final statement of the paper characterizing abelian groups with a minimal generating set.

Theorem 6.6. *Let A be an infinitely generated abelian group. The group A has a minimal generating set iff at least one of the following conditions is satisfied:*

(1) $\mathrm{card}(\mathrm{Spec}(A,=)) \geq 2$*;*
(2) $\dim(\tau A/p\tau A) = \mathrm{gen}(A)$ *for some prime number p;*
(3) $\sum_{p \in \mathrm{Spec}(A,<)} \dim(A/(pA+B)) = \mathrm{gen}(A)$ *for every finitely generated subgroup B of A.*

Moreover, if the group A is uncountable, property (3) *can be simplified to*

(3') $\sum_{p \in \mathrm{Spec}(A,<)} \dim(A/pA) = \mathrm{gen}(A)$*,*

and if the cardinality of the group A has uncountable cofinality, then A has a minimal generating set iff any of properties (1) *and* (2) *is satisfied.*

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