CHARACTERIZATION OF ABELIAN GROUPS WITH A MINIMAL GENERATING SET

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Abstract. We characterize abelian groups with a minimal generating set: Let $\tau A$ denote the maximal torsion subgroup of $A$. An infinitely generated abelian group $A$ of cardinality $\kappa$ has a minimal generating set iff at least one of the following conditions is satisfied:

1. $\dim(A/pA) = \dim(A/qA) = \kappa$ for at least two different primes $p, q$;
2. $\dim(\tau A/prA) = \kappa$ for some prime number $p$;
3. $\sum \{\dim(A/(pA + B)) \mid \dim(A/(pA + B)) < \kappa\} = \kappa$ for every finitely generated subgroup $B$ of $A$.

Moreover, if the group $A$ is uncountable, property (3) can be simplified to $\sum \{\dim(A/pA) \mid \dim(A/pA) < \kappa\} = \kappa$, and if the cardinality of the group $A$ has uncountable cofinality, then $A$ has a minimal generating set iff any of properties (1) and (2) is satisfied.

1. Introduction

For the notion of $S$-independence we refer to [3, pages 26 and 46]. In particular, a subset $X$ of a universal algebra $\mathfrak{A}$ is called $S$-independent provided that $x$ is not in the subalgebra generated by $X \setminus \{x\}$ for all $x \in X$. A subset $X$ of $\mathfrak{A}$ which is both $S$-independent and generating is called a minimal generating set of $\mathfrak{A}$. One should notice right away that minimal generating sets correspond to generating sets minimal with respect to inclusion.

In general, unless the algebra $\mathfrak{A}$ is finitely generated, the existence of a minimal generating set is not guaranteed. The question of the existence of a minimal generating set for various concrete algebraic structures, e.g. groups, rings, fields, was studied in [1] and [4]. A deeper insight into this problem restricted to abelian groups is in [6]. There the question of the existence of a minimal generating set is decided for torsion and partially for torsion free abelian groups, and some non-trivial examples are shown.

More should be said about [6]. The first author of this paper noticed that [6, Theorem 3.1], [6, Lemma 5.3] (and consequently [6, Theorem 5.5]) are not correct. In [6, Theorem 3.1] we have to restrict to torsion abelian groups while in [6, Lemma 5.3] some additional assumptions need to be added. In particular, there should not be a single prime $p$ such that all but finitely many torsion-free groups of rank 1 in the direct sum decomposition are divisible by all primes $q \neq p$. In both cases we wrongly applied [6, Proposition 1.5] as we assumed that if $D$ is a divisible abelian


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group and $A$ is an abelian group with a minimal generating set, say $X$, such that $\text{gen}(A) \geq \text{gen}(D)$, then the direct sum $A \oplus D$ has a minimal generating set which lifts $X$ over $D$ (following our terminology below). This holds only under some additional assumptions (see Corollary 3.12), e.g., if the group $A$ is torsion.

In this paper we correct both the results and we extend [6] to reach at the end the complete description of abelian groups with a minimal generating set. Let us outline the structure of the paper. This introduction is followed by Basic concept where we sum up necessary definitions and terminology. In Section 3 we reprove and generalize [6, Proposition 1.5] and [6, Lemma 2.1] which are main tools in [6] and prove some other general statements which will be applied in the sequel. Most of the time we restrict ourself to abelian groups, even though we believe that some of these results permit further generalizations for modules over a ring. In Section 4 we redo the characterization of torsion abelian groups with a minimal generating set. The last two sections deal with torsion-free and general abelian groups, respectively.

2. Basic concept

By card($X$) we denote the cardinality of a set $X$. By cf($\kappa$) we denote the cofinality of a cardinal $\kappa$. Given a collection $\{\kappa_i \mid i < I\}$ of cardinals, we denote by $\sum_{i \in I} \kappa_i$ the cardinality of its disjoint union. Given an ordinal $\sigma$ and maps $f, g : \sigma \to \sigma$ we use $g \leq f$ to denote that $g(\alpha) \leq f(\alpha)$ for every $\alpha < \sigma$.

We denote by $\omega$ the first infinite ordinal, that is $\omega = \{0, 1, 2, \ldots\}$. Finite ordinals will also be called natural numbers, in particular, 0 is a natural number. We will unify each ordinal with the set of its predecessors, e.g., $2 = \{0, 1\}$. By $\mathbb{P}$ we denote the set of all prime numbers and we put $\hat{p} = \mathbb{P} \setminus \{p\}$ for all $p \in \mathbb{P}$.

Let $X$ be a set and let $f : X \to Y$ be a map. Given $V \subseteq Y$, we denote by $[V]f^{-1}$ the $f$-preimage of $V$, i.e., $[V]f^{-1} = \{x \in X \mid f(x) \in V\}$. Similarly, given $U \subseteq X$, we denote by $f[U]$ the $f$-image of $U$, i.e., $f[U] = \{f(u) \mid u \in U\}$.

Let $A$ be an abelian group. We denote by $\tau A$ the maximal torsion subgroup of $A$ and by $\phi A$ the torsion free quotient $A/\tau A$. Given a prime number $p$, we denote by $\tau_p A$ the $p$-primary component of $\tau A$ and we set $\tau_p A = \bigoplus_{q \in \hat{p}} \tau_q A$. Similarly, given a domain $R$ and an $R$-module $A$ we denote by $\tau A$ the maximal torsion submodule of $A$ and by $\phi A$ the torsion free quotient $A/\tau A$.

All rings are supposed to be commutative. Given a ring $R$, an $R$-module $A$, and a subset $X \subseteq A$, we denote by $\text{Span}(X)$ the submodule generated by the set $X$. Further, we denote by $\pi_X : A \to A/\text{Span}(X)$ the canonical projection (sending $a \mapsto a + \text{Span}(X)$ for all $a \in A$). We will use the notation $B \leq A$ to say that $B$ is a submodule of $A$. By $\text{gen}(A)$ we denote the minimal cardinality of a generating set of $A$.

Let $R$ a ring and let $I$ be an ideal of $R$. An $R$-module $A$ is said to be $I$-divisible provided that $IA = A$. In particular, given $a \in \mathbb{Z}$, an abelian group $A$ is said to be $a$-divisible provided that $aA = A$.

3. General principles

Most of the results in [6] claiming the existence of a minimal generating set in a certain class of abelian groups are based on two statements, namely [6, Proposition 1.5] and [6, Lemma 2.1]. In this section we closely examine and generalize them (although we restrict ourselves to abelian groups). In particular we show that both
these results are based on the same simple property of linear maps (Corollary 3.2 below). On top of that we prove some of their consequences which will be applied in the rest of the paper.

We start with a nearly trivial observation. We leave its proof as an exercise.

**Lemma 3.1.** Let $R$ be a ring, let $A, B$ be $R$-modules, and let $C, D$ be submodules of the module $A$, and let $\gamma, \delta \in \text{hom}_R(A, B)$. Let $D \leq \ker \delta$ and $C \leq \ker \gamma$. Then $(\gamma + \delta)[C + D] = \gamma[D] + \delta[C]$.

**Corollary 3.2.** Let $R$ be a ring, let $A, B$ be $R$-modules, and let $\gamma, \delta \in \text{hom}_R(A, B)$. If $\ker \gamma + \ker \delta = A$, then $(\gamma + \delta)[A] = \gamma[A] + \delta[A]$.

**Proof.** By the assumption $\ker \gamma + \ker \delta = A$, hence $\gamma[A] = \gamma[\ker \delta]$ and $\delta[A] = \delta[\ker \gamma]$. Applying Lemma 3.1 we get that $(\gamma + \delta)[A] = (\gamma + \delta)[\ker \gamma + \ker \delta] = \gamma[\ker \delta] + \delta[\ker \gamma] = \gamma[A] + \delta[A]$.


**Definition.** Let $R$ be a ring, let $A, B$ be $R$-modules, and let $\alpha: A \rightarrow B$ be an $R$-linear map. Let $X$, resp. $Y$ be a subset of $A$, resp. $B$. We say that $X$ is $\mathcal{S}$-independent over $Y$ via $\alpha$ provided that $x \notin \text{Span}(X \setminus \{x\}) + \text{Span}(Y \backslash \alpha^{-1}(x))$ for all $x \in X$. We say shortly that $X$ is $\mathcal{S}$-independent over $Y$ in case $\alpha$ is the identity map. Similarly, we say that $X$ is $\mathcal{S}$-independent (via $\alpha$) provided that $Y = \emptyset$.

Let $Z$ be a subset of $B/\text{Span}(Y)$. We say $X$ lifts $Z$ over $Y$ via $\alpha$ provided that $\pi_Y \circ \alpha[X] = Z$ and the restriction $\pi_Y \circ \alpha \mid X$ is one-to-one. We say shortly that $X$ lifts $Z$ over $Y$ provided that $\alpha$ is the identity map and we say that $X$ lifts $Z$ (over $Y$) if $Y = \emptyset$.

Let $X' \subseteq A$. We say that $X$ and $X'$ are $\mathcal{S}$-equivalent via $\alpha$ over $Y$ provided that both the sets are $\mathcal{S}$-independent via $\alpha$ over $Y$ and $(\pi_Y \circ \alpha)[X] = (\pi_Y \circ \alpha)[X']$ (i.e., they lift the same set via $\alpha$ over $Y$). We say simply that $X$ and $X'$ are $\mathcal{S}$-equivalent over $Y$ provided that the map $\alpha$ is the identity map.

**Lemma 3.3.** Let $R$ be a ring, let $A, B$ be $R$-modules and let $\alpha: A \rightarrow B$ be an $R$-linear map. Let $\{X_i \mid i \in I\}$ be a collection of subsets of $A$. If $X_j$ is $\mathcal{S}$-independent over $\alpha[\bigcup_{i \neq j} X_i]$ for all $j \in I$, then $\bigcup_{i \in I} X_i$ is $\mathcal{S}$-independent via $\alpha$.

**Proof.** Let $j \in I$ and let $x \in X_j$. According to our assumption, we have that
\[
x \notin \text{Span}(X_j \setminus \{x\}) + \text{Span}(\alpha[\bigcup_{i \neq j} X_i])\alpha^{-1}
\]
\[
= \text{Span}(X_j \setminus \{x\}) + \text{Span}(\bigcup_{i \neq j} X_i) + \ker \alpha
\]
\[
\supseteq \text{Span}(\bigcup_{i \in I} X_i \setminus \{x\}) + \ker \alpha.
\]
This means that $\bigcup_{i \in I} X_i$ is $\mathcal{S}$-independent via $\alpha$.

We are going to derive a couple of corollaries of this simple lemma, not in full generality but in formulations allowing direct applications.

**Corollary 3.4.** Let $A$ be an abelian group, let $p_0, p_1$ be a couple of different primes and let $X_i$, for $i \in \mathbb{Z}$, be a subset of $p_i A$ which lifts a basis of $A/p_i A$ over $p_i A$. Then $X_0 \cup X_1$ is an $\mathcal{S}$-independent subset of $A$. 

Proof. Since $X_i$ lifts a basis of $A/p_iA$ over $p_iA$ and $X_i \subseteq p_{i+1}A$ we see that the set $X_i$ is 8-independent over $X_{i-1}$ for all $i \in 2$. By Lemma 3.3 we conclude that the set $X_0 \cup X_1$ is 8-independent. □

Corollary 3.5. Let $A$ be an abelian group. Let $\{p_i \mid i \in \omega\}$ be a set of primes and $\{X_i \mid i \in \omega\}$ a set of subsets of $A$ such that for all $j \in \omega$, $X_j \subseteq q_jA$, where $q_j = p_0 \cdots p_{j-1}$ and it lifts a basis of $A/(B_j + p_jA)$ over $B_j + p_jA$, where $B_j = \text{Span}(\bigcup_{i<j} X_i)$. Then the set $X = \bigcup_{i \in \omega} X_i$ is 8-independent.

Proof. Fix $j \in \omega$. Observe $X_k \subseteq q_kA \subseteq p_jA$ for all $k > j$. It follows that $\bigcup_{i \neq j} X_i$ is contained in $B_j + p_jA$ and so $X_j$ is 8-independent over $\bigcup_{i \neq j} X_i$. Applying Lemma 3.3, we conclude that the set $X = \bigcup_{i \in \omega} X_i$ is 8-independent. □

From now on, with a few exceptions, we restrict ourselves to abelian groups aiming to complete the characterization of abelian groups with a minimal generating sets. However, we believe that most of the results obtained on the way can be generalized for wider classes of modules over commutative or even non-commutative rings.

Lemma 3.6. Let $A$ be an abelian group, let $Y$ be a subset of $A$ and let $p_0, p_1$ be a couple of different primes such that $\text{card}(Y) \leq \dim(A/p_iA)$ for all $i \in 2$. Then there are subsets $X_i \subseteq p_{i+1}A$, $i \in 2$, such that $X_i$ lifts a basis of $A/p_iA$ over $p_iA$ for all $i \in 2$ and $Y \subseteq \text{Span}(X_0 \cup X_1)$.

Proof. We can without loss of generality assume that $\dim(A/p_0A) \leq \dim(A/p_1A)$.

Claim 1. There are subsets $Z_0 \subseteq Z_1 \subseteq A$ such that $Z_i$ lifts a basis of $A/p_iA$ over $p_iA$ for all $i \in 2$ and $Z_1 \setminus Z_0 \subseteq p_0A$.

Proof of Claim 1. Put $\lambda_i = \dim(A/p_iA)$, $i \in 2$, and let $\{y_{i,i} \mid \iota < \lambda_i\}$ lift a basis of $A/p_iA$ for all $i \in 0, 1$. For all $\iota < \lambda_1$ put

$$z_\iota = \begin{cases} p_1y_{\iota,0} + p_0y_{\iota,1} & \text{if } \iota < \lambda_0, \\ p_0y_{\iota,1} & \text{if } \lambda_0 \leq \iota < \lambda_1. \end{cases}$$

It is straightforward to verify that the sets $Z_i = \{z_\iota \mid \iota < \lambda_i\}$, $i \in 2$, satisfy the desired properties. □

Since $Z_1$ lifts a basis of $A/p_1A$, we have that $A = \text{Span}(Z_1) + p_1A$ for all $i \in 2$. Since $Z_0 \subseteq Z_1$, we conclude that $A = \text{Span}(Z_1) + p_0p_1A$. Thus there is a map $f: Y \rightarrow p_0p_1A$ such that $y \in \text{Span}(Z_1) + f(y)$ for all $y \in Y$. Since $\text{card}(Y) \leq \dim(A/p_0A) = \text{card}(Z_0)$, there is a projection $g: Z_0 \rightarrow Y$. Put $h = f \circ g$ and observe that $h: Z_0 \rightarrow p_0p_1A$ is a map such that $Y \subseteq \text{Span}(Z_1) + h(Z_0)$.

Let $F$ denote a free abelian group with a basis $Z_0 \times 2$. Note that, since $p_0, p_1$ are different primes, $a_0p_0 + a_1p_1 \equiv 1$ for some integers $a_0, a_1$. Let $\gamma': Z_0 \times 2 \rightarrow A$ be a map defined by the correspondence $(z, i) \mapsto a_1 - p_1z$. Let $\gamma: F \rightarrow A$ be a unique extension of $\gamma'$ to a group homomorphism. Observe that $\gamma[F] = \text{Span}(Z_0)$ and $\ker \gamma \cong \langle (a_0p_0(z, 0) - a_1p_1(z, 1) \mid z \in Z_0) \rangle$.

Let $\delta': Z_0 \times 2 \rightarrow p_0p_1A$ be a map defined by the correspondence $(z, i) \mapsto (-1)^i h(z)$). Let $\delta: F \rightarrow A$ be its unique extension to a group homomorphism. Observe that $\delta[F] = \text{Span}(h[Z_0])$ and $\ker \delta \subseteq \{(z, 0) + (z, 1) \mid z \in Z_0\}$.

Claim 2. The equality $F = \ker \gamma + \ker \delta$ holds.
Proof of Claim 2. Let $z \in Z_0$. Since $a_0p_0 + a_1p_1 = 1$, we have that $(z, 0) = (a_0p_0 + a_1p_1)(z, 0) = (a_0p_0(z, 0) - a_1p_1(z, 1)) + a_1p_1((z, 0) + (z, 1)) \in \ker \gamma + \ker \delta$. Now $(z, 1) = ((z, 1) + (z, 0)) - (z, 0) \in \ker \delta + (\ker \gamma + \ker \delta) = \ker \gamma + \ker \delta$. □Claim 2

Put $\beta = \gamma + \delta$. By Corollary 3.2 and Claim 2 we have that $\beta[F] = \text{Span}(Z_0) + \text{Span}(h[Z_0])$. Put $X_0 = \beta[Z_0 \times \{0\}]$ and $X_1 = \beta[Z_0 \times \{1\}] \cup (Z_1 \setminus Z_0)$. Then $\text{Span}(X_0 \cup X_1) = \text{Span}(\beta[Z_0 \times 2]) + \text{Span}(Z_1 \setminus Z_0) = \beta[\text{Span}(Z_0 \times 2)] + \text{Span}(Z_1 \setminus Z_0) = \beta[F] + \text{Span}(\text{Span}(Z_0) + \text{Span}(h[Z_0])) + \text{Span}(Z_1 \setminus Z_0) = (\text{Span}(Z_0) + \text{Span}(Z_1 \setminus Z_0)) + \text{Span}(h[Z_0]) = \text{Span}(Z_1) + \text{Span}(h[Z_0]) \supseteq Y$.

It is clear from the definition of the maps $\gamma, \delta$ that $\gamma[Z_0 \times \{i\}] \subseteq p_{i-1}A$, for all $i \in \mathbb{Z}$, and $\delta[F] \subseteq p_0p_1A$. Thus $\beta[Z_0 \times \{i\}] \subseteq p_{i-1}A$ for all $i \in \mathbb{Z}$. It follows that $X_0 \subseteq p_1A$. Since by Claim 1 we have that $Z_1 \setminus Z_0 \subseteq p_0A$, we infer that also $X_1 \subseteq p_0A$.

Let $z \in Z_0$ and $i \in \mathbb{Z}$. Then $\beta((z, i)) = \gamma((z, i)) + \delta((z, i)) = a_1\gamma i p_1z + (\gamma 1)i h(z) = (a_1p_1 + a_1p_1)z - a_1p_1z + (\gamma 1)i h(z) = z + (a_1p_1z + (\gamma 1)i h(z)) \in z + p_1A$. Thus $X_0 = \beta[Z_0 \times \{0\}]$ is $S$-equivalent to $Z_0$ over $p_0A$ and $X_1 = \beta[Z_0 \times \{1\}] \cup (Z_1 \setminus Z_0)$ is $S$-equivalent to $Z_1$ over $p_1A$. In particular, $X_i$ lifts a basis of $A/p_iA$ over $p_iA$ for all $i \in \mathbb{Z}$. □

Notation. Let $A$ be an abelian group. We will use the following notation:

$\text{Spec}(A, <) = \{p \in \mathbb{P} \mid 0 < \dim(A/pA) < \dim(A)\}$,
$\text{Spec}(A, =) = \{q \in \mathbb{P} \mid \dim(A/qA) = \dim(A)\}$.

**Proposition 3.7** ([6, Lemma 2.1]). Let $A$ be an abelian group. If $\text{card}(\text{Spec}(A, =)) \geq 2$, then $A$ has a minimal generating set.

**Proof.** Let $p_i, i \in \mathbb{Z}$, be a couple of different primes from $\text{Spec}(A, =)$ and let $Y$ be a generating set of $A$ of a minimal cardinality. Then $\text{card}(Y) \leq \dim(A/p_iA)$ for all $i \in \mathbb{Z}$ and, applying Lemma 3.6, we get $X_i \subseteq A$ such that $X_i \subseteq p_{i-1}A$. $X_i$ lifts a basis of $A/p_iA$ over $p_iA$, for all $i \in \mathbb{Z}$, and $Y \subseteq \text{Span}(X_0 \cup X_1)$. Since $A = \text{Span}Y$, we conclude that $A = \text{Span}(X_0 \cup X_1)$. It follows from Corollary 3.4 that the set $X_0 \cup X_1$ is $S$-independent. □

**Proposition 3.8.** Let $A$ be a countable abelian group. If $\text{Spec}(A/B, <) \neq \emptyset$ for every finitely generated subgroup $B$ of $A$, then $A$ has a minimal generating set.

**Proof.** Let $Y = \{y_i \mid i \in \omega\}$ be a generating set of $A$ (note that since $A$ is countable, we can as well put $Y = A$).

**Claim 1.** There is a sequence $p_0, p_1, p_2, \ldots$ of primes and a sequence $X_0, X_1, X_2, \ldots$ of finite sets such that, putting $q_0 = 1$, $q_j = \prod_{i=0}^{j-1} p_i$, and $B_j = \text{Span}(\bigcup_{i=0}^{\omega} X_i)$, the following properties are satisfied for all $j \in \omega$: $X_j \subseteq q_j A$, $X_j$ lifts a basis of $A/(p_jA + B_j)$, and $y_j \in B(2^{j+1})$.

**Proof of Claim 1.** First observe that the assumptions of Proposition 3.8 imply that $\text{Spec}(A/B, <)$ is infinite for every finitely generated subgroup $B$ of $A$. Indeed, since the abelian group $A$ is countable, there is a finitely generated subgroup $B_p$ of $A$ with $p(A/(B + B_p)) = A/(B + B_p)$, for every prime $p \in \text{Spec}(A/B, <)$. Thus if $\text{Spec}(A/B, <)$ was finite, $B' = B + \sum_{p \in \text{Spec}(A/B, <)} B_p$ would have been a finitely generated subgroup of $A$ with $\text{Spec}(A/B', <) = \emptyset$. (Note that $\text{Spec}(A, =) = \emptyset$.)
for every finitely generated subgroup $B''$ of $A$.) This contradicts our assumptions.

Let $p_0, p_1$ be two different primes from $\text{Spec}(A, \prec)$. By Lemma 3.6 there are subsets $X_i, j \in 2$, such that $X_i \subseteq p_i A$, $X_i$ lifts a basis of $A/p_i A$ over $p_i A$, for all $i \in 2$ and $y_0 \in \text{Span}(X_0 \cup X_1)$. Since $p_i \in \text{Spec}(A, \prec)$, $\dim(A/p_i A) < q_0$, for all $i \in \omega$, and so both the sets $X_0, X_1$ are finite. Since $q_0 = 1$, we have that $X_0 \subseteq q_0 A$. Since $q_1 = q_0$ and $X_1 \subseteq p_0 A$, we have that $X_1 \subseteq q_1 A$. Since $B_0 = \text{Span}(\emptyset) = 0$, $X_0$ lifts a basis of $A/(p_0 A + B_0)$ over $p_0 A + B_0 = p_0 A$. Since $X_0 \subseteq p_1 A$, $B_1 = \text{Span}(X_0) \subseteq p_1 A$, and so $p_1 A + B_1 = p_1 A$. Thus $X_1$ lifts a basis of $A/(p_1 A + B_1)$ over $p_1 A + B_1$. Since $B_2 = \text{Span}(X_0 \cup X_1)$, $y_0 \in B_2$.

Let $j \in \omega$ and suppose that we have already picked prime numbers $p_0, \ldots, p_{2j-1}$ and constructed finite sets $X_0, \ldots, X_{2j-1}$ satisfying the desired properties. Put $A' = A/B_{2j}$ and $y_j' = y_j + B_{2j} \in A'$. Observe that $A = p_j A + B_{2j}$ for every $i \in 2j$. It follows that $A = q_{2j} A + B_{2j}$, whence $A' = q_{2j} A'$. Since the constructed sets are finite, the group $B_{2j}$ is finitely generated, hence, by our assumption, $\text{Spec}(A', \prec)$ is not empty. As noted above, this set is in fact infinite. Pick a couple $p_{2j}, p_{2j+1}$ of different primes from $\text{Spec}(A', \prec)$. By Lemma 3.6, there are sets $X_i', j \in 2$, such that $X_i' \subseteq p_{2j+1} A'$, $X_i'$ lifts a basis of $A'/p_{2j+1} A'$ over $p_{2j+1} A'$, for all $i \in 2$, and $y_j' \in \text{Span}(X_0' \cup X_1')$. Since $A = q_{2j} A + B_{2j}$, there is $X_{2j+1} \subseteq p_{2j+1} - q_{2j} A$ lifting the set $X_i'$ over $B_{2j}$, for all $i \in 2$. Clearly $X_{2j} \subseteq q_{2j} A$ and, since $X_0'$ lifts a basis of $A'/p_{2j+1} A'$ over $p_{2j+1} A'$, $X_{2j}$ lifts a basis of $A/(p_{2j} A + B_{2j})$ over $p_{2j} A + B_{2j}$. Since $q_{2j+1} = p_{2j}/p_{2j}$, $X_{2j+1} \subseteq q_{2j+1} A$ and since $X_0'$ lifts a basis of $A'/p_{2j+1} A'$ over $p_{2j+1} A'$, we get that $X_{2j+1}$ lifts a basis of $A/(p_{2j+1} A + B_{2j})$ over $p_{2j+1} A + B_{2j}$.

We conclude that $X_{2j+1}$ lifts a basis of $A/(p_{2j+1} A + B_{2j+1})$ over $p_{2j+1} A + B_{2j+1}$. Finally, since $y_j' \in \text{Span}(X_0' \cup X_1')$, $y_j \in \text{Span}(X_{2j} \cup X_{2j+1}) + B_{2j} = B_{2(j+1)}$.

Claim 1

Put $X = \bigcup_{j \in \omega} X_j$ and observe that the set $X$ is $S$-independent by Corollary 3.5. Since $y_j \in B_{2j} \subseteq \text{Span}(X)$ for all $j \in \omega$, $Y \subseteq \text{Span}(X)$, and so $X$ generates $A$. It follows that $X$ is a minimal generating set of $A$. 

Proposition 3.10 below guarantee the existence of a minimal generating set in an uncountable abelian group under hypothesis similar to those of Proposition 3.8.

In order to demonstrate closer similarity of both the statements we reformulate Proposition 3.8 as follows:

**Corollary 3.9.** Let $A$ be a countable abelian group. If for every finitely generated subgroup $B$ of $A$

$$\sum_{p \in \text{Spec}(A, \prec)} \dim(A/(pA + B)) = \text{gen}(A),$$

then $A$ has a minimal generating set.

Now the promised proposition:

**Proposition 3.10.** Let $A$ be an uncountable abelian group. Suppose that

$$\sum_{p \in \text{Spec}(A, \prec)} \dim(A/pA) = \text{gen}(A).$$

Then $A$ has a minimal generating set.
Proof. Put \( x = \text{gen}(A) \). Observe that the equality (3.1) can be satisfied only if \( cf(x) = \kappa_0 \). In this case there is a sequence \( p_0, p_1, p_2, \ldots \) of primes from \( \text{Spec}(A, \prec) \) such that, having denoted \( \lambda_i = \dim(A/p_i A) \), we get an increasing sequence \( \lambda_0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots \) of cardinals whose supremum is \( \kappa \). Similarly as in Proposition 3.8 set \( q_0 = 1 \) and \( q_j + 1 = q_j p_j \) for all \( j \in \omega \). Let \( Y = \{ y_i \mid i < \kappa \} \) be a generating set of \( A \). For each \( i \in \omega \) put \( Y_i = \{ y_i \mid i < \lambda_i \} \).

From now on the proof closely follows the proof of Proposition 3.8.

**Claim 1.** There is a collection \( \{ X_j \mid j \in \omega \} \) of subsets of \( A \) such that \( X_j \subseteq q_j A \), for all \( j \in \omega \) and, putting \( B_j = \text{Span}(\bigcup_{i=0}^{j-1} X_i) \), the set \( X_j \) lifts a basis of \( A/(p_j A + B_j) \) over \( p_j A + B_j \) and \( \cap Y_j \subseteq B_2(j+1) \), for all \( j \in \omega \).

**Proof of Claim 1.** Repeatedly applying Lemma 3.6, we will construct inductively the sets \( X_j \), \( j \in \omega \), adding a couple of them in each step. By Lemma 3.6, there are subsets \( X_0 \subseteq p_1 A \) and \( X_1 \subseteq p_0 A \) such that \( X_i \) lifts a basis of \( A/p_i A \) over \( p_i A \) for all \( i \in 2 \) and \( Y_0 \subseteq B_2 = \text{Span}(X_0 \cup X_1) \). Notice that \( X_1 \subseteq p_0 A = q_1 A \) and \( B_1 = \text{Span}(X_0) \subseteq p_1 A \), hence \( p_1 A + B_1 = p_1 A \). Thus the sets \( X_0, X_1 \) satisfy the required properties.

Let \( 1 \leq j \in \omega \) and suppose that we have constructed sets \( X_0, \ldots, X_{j-1} \) so that the required properties are satisfied. Put \( Y' = \{ y_j + B_2j \mid i < \lambda_j \} \) and \( A' = A/B_2j \). Since \( A = p_i A + B_2j \) for each \( i \in 2j \), we have \( A = q_2j A + B_2j \) and thus \( A' = q_2j A' \). Observe that \( \dim(A'/p_k A') = \lambda_k \) for all \( 2j \leq k \in \omega \). Indeed, \( \dim(A'/p_k A') \leq \dim(A/p_k A) \leq \dim(A'/p_k A') + \text{gen}(B_2j) \), and, since \( \text{gen}(B_2j) \leq \sum_{i=0}^{2j-1} \lambda_i < \lambda_{2j} \), we get that \( \lambda_k = \dim(A/p_k A) = \dim(A'/p_k A') \).

By Lemma 3.6 there are \( X'_i \subseteq p_{2j+1-i} A' \) such that \( X'_i \) lifts a basis of \( A'/p_{2j+1} A' \) over \( p_{2j+1} A' \), for all \( i \in 2 \), and \( Y' \subseteq \text{Span}(X'_0 \cup X'_j) \). Since \( A' = q_2j A' \), there are \( X_{2j+1} \subseteq p_{2j+1-i} A \) lifting \( X'_i \) over \( B_2j \), for \( i \in 2 \). Clearly \( X_2j \subseteq q_2j A \) and \( X_{2j+1} \subseteq p_2q_2j A = q_2j+1 A \). Since \( X'_0 \) lifts a basis of \( A'/p_{2j} A' \) over \( p_{2j} A' \), \( X_{2j} \) lifts a basis of \( A/(p_{2j} A + B_2j) \) over \( p_{2j} A + B_2j \). Since \( X'_1 \) lifts a basis of \( A'/p_{2j+1} A' \) over \( p_{2j+1} A' \), \( X_{2j+1} \) lifts a basis of \( A/(p_{2j+1} A + B_2j) \) over \( p_{2j+1} A + B_2j \) and \( X_{2j} \subseteq p_{2j+1} A \), it follows that \( X_{2j+1} \) lifts a basis of \( A/(p_{2j+1} A + B_2(j+1)) \) over \( p_{2j+1} A + B_2(j+1) \). Finally, since \( Y' \subseteq \text{Span}(X'_0 \cup X'_j) \), we conclude that \( Y_j \subseteq \text{Span}(X_{2j} \cup X_{2j+1}) + B_2j = B_2(j+1). \)

Set \( X = \bigcup_{i \in \omega} X_i \). It follows from Claim 1 that \( Y_j \subseteq \text{Span}(X) \) for all \( j \in \omega \), hence \( Y \subseteq \text{Span}(X) \), whence \( \text{Span}(X) = A \). The set \( X \) is \( \kappa \)-independent by Corollary 3.5. It follows that \( X \) is a minimal generating set of \( A \).

**Lemma 3.11 ([6, Proposition 1.5]).** Let \( A \) be an abelian group with a minimal generating set and let \( B \) be a group such that \( \text{Ext}(A,B) = 0 \). If \( \text{gen}(A) \geq \text{gen}(B) \), then \( A \oplus B \) has a minimal generating set.

**Proof.** It is clear if \( A \) is finitely generated. Suppose otherwise and let \( X \) be a minimal generating set of \( A \). Let \( F \) be a free abelian group with a basis \( X \) and let \( \gamma: F \to A \) be a unique projection extending the identity map on \( X \). Put \( K = \ker \gamma \). Since a subgroup of a free abelian group is free [5, Theorem 10.18], \( K \) is free. We will consider two cases. First suppose that \( \text{gen}(K) < \text{gen}(A) = \text{card}(X) \) and pick any prime \( p \). Then \( \dim(A/pA) = \dim(F/pF + K) = \text{card}(X) = \text{gen}(A) \), indeed, \( \text{gen}(K) < \text{card}(X) \). It follows that \( A \) satisfies the assumptions of Proposition 3.7, and so it has a minimal generating set. It remains to assume that \( \text{gen}(K) = \text{gen}(A) \). Since \( \text{gen}(A) \geq \text{gen}(B) \) and \( K \) is free, there is a projection \( \delta: K \to B \). Since
Ext(A, B) = 0, the projection δ' extends to a homomorphism δ: F → B. Since δ[K] = δ[F], K + ker(δ) = F. Since K = ker(γ), we conclude that F = ker(γ) + ker(δ). By Corollary 3.2 (γ + δ)[F] = γ[F] + δ[F] = A ⊕ B. Since F = Span(X), we get that Span((γ + δ)[X]) = A ⊕ B. Since X (as a basis of F) is S-independent via γ, it is S-independent via γ + δ over B (note that |B|(γ + δ)^{-1} = K), and so it is S-independent via γ + δ. It follows that (γ + δ)[X] is a minimal generating set of A ⊕ B.

Note that in the second case the minimal generating set lifts X over B. Some obvious properties of A would guarantee that gen(K) = gen(A).

**Corollary 3.12.** Let A be an abelian group with an infinite minimal generating set X and let B be an such that gen(B) ≤ card(X) and Ext(A, B) = 0. Suppose that either card(τ A) = card(X) or A has no free direct summand of rank equal card(X). Then A ⊕ B has a minimal generating set which lifts X over B.

4. TORSION GROUPS

Starting with torsion abelian groups, we correct [6, Theorem 3.1], formulated in [6] for all uncountable abelian groups. The theorem holds under the additional assumption that the group in question is torsion but it is not true in general as follows from Corollary 5.3. Then we rephrase the characterization of countable torsion abelian groups with a minimal generating set (see [6, Lemma 4.3]) and combining the countable and uncountable cases we reformulate properties characterizing torsion abelian groups with a minimal generating set [6, Theorem 4.4].

**Lemma 4.1.** Let A be an infinitely generated abelian group. If

\[ \sum_{p \in P} \dim(A/pA) < \text{card}(A), \]

then A has not a minimal generating set.

**Proof.** Note that since the group A is infinitely generated, card(A) = gen(A). For every prime number p there is a subset X_p of A such that card(X_p) = dim(A/pA) and A = Span(X_p)+pA. Put B = Span(∪_{p \in P} X_p). Since gen(B) ≤ \sum_{p \in B} dim(A/pA) < gen(A) and gen(A) is infinite, gen(B) < gen(A). It follows that gen(B) < gen(A/B) and since p(A/B) = A/B for every prime number p, the abelian group A/B is divisible. Then A has not a minimal generating set by [6, Lemma 1.3].

**Lemma 4.2.** Let λ ≤ ω be infinite cardinals and let X be a subset of ω of cardinality ω. Then there is a map h: X → ω such that h(ξ) ≤ ξ and card([|ξ|]h^{-1}) = λ for all ξ ∈ X.

**Proof.** Let \{X_α | α < λ\} be a partition of the set X into λ pairwise disjoint subsets of cardinality ω. Let X_α = {x_{α,ξ} | ξ < ω}, where x_{α,β} < x_{α,γ} whenever β < γ < ω, for all α < λ. By induction we easily prove that ξ ≤ x_{α,ξ} for all α < λ and ξ < ω. Defining h(x_{α,ξ}) = ξ for all α < λ and all ξ < ω we get the map h with the desired properties.

**Lemma 4.3.** Let λ ≤ ω be infinite cardinals, let Y be a set of cardinality ω, and let f: Y → ω be a map such that card([|ξ|]f^{-1}) ≤ λ for every ξ < ω. Then there is a map g: Y → ω such that g ≤ f and card([|ξ|]g^{-1}) = λ for all ξ < ω.
Proof. Put $X = f[Y]$, let $h : X \to \kappa$ be a map from Lemma 4.2, and set $g = h \circ f$. The desired properties of the map $g$ follows readily from the properties of $h$. \qed

Lemma 4.4. Let $A$ be an uncountable abelian group. If $\dim(\tau A/p\tau A) = gen(A)$ for some prime number $p$, then $A$ has a minimal generating set.

Proof. Let $\kappa$ denote the cardinality of $A$. Let $U^\tau$ be a subset of $\tau A$ lifting a basis of $\tau A/p\tau A$ over $p\tau A$. Since $\tau A$ is pure subgroup of $A$ (e.g., see [2, p. 77]), $\tau A = \tau A \cap pA$. It follows that $U^\tau$ lifts a linearly independent subset of $A/pA$ over $pA$. Let $U^\tau \subseteq U$, where $U$ lifts a basis of $A/pA$ over $pA$. Let $U = \{u_\iota \mid \iota \in \kappa\}$ be some ordering of the set $U$. Put $I^\tau = \{\iota \in \kappa \mid u_\iota \in U^\tau\}$ and observe that $\text{card}(I^\tau) = \kappa$. Find pairwise disjoint subsets $I_q, q \in \hat{p}$, of $\kappa$ such that $\text{card}(I_q) = \dim(A/qA)$ for all $q \in \hat{p}$ and $\text{card}(I^\tau) = \kappa$, where we set $I^\tau = I^\tau \setminus \bigcup_{q \in \hat{p}} I_q$. For each $q \in \hat{p}$ pick a subset $Y_q = \{a_\iota \mid \iota \in I_q\}$ of $A$ such that $A = \text{Span}(Y_q) + qA$. Put $v_\iota = pa_\iota + qa_\iota$ for all $\iota \in I_q$ and all $q \in \hat{p}$, and $v_\iota = u_\iota$ for all $\iota \in \kappa \setminus I_q$. Set $V = \{v_\iota \mid \iota \in \kappa\}$ and $B = \text{Span}(V)$. Observe that $V$ lifts a basis of $A/pA$ over $pA$ and $A = B + qA$ for all $q \in \hat{p}$. It follows that $D = A/B$ is divisible.

Decompose $D = \oplus_{\alpha < \kappa} D_\alpha$ into a direct sum of at most countable divisible groups. For each $\alpha < \kappa$ pick a countable subgroup $E_\alpha$ of $A$ such that $D_\alpha = (E_\alpha + B)/B$ and a countable subset $J_\alpha$ of $\kappa$ such that $E_\alpha \cap B \subseteq \text{Span}(V_\alpha)$, where $V_\alpha = \{v_\iota \mid \iota \in J_\alpha\}$. Let $f : \kappa \to \kappa$ be a map defined by

$$f(\xi) = \begin{cases} \text{min}\{\alpha \mid \xi \in J_\alpha\} & \text{if } \xi \in J_\alpha \text{ for some } \alpha \in \kappa; \\ \xi & \text{otherwise.} \end{cases}$$

Put $f^* = f \upharpoonright I^\tau$. Observe that $[(\alpha)]^{f^{-1}} \subseteq J_\alpha \cup \{\alpha\}$, in particular, $[(\alpha)]^{f^{-1}}$ is at most countable for every $\alpha \in \kappa$. Consequently, $[(\alpha)]^{(f^*)^{-1}}$ is at most countable, for every $\alpha \in \kappa$. By Lemma 4.3, there is a map $g^* : I^\tau \to \kappa$ such that $g^* \leq f^*$ and $\text{card}([(\alpha)]^{(g^*)^{-1}}) = \aleph_0$ for all $\alpha \in \kappa$. Let $g : \kappa \to \kappa$ be a map such that $g \upharpoonright I^\tau = g^*$ and $g \upharpoonright (\kappa \setminus I^\tau) = f \upharpoonright (\kappa \setminus I^\tau)$. Observe that $\xi \in J_\alpha$ implies that $f(\xi) \leq \alpha$, for all $\xi$ and $\alpha$ from $\kappa$. It follows that $J_\alpha \subseteq [\alpha + 1]^{g^{-1}}$ for all $\alpha \in \kappa$.

Put $G_\alpha = \{v_\iota \mid \iota \in [(\alpha)]^{g^{-1}}\}$, resp. $G_{<\alpha} = \{v_\iota \mid \iota \in [\alpha]^{g^{-1}}\}$, and set $B_\alpha = \text{Span}(G_\alpha)$, resp. $B_{<\alpha} = \text{Span}(G_{<\alpha})$, for all $\alpha \in \kappa$. Observe that, since $\text{card}([(\alpha)]^{(g^*)^{-1}}) = \aleph_0$, we have that $\text{card}(G_\alpha \cap \tau A) = \aleph_0$, for all $\alpha \in \kappa$. Now put

$$C_\alpha = B_{<\alpha + 1}/B_{<\alpha} \simeq B_\alpha/(B_\alpha \cap B_{<\alpha}),$$

for all $\alpha \in \kappa$. Since $J_\alpha \subseteq [\alpha + 1]^{g^{-1}}$, we have that $V_\alpha \subseteq G_{<\alpha + 1}$, hence $E_\alpha \cap B \subseteq B_{<\alpha + 1}$, for all $\alpha \in \kappa$. It follows that

$$D_\alpha = (E_\alpha + B)/B \cong E_\alpha/(E_\alpha \cap B) \cong (E_\alpha \cap B_{<\alpha + 1})/(E_\alpha \cap B_{<\alpha}) \cong (E_\alpha + B_{<\alpha + 1})/B_{<\alpha},$$

for all $\alpha < \kappa$. Consequently, $D_\alpha \oplus C_{\alpha + 1} \cong (E_\alpha + B_{<\alpha + 2})/B_{<\alpha + 1}$, for all $\alpha \in \kappa$. The set $V$ lifts a basis of $A/pA$ over $pA$, in particular, it is $\delta$-independent. It follows that $G_{\alpha + 1}$ lifts a minimal generating set of $C_{\alpha + 1}$ over $B_{<\alpha + 1}$, for all $\alpha \in \kappa$. Denote this set by $H_{\alpha + 1}$ (note that $H_{\alpha + 1} = \pi(B_{<\alpha + 1}/G_{\alpha + 1})$ for all $\alpha \in \kappa$).

Since $\text{card}([(\alpha)]^{(g^*)^{-1}}) = \aleph_0$, we have that $\text{card}(G_{\alpha + 1} \cap \tau A) = \aleph_0$, we infer that $\text{card}(H_{\alpha + 1} \cap \tau (A/B_{\leq \alpha}))) = \aleph_0$, for all $\alpha \in \kappa$. It follows that, for all $\alpha \in \kappa$, the direct sum $D_\alpha \oplus C_{\alpha + 1}$ has a minimal generating set, say $W_{\alpha + 1}$, which lifts $H_{\alpha + 1}$ over $D_\alpha$ by Corollary 3.12. Since $D_\alpha$ is divisible, the set $W_{\alpha + 1}$ is formed by suitable elements $v_\iota + pe_\iota + B_{<\alpha + 1}$, where $\iota \in [(\alpha + 1)]^{g^{-1}}$ and $e_\iota \in E_\alpha$ for all
$\iota \in [(\alpha + 1)]g^{-1}$. Given $\alpha \in \kappa$ and $\iota \in [(\alpha)]g^{-1}$, we define

$$x_\iota = \begin{cases} v_\iota & \text{provided that ordinal } \alpha \text{ has no predecessor}, \\ v_\iota + pe_\iota & \text{otherwise,} \end{cases}$$

and put $X = \{x_\iota \mid \iota \in \kappa\}$.

**Claim 1.** The set $X$ forms a minimal generating set of $A$.

**Proof of Claim 1.** It is straightforward to see that the set $X$ is $S$-independent, indeed, $x_\iota + pA = v_\iota + pA$ for every $\iota \in \kappa$, whence $X$ lifts a basis of $A/pA$ over $pA$.

It remains to verify that $A = \text{Span}(X)$. Put $X_\alpha = \{x_\iota \mid \iota \in [(\alpha)]g^{-1}\}$, resp. $X_\alpha = \{x_\iota \mid \iota \in [\alpha]g^{-1}\}$, and set $A_\alpha = \text{Span}(X_\alpha)$, resp. $A_\alpha = \text{Span}(X_\alpha)$, for all $\alpha \in \kappa$. By transfinite induction on $\beta \in \kappa$ we prove simultaneously that $B_{< \beta} \subseteq A_{< \beta}$ and $E_\alpha \subseteq A_{< \beta}$, whenever $\beta = \alpha + 2$.

First, observe that $B_{< 0} = A_{< 0} = \text{Span}(\emptyset) = \{0\}$. Now let $0 < \beta \in \kappa$, and suppose that the assertion holds for all $\alpha \in \beta$. If $\beta$ is a limit ordinal, then $B_{< \beta} = \bigcup_{\alpha < \beta} B_\alpha \subseteq \bigcup_{\alpha < \beta} A_\alpha = A_{< \beta}$, by the induction hypothesis. Let $\beta = \alpha + 1$, where the ordinal $\alpha$ has no predecessor. Then $B_{< \alpha} \subseteq A_{< \alpha}$ by the induction hypothesis. Since $\alpha$ has no predecessor, it follows from (4.1) that $G_{\alpha} = V_\alpha$, hence $A_\alpha = B_\alpha$, whence $B_{< \alpha + 1} \subseteq A_{< \alpha + 1}$. Finally suppose that $\beta = \alpha + 2$ for some $\alpha \in \kappa$. Then $X_{\alpha + 1}$ lifts $W_{\alpha + 1}$ over $B_{< \alpha + 1}$ by (4.1) and $B_{< \alpha + 1} \subseteq A_{< \alpha + 1}$ by the induction hypothesis. Since $W_{\alpha + 1}$ forms a minimal generating set of $D_\alpha \oplus C_{\alpha + 1}$, we conclude that $E_\alpha + B_{< \alpha + 2} \subseteq A_{< \alpha + 2}$.

This verification concludes the proof of the statement. $\square$

**Proposition 4.5.** Let $A$ be a torsion abelian group. If $\text{Spec}(A, \neq) \neq \emptyset$, then $A$ has a minimal generating set.

**Proof.** If $A$ is uncountable, then the proposition follows from Lemma 4.4. Suppose that the group $A$ is countable and decompose it into $A = R \oplus D$, where $R$ is reduced and $D$ is divisible. Then $\dim(R/pR) = \dim(A/pA) = \aleph_0$, for some prime number $p$. In particular, $R$ is infinite. Since $R$ is torsion, it cannot be finitely generated, hence it has an infinite minimal generating set by [6, Lemma 4.3]. Applying Lemma 3.11, we conclude that $A$ has a minimal generating set. $\square$

**Lemma 4.6.** Let $A$ be a countable torsion abelian group. If $\text{Spec}(A, <)$ is infinite, then $A$ has a minimal generating set.

**Proof.** Let $B$ a finitely generated subgroup of $A$. Since $B$ is torsion, it is finite, hence $pB = B$ for all but finitely many primes. Thus there is a finite subset $F \subseteq \mathbb{P}$ such that $\dim(A/(pA + B)) = \dim(A/pA)$ for all $p \in \mathbb{P} \setminus F$. It follows that $\text{Spec}(A, <) \setminus F \subseteq \text{Spec}(A/B, <)$, thus $\text{Spec}(A/B, <)$ is infinite, in particular, it is non-empty. By Proposition 3.8, $A$ has a minimal generating set. $\square$

**Lemma 4.7.** Let $A$ be an abelian $p$-group and let $B$ be its basic subgroup. There is a minimal generating set $X$ of $B$ which lifts a basis of $A/pA$ over $pA$.

**Proof.** The group $B$ is a direct sum of cyclic $p$-groups, say $B = \bigoplus_{x \in X} C_x$, where $X$ denotes a set of generators of the cyclic summands in the decomposition and $C_x$ is a finite cyclic group generated by $x$ for all $x \in X$. Obviously, the set $X$ lifts a basis of $B/pB$ over $pB$. Since $B$ is a basic subgroup of $A$, the factor group $A/B$ is
divisible, whence \( A = B + pA \), and \( B \) is pure subgroup of \( A \), whence \( pB = B ∩ pA \). It follows that there is a natural isomorphism

\[
B/pB = B/(pA ∩ B) \cong (B + pA)/pA = A/pA,
\]
given by the correspondence \( x + pB \mapsto x + pA \), \( x \in X \). We conclude that \( X \) lifts a basis of \( A/pA \) over \( pA \). □

**Theorem 4.8.** Let \( A \) be a torsion abelian group of an infinite cardinality \( \kappa \) and let \( B \) be its basic subgroup. Then the following are equivalent.

1. \( A \) has a minimal generating set.
2. \( \text{card}(B) = \kappa \).
3. \( \sum_{p \in \mathbb{P}} \text{dim}(A/pA) = \kappa \).

**Proof.** (1 \( \Rightarrow \) 2) Since \( B \) is a basic subgroup of \( A \), the factor group \( A/B \) is divisible and the implication follows from [6, Lemma 1.3]. (2 \( \Rightarrow \) 3) By Lemma 4.7, \( \tau_pB \) has a minimal generating set \( X_p \) which lifts a basis of \( \tau_pA/\tau_pA \) over \( \tau_pA \), for every \( p \in \mathbb{P} \). Since \( \tau_pA = \tau_pA \), we get that \( \text{dim}(A/pA) = \text{dim}(\tau_pA/\tau_pA) = \text{card}(X_p) \), for all \( p \in \mathbb{P} \). Since \( B \) is an infinite torsion group,

\[
\text{card}(B) = \text{gen}(B) = \text{card}(\bigcup_{p \in \mathbb{P}} X_p) = \sum_{p \in \mathbb{P}} \text{card}(X_p) = \sum_{p \in \mathbb{P}} \text{dim}(A/pA).
\]

(3 \( \Rightarrow \) 1) If \( A \) is countable, then (3) implies that either \( \text{Spec}(A, =) \neq \emptyset \) or \( \text{Spec}(A, <) \) is infinite and we infer that \( A \) has a minimal generating set by Proposition 4.5 or by Lemma 4.6, respectively. If \( A \) is uncountable, then (3) implies that either \( \text{Spec}(A, =) \neq \emptyset \) or \( \sum_{p \in \text{Spec}(A, <)} \text{dim}(A/pA) = \kappa \). Then \( A \) has a minimal generating set by Proposition 4.5 or by Proposition 3.10, respectively. □

Theorem 4.8 can be simplified in case the cofinality of the cardinality of the group \( A \) is uncountable. Indeed, in this case \( \sum_{p \in \text{Spec}(A, <)} \text{dim}(A/pA) < \text{gen} A \), and we can simplify the statement of the previous theorem as follows:

**Corollary 4.9.** Let \( A \) be a torsion abelian group such that \( \text{cf}(\text{card}(A)) > \aleph_0 \). Then \( A \) has a minimal generating set iff \( \text{dim}(A/pA) = \kappa \) for some prime number \( p \) (i.e., \( \text{Spec}(A, =) \neq \emptyset \)).

5. **Torsion free abelian groups**

In [6] we did not succeed to characterize torsion free abelian groups with a minimal generating set. Here we complete this characterization. What in [6] was missing is Lemma 5.2. Roughly saying, it states that for a torsion free abelian group \( A \) to have a minimal generating set one prime in \( \text{Spec}(A, =) \) is not enough. We prove its more general version applicable also for mixed groups.

**Definition.** Let \( A \) be an abelian group and let \( X \subseteq A \). We say that a \( X \) is \( \mathbb{Z} \)-linearly independent if the only vanishing linear combination of elements of \( X \) with integer coefficients is a trivial combination.

The next lemma is well-known, we leave the proof to the reader.

**Lemma 5.1.** Let \( A \) be a torsion free abelian group and let \( p \) be a prime number. Then every \( X \subseteq A \) which lifts a linearly independent subset of \( A/pA \) over \( pA \) is \( \mathbb{Z} \)-linearly independent.
Lemma 5.2. Let $A$ be an abelian group. Suppose that there is a prime number $p$ and a subset $U$ of $A$ with $\text{card}(U) < \text{gen}(A)$ such that $A = qA + \text{Span}(U)$ for all $q \in \mathbb{P}$ and $\tau A \subseteq pA + \text{Span}(U)$. Then $A$ has not a minimal generating set.

Proof. We start with proving that the abelian group $A$ is not finitely generated. Suppose otherwise. Since a finitely generated abelian group is a direct sum of cyclic groups [5, Corollary 10.22], we observe that $\bigcap_{p \in \mathbb{P}} pA = 0$. It follows from our assumptions that $\tau A \subseteq pA + \text{Span}(U)$ for all $p \in \mathbb{P}$, hence $\tau A \subseteq \text{Span}(U)$. Since the group $\text{Span}(U)$ is finitely generated, we get that $\text{Span}(U) \simeq \tau A \oplus \phi \text{Span}(U)$.

The equality $\phi \text{Span}(U) = \text{rank } A$ would imply that $\text{Span}(U) \simeq A$ which is not the case, since $\text{gen}(\text{Span}(U)) \leq \text{card } U < \text{gen } (A)$. Applying [5, Exercise 10.15], we get that the group $\phi \text{Span}(U)$ has finite index in $\phi A$. It follows that the factor group $A/\text{Span}(U)$ is not finite. Since it is finitely generated, it has a free direct summand. But then $A/\text{Span}(U)$ is not divisible by any prime number, which contradicts our assumptions.

For the rest of the proof assume that $A$ is not finitely generated. Towards a contradiction, suppose that $A$ has a minimal generating set $X$. Pick a finite $X_0 \subseteq X$ such that $u \in \text{Span}(X_0)$, for every $u \in U$, and put $Y = \bigcup_{u \in X_0} X_u$. Observe that either $U$ is finite and then $Y$ is finite as well or $\text{card} (X) = \text{card} (Y)$. Put $Z = X \setminus Y$. Since the group $A$ is not finitely generated and $\text{card} (U) < \text{gen } (A) = \text{card} (X)$, we infer that $\text{card} (Y) < \text{card } (X)$, and so $\text{card} (Z) = \text{card } (X)$. Further deduce from the properties of the set $U$ that $A = qA + \text{Span}(Y)$ for all $q \in \mathbb{P}$ and $\tau A \subseteq pA + \text{Span}(Y)$.

Claim 1. The set $Z$ lifts a linearly independent subset of $A/(pA + \text{Span}(Y))$ over $pA + \text{Span}(Y)$.

Proof of Claim 1. Let $Z$ denote the collection of all subsets of $X$ which are $S$-independent over $pA + \text{Span}(Y)$, i.e., those subsets, which lift a linearly independent subset of $A/(pA + \text{Span}(Y))$ over $pA + \text{Span}(Y)$. The set $Z$ has a maximal element, say $Z'$, by Zorn’s lemma. Suppose that $Z' \neq Z$ and put $X' = Z' \cup Y$. Then $A/\text{Span}(X')$ is a nontrivial divisible group with a minimal generating set (lifted by nonempty $X \setminus X'$), which cannot be the case. □

Since $\tau A \subseteq pA + \text{Span}(Y)$, the set $Z$ is $S$-independent over $\tau A$. Put $Z_\tau = \pi_{\tau A}(Z)$. The set $Z$, and so the set $Z_\tau$ as well, lifts a linearly independent subset of $A/(pA + \text{Span}(Y))$ over $pA + \text{Span}(Y)$ by Claim 1. Since $\tau A \subseteq pA + \text{Span}(Y)$, we get that $Z_\tau$ lifts a linearly independent subset of $A/(pA + \tau A) \simeq (A/\tau A)/p(A/\tau A)$. By Lemma 5.1 we get that $Z_\tau$ is a $Z$-linearly independent subset of $A/\tau A$. That is, $\text{Span}(Z_\tau) = \text{Span}(Z) + \tau A/\tau A$ is a free subgroup of $A/\tau A$. Put $Y_\tau = \pi_{\tau A}(Y)$ and note that $\text{Span}(Y_\tau) = (\tau A + \text{Span } Y)/\tau A$ and $\text{Span}(Z_\tau) + \text{Span}(Y_\tau) = A/\tau A$.

Since $\text{card} (Y_\tau) \leq \text{card} (Y) < \text{card} (Z) = \text{card} (Z_\tau)$ and the group $\text{Span}(Z_\tau)$ is free of rank $\text{card} (Z_\tau)$, we infer that the group

$$A/(\tau A + \text{Span}(Y)) \simeq (A/\tau A)/\text{Span}(Y_\tau) = (\text{Span}(Z_\tau + \text{Span}(Y_\tau))/\text{Span}(Y_\tau))$$

has a nontrivial free direct summand. But this is impossible, since the group $A/(\tau A + \text{Span}(Y))$ is divisible by every $q \in \mathbb{P}$. □

Notice that Lemma 5.2 generalizes [6, Lemma 1.3]. For a torsion free abelian group we have its following corollary:
Corollary 5.3. Let $A$ be a torsion free abelian group. Suppose that there is a prime number $p$ and a subset $Y \subseteq A$ of cardinality less than $\text{gen}(A)$ such that $A = qA + \text{Span}(Y)$ for every prime $q \neq p$. Then $A$ has not a minimal generating set.

Theorem 5.4. A torsion free abelian group $A$ has a minimal generating set iff either $\text{card}(\text{Spec}(A,=)) \geq 2$ or
\[(5.1) \sum_{p \in \text{Spec}(A,<)} \dim(A/(pA + B)) = \text{gen}(A),\]
for every finitely generated subgroup $B$ of $A$.

Proof. ($\Leftarrow$) If $\text{card}(\text{Spec}(A,=)) \geq 2$, then the group $A$ has a minimal generating set due to Proposition 3.7, while if $(5.1)$ is satisfied, then $A$ is not finitely generated and it has a minimal generating set by Proposition 3.10, resp. Proposition 3.8 in case that $\text{gen}(A) > \aleph_0$, resp. $\text{gen}(A) = \aleph_0$. Note that if $\text{gen}(A) > \aleph_0$, then the equation $(5.1)$ can be simplified to $(3.1)$. ($\Rightarrow$) A finitely generated torsion free abelian group $A$ is free, in which case $\text{Spec}(A,=) = \mathbb{P}$. Thus we can assume that $A$ is not finitely generated. Suppose that $\text{card}(\text{Spec}(A,=)) \leq 1$. Then $\dim(A/pA) < \text{gen}(A)$ for all but a single prime, say $p$. Suppose that there is a subgroup $B$ of $A$ generated by a finite set $Y_0$ such that $\sum_{p \in \text{Spec}(A,<)} \dim(A/(pA + B)) < \text{gen}(A)$. It follows that there is a subset $Y$ of $A$ containing $Y_0$ such that $\text{card}(Y) < \text{gen}(A)$ and $\text{Span}(Y) + qA = A$ for every prime $q \neq p$. We conclude that $A$ has not a minimal generating set by Corollary 5.3. □

In case the cardinality of $A$ is of an uncountable cofinality, we can remove $(5.1)$ from the previous statement.

Corollary 5.5. Let $A$ be a torsion free abelian group. If the cardinality of $A$ has uncountable cofinality, then the group $A$ has a minimal generating set iff $\text{card}(\text{Spec}(A,=)) \geq 2$.

6. Mixed Groups - General Case

One would expect that combining the characterization in torsion and torsion-free case would suffice to characterize all abelian groups with a minimal generating set. It is quite true in the uncountable case while for countable abelian groups we need one more result, namely Lemma 6.4. Thus we will treat the uncountable and countable case separately and combine both the cases to gain the final characterization in Theorem 6.6.

6.1. Uncountable abelian groups.

Theorem 6.1. Let $A$ be an abelian group of an uncountable cardinality $\kappa$. Then $A$ has a minimal generating set iff at least one of the following conditions is satisfied:

1. $\text{card}(\text{Spec}(A,=)) \geq 2$;
2. $\dim(\tau A/prA) = \kappa$ for some prime number $p$;
3. $\sum_{p \in \text{Spec}(A,<)} \dim(A/pA) = \kappa$.

Proof. ($\Leftarrow$) If $\text{Spec}(A,=)$ has at least two elements, then $A$ has a minimal generating set by Proposition 3.7, if $\dim(\tau A/prA) = \kappa$, for some prime number $p$, then $A$ has a minimal generating set by Lemma 4.4, and, finally, if $\sum_{p \in \text{Spec}(A,<)} \dim(A/pA) = \kappa$, then $A$ has a minimal generating set by Proposition 3.10. ($\Rightarrow$) Suppose that
\[
\sum_{p \in \text{Spec}(A, <)} \dim(A/pA) < \infty. \quad \text{If Spec}(A, =) = \emptyset, \text{then} \sum_{p \in \mathbb{Z}} \dim(A/pA) < \infty \quad \text{and} \quad \text{the group} \ A \ \text{has not a minimal generating set by Lemma 4.1. Let Spec}(A, =) = \{p\} \ \text{for a single prime} \ p. \ \text{Then there is a subset} \ Y \subseteq A \ \text{with} \ \text{card}(Y) < \infty \ \text{such that} \ A = qA + \text{Span}(Y) \ \text{for all} \ q \in \mathbb{Z}. \ \text{If, moreover,} \ \dim(\tau A/prA) < \infty, \ \text{then} \ \tau A \subseteq pA + \text{Span}(Z) \ \text{for some} \ Z \subseteq A \ \text{with card}(Z) < \infty. \ \text{Putting} \ U = Y \cup Z, \ \text{we conclude that} \ A \ \text{has not a minimal generating set by Lemma 5.2}. \]

**Corollary 6.2.** Let \( A \) be an abelian group and suppose that cardinality of \( A \) is of an uncountable cofinality. Then \( A \) has a minimal generating set if either \( \text{card}(\text{Spec}(A, =)) \geq 2 \) or \( \dim(\tau A/prA) = \text{card}(A) \) for some prime number \( p \).

Combining Theorem 6.1 with Theorem 4.8 and Theorem 5.4 we get readily another of its corollaries.

**Corollary 6.3.** Let \( A \) be an uncountable abelian group. Then \( A \) has a minimal generating set if either \( \tau A \) has a minimal generating set and \( \text{card}(\tau A) = \text{card}(A) \) or \( A/\tau A \) has a minimal generating set and \( \text{card}(A/\tau A) = \text{card}(A) \).

### 6.2. Countable abelian groups and the final statement.

**Lemma 6.4.** Let \( A \) be a countable abelian group such that \( \dim(\tau A/prA) = \aleph_0 \) for some prime \( p. \) Then \( A \) has a minimal generating set.

**Proof.** If there is a prime \( q \neq p \) such that \( \dim(A/qA) = \aleph_0 \), then \( \text{Spec}(A, =) \geq 2 \) and \( A \) has a minimal generating set by Proposition 3.7. Thus we can suppose that \( \text{Spec}(A, =) = \{p\} \). If \( \text{Spec}(A/B, <) \neq \emptyset \) for every finitely generated subgroup \( B \) of \( A \), then \( A \) has a minimal generating set by Proposition 3.8. So assume that there is a finitely generated subgroup \( B \) of \( A \) such that \( A = qA + B \) for all \( q \in \mathbb{Z} \). Since the subgroup \( B \) is finitely generated, \( \dim(\tau A/B) = \text{card}(\tau A) = \text{card}(A) \) and \( A \) has a minimal generating set if \( A/B \) has a minimal generating set by [6, Lemma 5.1]. Observe that the factor group \( A/B \) is divisible by every \( q \in \mathbb{Z} \), thus replacing \( A \) with \( A/B \), we can without loss of generality assume that \( qA = A \) for all \( q \in \mathbb{Z} \).

Put \( T = \tau A, \ \Phi = \phi A, \) and \( A' = A/pT \). Observe that \( \tau A' = T/pT \) is a bounded subgroup of \( A' \), hence it is its direct summand by [5, Corollary 10.42]. Therefore \( A' = \tau A' + \phi A' \simeq (T/pT) + \Phi \). Let \( Z' \subseteq \Phi \) lift a basis of \( \Phi/p\Phi \). Put \( \Phi' = \Phi/\text{Span}(Z') \). Recall that \( \dim(T/pT) = \aleph_0 \), and so we can pick linearly independent \( Y'' \subseteq T/pT \) such that \( \dim(\text{Span}(Y'')) = \text{codim}(\text{Span}(Y'')) = \aleph_0 \). Put \( A'' = \text{Span}(Y'') \oplus \Phi' \). Observe that the group \( \Phi' \) is divisible, hence \( A'' \) has a minimal generating set \( Y' \) which lifts \( Y'' \) over \( \Phi' \) by Corollary 3.12. Let \( Z \) be a subset of \( A \) which lifts \( Z' \) over \( pT \), and let \( Y \) be a subset of \( A \) which lifts \( Y' \) over \( \text{Span}(Z) + pT \). Denote by \( C \) the subgroup of \( A \) generated by \( Y \cup Z \). Observe that \( A = C + T \), hence the group \( T' = A/C \simeq T/(C \cap T) \) is torsion. Since \( \dim(T'/pT') = \text{codim}(\text{Span}(Y'')) = \aleph_0 \), the group \( T' \) has a minimal generating set \( X' \) by Proposition 4.5. Note that \( X' \) lifts a basis of \( T'/pT' \) over \( pT' \), indeed, \( T' = \text{Span}(X') + pT' \) and if \( T' = \text{Span}(X'') + pT' \) for some \( X'' \subseteq X' \), we would get a nontrivial divisible group \( T'/\text{Span}(X'') \) with a minimal generating set (corresponding to the canonical image of \( X' \odot X'' = \emptyset \)).

Let \( X \) lift \( X' \) over \( C \). Note that \( Y \) lifts a linearly independent subset of \( A/(pA + \text{Span}(Z)) \) over \( pA + \text{Span}(Z) \) and \( X \) lifts a linearly independent subset of \( A/(pA + C) = A/(pA + \text{Span}(Y \cup Z)) \) over \( pA + \text{Span}(Y \cup Z) \). It follows that \( X \cup Y \cup Z \) lifts a linearly independent subset of \( A/pA \), in particular, the union \( X \cup Y \cup Z \) forms an \( S \)-independent subset of \( A \). Since \( A = \text{Span}(X) + C \) and \( C = \text{Span}(Y \cup Z) \), we have
that $A = \text{Span}(X \cup Y \cup Z)$. We conclude that $X \cup Y \cup Z$ is a minimal generating set of $A$. □

**Theorem 6.5.** Let $A$ be an infinitely generated countable abelian group. Then $A$ has a minimal generating set iff at least one of the following conditions is satisfied:

1. $\text{card}(\text{Spec}(A, =)) \geq 2$;
2. $\dim(\tau A/pr A) = \aleph_0$ for some prime number $p$;
3. $\text{Spec}(A/B, <) \neq \emptyset$ for every finitely generated subgroup $B$ of $A$.

**Proof.** $(\Leftarrow)$ If $\text{Spec}(A, =) \geq 2$, then $A$ has a minimal generating set by Proposition 3.7, if $\dim(\tau A/pr A) = \aleph_0$ for some prime number $p$, then has a minimal generating set by Lemma 6.4, and if $\text{Spec}(A/B, <) \neq \emptyset$ for every finitely generated subgroup $B$ of $A$, then the existence of a minimal generating set of $A$ follows from Proposition 3.8. $(\Rightarrow)$ Suppose that $\text{Spec}(A/B, <) = \emptyset$ for some finitely generated subgroup $B$ of $A$. Note that $\text{Spec}(A, =) = \text{Spec}(A/B, =)$ and $\dim(\tau A/pr A) = \aleph_0$ iff $\dim(\tau A/B/pr A) = \aleph_0$ for each prime number $p$. By [6, Lemma 5.1], the group $A$ has minimal generating set iff the factor-group $A/B$ has a minimal generating set. Thus, replacing $A$ by $A/B$, we can without loss of generality assume that $\text{Spec}(A, <) = \emptyset$. If $\text{Spec}(A, =) = \emptyset$, then $A$ is divisible and so it has not a minimal generating set. If $\text{Spec}(A, =) = \{p\}$ for a single prime $p$ and $\dim(\tau A/pr A) < \aleph_0$, then $A$ has not a minimal generating set by Lemma 5.2. □

Note that applying Corollary 3.9, we can replace property (3) in Theorem 6.5 by requiring that $\sum_{p \in \text{Spec}(A, <)} \dim(A/(pA + B)) = \text{gen}(A)$ for every finitely generated subgroup $B$ of $A$. Combining Theorem 6.1 and Theorem 6.5 treating uncountable and countable case, respectively, we get the final statement of the paper characterizing abelian groups with a minimal generating set.

**Theorem 6.6.** Let $A$ be an infinitely generated abelian group. The group $A$ has a minimal generating set iff at least one of the following conditions is satisfied:

1. $\text{card}(\text{Spec}(A, =)) \geq 2$;
2. $\dim(\tau A/pr A) = \text{gen}(A)$ for some prime number $p$;
3. $\sum_{p \in \text{Spec}(A, <)} \dim(A/(pA + B)) = \text{gen}(A)$ for every finitely generated subgroup $B$ of $A$.

Moreover, if the group $A$ is uncountable, property (3) can be simplified to

3'. $\sum_{p \in \text{Spec}(A, <)} \dim(A/pA) = \text{gen}(A)$,

and if the cardinality of the group $A$ has uncountable cofinality, then $A$ has a minimal generating set iff any of properties (1) and (2) is satisfied.

**References**
