TRANSFINITE ALGORITHMS SEARCHING FOR MAXIMAL WEAKLY INDEPENDENT SUBSETS

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Abstract. We study a question whether a generating set of subspaces of a right vector space of a bounded finite dimension contains a generating subset minimal with respect to inclusion. We obtain partial positive results as well as a complete answer to the set theoretic analogy of the question.

1. Introduction

All rings are supposed to be associative with a unit element. By R-modules we will mean right modules over a ring R. Let $\operatorname{Sub} M$, resp. $\operatorname{Sub}_k M$ denote the set of all finitely generated submodules of an R-module M, resp. the set of all submodules of M generated by at most k-elements.

Given a ring R, an R-module M, and a subset X of M we denote by $\operatorname{Span}_R(X)$ the submodule of M generated by X. However, for a set $\mathcal{F}$ of submodules we will often prefer the notation $\sum \mathcal{F}$ for a submodule generated by $\bigcup \mathcal{F}$.

Let R be a ring, let M be an R-module and let N be a submodule of M. We say that a set X of elements of M is *weakly independent* over N provided that

$$x \notin N + \operatorname{Span}(X \setminus \{x\}),$$

for all $x \in X$. We say that the set X is *weakly independent* if it is weakly independent over the zero submodule. If $M = N + \operatorname{Span}(X)$ and X is weakly independent over N we will call $\mathcal{F}$ a *weak basis* of M over N. A weak basis of M over a zero submodule will be called a *weak basis* of M. A left R-module possessing a weak basis is called *weakly based* and the module is called *regularly weakly based* providing each of generating sets of the module contains a weak basis.

We will apply a similar concept of the weak independence on sets of submodules of a module. In particular, given a ring R, a left R-module M, and its submodule N, we say that a set $\mathcal{F}$ of submodules of M is *weakly independent* over N provided that

$$S \nsubseteq N + \sum (\mathcal{F} \setminus \{S\}),$$

for all $S \in \mathcal{F}$. We say plainly that $\mathcal{F}$ is *weakly independent* if it is weakly independent over the zero submodule. Further, the set $\mathcal{F}$ is a *weak basis* of M provided that $\mathcal{F}$ is weakly independent and $\sum \mathcal{F} = M$. The set $\mathcal{F}$ of submodules of M is *weakly based*

Date: 27/3/2015.

2000 *Mathematics Subject Classification.* 97E60, 15A03, 06C99, 16L30.

Key words and phrases. Minimal cover, weak basis, regularly weakly based, perfect rings, distributivity.

The first author was partially supported ... The second author was partially supported by the project SVV-2012-265317 of Charles University in Prague. The second and the third authors were partially supported by the Grant Agency of the Czech Republic under the grant no. GACR 14-15479S..

providing it contains a weak basis and it is *regularly weakly based* if each $\mathcal{F}' \subseteq \mathcal{F}$ with $\sum \mathcal{F}' = M$ contains a weak basis.

Compare the above defined concept of weak independence to the usual notion of independence of modules: In particular, a set $\mathcal{F}$ of submodules of M is *independent* over N provided that

$$S \cap \sum (\mathcal{F} \setminus \{S\}) \subseteq N,$$

for all $S \in \mathcal{F}$, and $\mathcal{F}$ is *independent* provided that it is independent over the zero submodule.

Nashier and Nichols in [8] asked to characterize rings over which all right modules are regularly weakly based and proved that such rings must be right perfect. Specifically, they asked whether all modules over right perfect rings are regularly weakly based. In [5] the authors reduced the question to: "Are all modules over semisimple rings regularly weakly based?" By well-known Wedderburn-Artin theorem, semisimple rings correspond to finite direct products of simple artinian rings.

Being aware of this, it seems natural to decompose the original problem into two special cases:

(1) Is every right module over a simple artinian ring regularly weakly based?
(2) Is every module over a finite product of division rings regularly weakly based?

In the paper we concentrate on the first one, proving some partial results and extracting a particular question whose answer should give a decisive clue to above stated problems. Yet before that, let us discuss briefly the second case. It is connected to the problem of characterization of regularly weakly based modules over Dedekind domains that the authors aimed in [5]. They proved that a regularly weakly based module over a Dedekind domain is isomorphic to a direct sum $F \oplus B$ of a finitely generated free module F and a bounded torsion module B. This, together with [5, Lemma 3.1], reduces the problem to bounded torsion modules which, in fact, correspond to modules over non trivial factors of Dedekind domains. Furthermore, we can factor out the multiple by the Jacobson radicals of the factors, which is nilpotent (see [5, Lemma 2.3]), and even reduce the problem to modules over finite direct product of fields. This, indeed, corresponds to the commutative version of the second case.

Let R be a ring, n a positive integer, and let M be a right module over the full matrix ring $S = \mathbb{M}_n(R)$. We denote by $\boldsymbol{e}_{ij}$, $i, j = 1, \ldots, n$, the matrix units of S. Then there is a one-to-one correspondence, say ϕ, between cyclic submodules of M and submodules of the right R-module $M\boldsymbol{e}_{11}$ generated by at most n elements given by

$$xS \mapsto \mathrm{Span}_R\{x\boldsymbol{e}_{11}, x\boldsymbol{e}_{21}, \ldots, x\boldsymbol{e}_{n1}\}.$$

Moreover, $x \in \mathrm{Span}_S(Y)$, for a given $x \in M$ and $Y \subseteq M$ if and only if $\phi(xS) \subseteq \sum \phi(Y)$. This is folklore. It follows that elements $x_1, \ldots, x_n \in M$ are weakly independent if and only if the corresponding submodules $\phi(x_1S), \ldots, \phi(x_nS)$ of an R-module $M\boldsymbol{e}_{11}$ are. Therefore the S-module M is weakly based if and only if the set $\mathrm{Sub}_n(M\boldsymbol{e}_{11})$ is weakly based.

A ring R is simple right artinian ring if and only if it has a simple right generator [1, Proposition 13.5], say T. As T is simple, its endomorphism ring, D, is a division ring. If these holds true, then R is isomorphic to the full matrix ring $\mathbb{M}_n(D)$ for some positive integer n [1, Theorem 13.4]. Following the discussion above a right

R-module M is weakly based if and only if the set $\mathrm{Sub}_n(M\boldsymbol{e}_{11})$ of right D-vector spaces is weakly based. Thus the problem whether all right artinian rings are regularly weakly based is equivalent to the following:

Problem 1.1. *Given a positive integer n and a vector space V over a division ring, is the set $\mathrm{Sub}_n(V)$ regularly weakly based?*

We will attack this problem, first proving its set theoretic analogy, then proving that every $\mathcal{F} \subseteq \mathrm{Sub}_n(V)$ such that $V = \sum \mathcal{F}$ and satisfying

$$S = \bigoplus_{i \in I} (S \cap U_i) \quad \text{for all} \quad S \in \mathcal{F}.$$

for some decomposition $V = \bigoplus_{i \in I} U_i$ into a direct sum of finitely dimensional subspaces contains a weak basis, and finally generalizing this to the case when one of the subspaces U_i, say W, is of an infinite dimension and $\dim(S \cap W) \leq 1$ for all $S \in \mathcal{F}$.

2. Set theoretic problems

We start our attack to Problem 1.1 from another perspective, solving a purely set theoretic version of the problem. Despite relative simplicity of the solution, it is not trivial, and we believe that it is noteworthy on its own. First we need to clarify the notation and develop terminology to state the problem and describe its solution.

We let **On** denote the ordered class of all ordinal numbers and ω the first infinite ordinal (corresponding with the set of all nonnegative integers). We identify each ordinal number with the set of all its predecessors.

Given a set X, we denote by $|X|$ its size. By $\mathcal{P}(X)$ we denote the set of all subsets of X and, given a positive integer n, we denote by $[X]^{<\omega}$, resp. $[X]^{\leq n}$, the set of all finite subsets of the set X, resp. the set of all subsets of X of size at most n.

Fix a couple of sets C, D, a binary relation $\Theta \subseteq C \times D$ and subsets $A \subseteq C$ and $B \subseteq D$. We define

$$A\Theta = \{d \in D \mid (a, d) \in \Theta \quad \text{for some} \quad a \in A\},$$
$$\Theta B = \{c \in C \mid (c, b) \in \Theta \quad \text{for some} \quad b \in B\}.$$

Given $a \in A$, $b \in B$, we will write $a\Theta$, Θb for $\{a\}\Theta$, $\Theta\{b\}$, respectively.

We say that A *covers* B (or A is a *cover of* B) whenever $B \subseteq A\Theta$. We say that A is a *cover* providing it covers the whole D.

We say that A is *minimal on* B provided that for all $a \in A$ there is $b \in B$ such that $(a, b) \in \Theta$ but $(a, b') \notin \Theta$ for all $b \neq b' \in B$. Observe that A is minimal on B iff there is $W \subseteq B$ such that $(A \times W) \cap \Theta$ forms a graph of a surjective map $W \to A_0$. Formally, we say that $W \subseteq B$ *witnesses* the minimality of A provided that $A \subseteq \Theta W$ and $|A \cap \Theta w| = 1$ for all $w \in W$, and A is minimal on B iff B contains a subset witnessing the minimality. The subset A is said to be *minimal* providing it is minimal on the whole D.

By a *minimal cover of* B we mean $A \subseteq C$ covering B minimal on B. Observe that a minimal cover of B is its cover minimal w.r.t. inclusion. A *minimal cover* is a minimal cover of D.

We say that A is *bounded by* a positive integer n on B provided that

$$|a\Theta \cap B| \leq n \tag{2.1}$$

for all $a \in A$. We say that A is *bounded* on B provided that it is bounded by n on B for some $n \in \mathbb{N}$. The set A is bounded by n, resp. bounded providing it is bounded by n on D, resp. bounded on D.

Proposition 2.1. *Let A be a cover of B bounded on B. Then A contains a minimal cover of B.*

Proof. We will proceed by induction on n such that A is bounded by n on B. When $n = 1$, we use that A covers B and pick for all $b \in B$ some $\theta(b) \in A$ such that $(b, \theta(b)) \in \Theta$. We put $M = \{\theta(b) \mid b \in B\}$ and observe that $\Theta \cap (M \times B)$ equals to the graph of θ. It follows that M is a minimal cover of B. Suppose that $1 < n$ and that the statement holds for all $n' < n$.

Applying Zorn's lemma we find $A_0 \subseteq A$ maximal with respect to the property

$$|B \cap a\Theta \cap a'\Theta| = \emptyset \tag{2.2}$$

for all $a \neq a'$ in A_0. We put $B_1 = B \setminus A_0\Theta$ and $A_1 = A \setminus A_0$. It follows from the maximality of A_0 that $a\Theta \cap A_0\Theta \neq \emptyset$ for all $a \in A_1$. From this we infer that A_1 is bounded by $n-1$ on B_1. By the induction hypothesis we find a minimal cover $M_1 \subseteq A_1$ of B_1. Let $W_1 \subseteq B_1$ be witnessing the minimality of A_1.

Put $W_0 = B \setminus M_1\Theta$ and $M_0 = \{a \in A_0 \mid a\Theta \cap W_0 \neq \emptyset\}$. It follows from (2.2) that M_0 is minimal on W_0 and W_0 witnesses the minimality. Since M_1 covers $B_1 = B \setminus A_0\Theta$, we have that $W_0 \subseteq A_0\Theta$, whence M_0 covers W_0. We conclude that $M = M_1 \cup M_2$ covers $B = B_1 \cup W_0$. By the definition, $M_1\Theta \cap W_0 = \emptyset$. Since $M_0 \subseteq A_0$, we have that $M_0\Theta \cap B_1 = \emptyset$, whence $M_0\Theta \cap W_1 = \emptyset$. It follows that M is minimal on B and $W = W_0 \cup W_1$ witnesses the minimality. We conclude that M is a minimal cover of B. □

Let $Y \subseteq X$ be sets and $\mathcal{A} \subseteq \mathcal{P}(X)$. We say that $\mathcal{A}$ is a cover of Y if $Y \subseteq \bigcup \mathcal{A}$. A *minimal cover* of Y is its cover minimal w.r.t. inclusion. Let n be a positive integer.

Applying Proposition 2.1 when Θ corresponds to the relation of "$\in$" on $X \times \mathcal{P}(X)$ we get the following corollary:

Corollary 2.2. *Let $Y \subseteq X$ be sets. Then every cover $\mathcal{A} \subseteq \mathcal{P}(X)$ of Y such that for some positive integer n, $|Y \cap S| \leq n$ for all $S \in \mathcal{A}$ contains a minimal cover.*

Let X be a set and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$. We say that $\mathcal{A}$ *saturates* $\mathcal{B}$ provided that $T \cap \bigcup \mathcal{A} \neq \emptyset$ for all $T \in \mathcal{B}$ and $\mathcal{A}$ is said to be *minimal saturating* $\mathcal{B}$ if $\mathcal{A}$ is minimal w.r.t. inclusion such that it saturates $\mathcal{B}$.

Putting

$$\Theta = \{(S, T) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid S \cap T \neq \emptyset\},$$

we get the following corollary of Proposition 2.1

Corollary 2.3. *Let X be a set and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$. If $\mathcal{A}$ saturates $\mathcal{B}$ and there is a positive integer n such that*

$$|\{T \in \mathcal{B} \mid S \cap T \neq \emptyset\}| \leq n$$

for all $S \in \mathcal{A}$, then $\mathcal{A}$ contains a subset which is minimal saturating on $\mathcal{B}$.

The next lemma, concluding the section, is closely related to Proposition 2.1.

Lemma 2.4. *Let $n \in \mathbb{N}$, let X be a set and $\mathcal{A} \subseteq [X]^n$, $\mathcal{B} \subseteq \mathcal{P}(X)$. If $\mathcal{A}$ saturates $\mathcal{B}$ then it contains a subset minimal saturating $\mathcal{B}$.*

Proof. We proceed by induction on n in the same fashion as in Proposition 2.1. □

3. Modular problem

Within this section fix a division ring D. By a vector space we mean a right vector space over D.

Definition. Let $\mathcal{F}$ be a collection of subspaces of a vector space V. We say that a subspace U of V is *$\mathcal{F}$-distributive* provided that

$$U \cap \sum \mathcal{G} = \sum_{T \in \mathcal{G}} (U \cap T) \tag{3.1}$$

for all $\mathcal{G} \subseteq \mathcal{F}$. Observe that U is $\mathcal{F}$-distributive iff (3.1) holds for all finite subsets $\mathcal{G}$ of $\mathcal{F}$.

Lemma 3.1. *Let V be a vector space, let $\mathcal{F} \subseteq \operatorname{Sub} V$, and let $\mathcal{H} \subseteq \mathcal{F}$. Further, let $\mathcal{G}$ be an independent subset of $\mathcal{F}$, let σ an ordinal number, and let $\langle \mathcal{G}_\alpha \mid \alpha < \sigma \rangle$ be a decreasing family (i.e. $\mathcal{G}_\alpha \supseteq \mathcal{G}_\beta$ for all $\alpha \le \beta < \sigma$) of subsets of $\mathcal{G}$. Finally, let U be a finitely generated $\mathcal{F}$-distributive subspace of V. Then*

$$U \cap \left(\sum \bigcap_{\alpha < \sigma} \mathcal{G}_a + \sum \mathcal{H} \right) = U \cap \left(\sum \mathcal{G}_\beta + \sum \mathcal{H} \right) \tag{3.2}$$

for some $\beta < \sigma$.

Proof. First observe that the assumptions that the family $\langle \mathcal{G}_\alpha \mid \alpha < \sigma \rangle$ is decreasing and the sets $\mathcal{G}_\alpha$ are independent implies

$$\sum \bigcap_{\alpha < \sigma} \mathcal{G}_\alpha = \bigcap_{\alpha < \sigma} \sum \mathcal{G}_\alpha \tag{3.3}$$

(a weaker assumption that the union $\bigcup_{\alpha < \sigma} \mathcal{G}_\alpha$ is independent would guarantee (3.3)). Applying (3.3) and the distributivity of U we get that

$$U \cap \left(\sum \bigcap_{\alpha < \sigma} \mathcal{G}_a + \sum \mathcal{H} \right) = \left(\bigcap_{\alpha < \sigma} \sum_{T \in \mathcal{G}_a} (U \cap T) \right) + \left(U \cap \sum \mathcal{H} \right). \tag{3.4}$$

For each $\alpha < \sigma$ put $U_\alpha = \sum_{T \in \mathcal{G}_a} (U \cap T)$. Then $\langle U_\alpha \mid \alpha < \sigma \rangle$ is a decreasing family of subspaces of U and, since U is finitely generated, there is $\beta < \sigma$ with $U_\beta = \bigcap_{\alpha < \sigma} U_\alpha$. Substituting this to (3.4) and using distributivity of U again, we conclude with (3.2). □

For $\mathcal{F} \subseteq \operatorname{Sub} V$ let $\Delta(\mathcal{F})$ denote the set of all finitely generated $\mathcal{F}$-distributive subspaces of $\sum \mathcal{F}$. It is easy to see from the definition that $U \cap \sum \mathcal{F}$ is $\mathcal{F}$-distributive for every $\mathcal{F}$-distributive $U \in \operatorname{Sub} V$. Thus

$$\Delta(\mathcal{F}) = \{ U \cap \sum \mathcal{F} \mid U \text{ is a finitely generated } \mathcal{F}\text{-distributive subspace of } V \}.$$

Corollary 3.2. *Let V be a vector space, $\mathcal{F} \subseteq \operatorname{Sub} V$, and let $\mathcal{G}$ be an independent subset of $\mathcal{F}$. Let β be an ordinal number and $\langle \mathcal{G}_\alpha \mid \alpha < \beta \rangle$ a decreasing family of subsets of $\mathcal{G}$. Suppose that $\mathcal{H} \subseteq \mathcal{F}$ satisfies that*

$$\sum \Delta(\mathcal{F}) \subseteq \sum \mathcal{G}_\alpha + \sum \mathcal{H}, \quad \textit{for all } \alpha < \beta.$$

Then

$$\sum \Delta(\mathcal{F}) \subseteq \sum \bigcap_{\alpha<\beta} \mathcal{G}_\alpha + \sum \mathcal{H}.$$

Given a set $\mathcal{F}$ of subspaces of a vector space V and a subspace X of V we will use the notation

$$\mathcal{F}^X = \{(S+X)/X \mid S \in \mathcal{F}\}.$$

Lemma 3.3. *Let $\mathcal{F}$ be a set of subspaces of a vector space V, let $\mathcal{F}_0 \subseteq \mathcal{F}$, and let $W = \sum \mathcal{F}_0$. If a subspace U of V is $\mathcal{F}$-distributive, then the factor $(U+W)/W$ is $\mathcal{F}^W$ distributive.*

Proof. We need to verify that for every $\mathcal{G} \subseteq \mathcal{F}$,

$$(U+W) \cap \sum_{T\in\mathcal{G}} (T+W) = \sum_{T\in\mathcal{G}} (U+W) \cap (T+W).$$

Applying modularity and $\mathcal{F}$-distributivity of U we get that

$$(U+W) \cap \sum_{T\in\mathcal{G}} (T+W) = \left(\sum_{T\in\mathcal{G}} U \cap (T+W) \right) + W.$$

Using modularity again we conclude that

$$\left(\sum_{T\in\mathcal{G}} U \cap (T+W) \right) + W = \sum_{T\in\mathcal{G}} (U+W) \cap (T+W).$$

□

Corollary 3.4. *Let $\mathcal{F}$ be a set of subspaces of a vector space V, let $\mathcal{F}_0 \subseteq \mathcal{F}$, and let $W = \sum \mathcal{F}_0$. Then*

$$\Delta(\mathcal{F})^W \subseteq \Delta(\mathcal{F}^W).$$

Theorem 3.5. *Let V be a vector space, let k be a positive integer, and let $\mathcal{F} \subseteq \mathrm{Sub}_k V$. Then there is a weakly independent $\mathcal{K} \subseteq \mathcal{F}$ with $\sum \Delta(\mathcal{F}) \subseteq \sum \mathcal{K}$.*

Proof. We proceed by induction on k (such that $\mathcal{F} \subseteq \mathrm{Sub}_k V$). The proof is elementary when $k = 1$. Suppose that $k > 1$ and that the statement holds whenever $\mathcal{F} \subseteq \mathrm{Sub}_{k-1} V$.

First, we find, applying Zorn's lemma, a maximal independent subset $\mathcal{G}$ of $\mathcal{F}$. Next, we are going to construct stepwise a decreasing family $\langle \mathcal{G}_i \mid i < \omega \rangle$ of subsets of $\mathcal{G}$ and an increasing family $\langle \mathcal{H}_i \mid i < \omega \rangle$ (i.e. $\mathcal{H}_i \subseteq \mathcal{H}_j$ for all $i \le j < \omega$) of subsets of $\mathcal{F}$ such that, for all $n < \omega$,

- (1_n) $\mathcal{G}_n$ is independent over $\sum \mathcal{H}_n$,
- (2_n) $\mathcal{H}_{n+1}$ is weakly independent over $\sum \mathcal{G}_n$,
- (3_n) $\sum \Delta(\mathcal{F}) \subseteq \sum \mathcal{H}_{n+1} + \sum \mathcal{G}_n$.

We put $\mathcal{H}_0 = \emptyset$ and $\mathcal{G}_0 = \mathcal{G}$; thus property (1_0) is trivially satisfied. Further, we set $W_0 = \sum \mathcal{G}_0$. Since $\mathcal{G}$ is maximal independent, $S \cap W_0 \ne 0$ for all $S \in \mathcal{F}$. It follows that $\mathcal{F}^{W_0} \subseteq \mathrm{Sub}_{k-1}(V/W_0)$. By the induction hypothesis, there is $\mathcal{H}_1 \subseteq \mathcal{F}$ weakly independent over W_0 with $\sum \Delta(\mathcal{F}^{W_0}) \subseteq \sum \mathcal{H}_1^{W_0}$; in particular, property (2_0) holds true. By Corollary 3.4 we have that $\Delta(\mathcal{F})^{W_0} \subseteq \Delta(\mathcal{F}^{W_0})$, hence $\Delta(\mathcal{F}) \subseteq \sum \mathcal{H}_1 + W_0 = \sum \mathcal{H}_1 + \sum \mathcal{G}_0$; this is (3_0).

Let $n < \omega$ and suppose that we have constructed sets $\mathcal{H}_n$, $\mathcal{H}_{n+1}$ and $\mathcal{G}_n$ so that properties $(1_n - 3_n)$ are satisfied. In order to take the next step, we use

Zorn's lemma to find $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n$ maximal independent over $\sum \mathcal{H}_{n+1}$. This ensures property (1_{n+1}). We put $W_{n+1} = \sum \mathcal{H}_{n+1} + \sum \mathcal{G}_{n+1}$. Since $\mathcal{G}_{n+1}$ is a maximal subset of $\mathcal{G}_n$ independent over $\sum \mathcal{H}_{n+1}$, the inclusion $\mathcal{G}_n^{W_{n+1}} \subseteq \mathrm{Sub}_{k-1}(V/W_{n+1})$ holds true. By the induction hypothesis, there is $\mathcal{H}'_{n+1} \subseteq \mathcal{G}_n$ weakly independent over W_{n+1} with $\Delta(\mathcal{G}_n^{W_{n+1}}) \subseteq \sum (\mathcal{H}'_{n+1})^{W_{n+1}}$. By Corollary 3.4 we have that $\Delta(\mathcal{G}_n)^{W_{n+1}} \subseteq \Delta(\mathcal{G}_n^{W_{n+1}})$. The inclusion $\mathcal{G}_n \subseteq \mathcal{F}$ implies that $\Delta(\mathcal{F}) \subseteq \Delta(\mathcal{G}_n)$, whence $\sum \Delta(\mathcal{F})^{W_{n+1}} \subseteq \Delta(\mathcal{G}_n)^{W_{n+1}}$. Altogether we get that $\sum \Delta(\mathcal{F})^{W_{n+1}} \subseteq \sum (\mathcal{H}'_{n+1})^{W_{n+1}}$, hence $\sum \Delta(\mathcal{F}) \subseteq \sum \mathcal{H}'_{n+1} + W_{n+1} = \sum \mathcal{H}'_{n+1} + \sum \mathcal{H}_{n+1} + \sum \mathcal{G}_{n+1} = \sum (\mathcal{H}'_{n+1} \cup \mathcal{H}_{n+1}) + \sum \mathcal{G}_{n+1}$. Put $\mathcal{H}_{n+2} = \mathcal{H}'_{n+1} \cup \mathcal{H}_{n+1}$. We readily see that property (3_{n+1}) is satisfied. It follows from the construction that $\mathcal{H}'_{n+1}$ is weakly independent over $\sum \mathcal{H}_{n+1} + \sum \mathcal{G}_{n+1}$ and that $\mathcal{G}_n \supseteq \mathcal{H}'_{n+1} \cup \mathcal{G}_{n+1}$. The latter together with (2_n) implies that $\mathcal{H}_{n+1}$ is weakly independent over $\sum \mathcal{H}'_{n+1} + \sum \mathcal{G}_{n+1}$. We conclude that $\mathcal{H}_{n+2}$ is weakly independent over $\sum \mathcal{G}_{n+1}$ which is the remaining property (3_{n+1}).

Put $\mathcal{G}_\omega = \bigcap_{n<\omega} \mathcal{G}_n$, $\mathcal{H}_\omega = \bigcup_{n<\omega} \mathcal{H}_n$ and $\mathcal{K} = \mathcal{G}_\omega + \mathcal{H}_\omega$. For each $n \in \omega$, property (1_n) implies that $\mathcal{G}_\omega$ is independent over $\sum \mathcal{H}_n$, whence we get that $\mathcal{G}_\omega$ is independent over $\sum \mathcal{H}_\omega$. Property (2_n) clearly implies that $\mathcal{H}_{n+1}$ is weakly independent over $\sum \mathcal{G}_\omega$, whence $\mathcal{H}_\omega$ is weakly independent over $\sum \mathcal{G}_\omega$. We conclude that the set $\mathcal{K}$ is weakly independent.

Property (3_n) implies $\sum \Delta(\mathcal{F}) \subseteq \sum \mathcal{G}_n + \sum \mathcal{H}_\omega$. Applying Corollary 3.2 we infer that $\sum \Delta(\mathcal{F}) \subseteq \sum \bigcap_{n<\omega} \mathcal{G}_n + \sum \mathcal{H}_\omega = \sum \mathcal{G}_\omega + \sum \mathcal{H}_\omega = \sum \mathcal{K}$. This concludes the proof. □

Corollary 3.6. *Let V be a vector space, let k be a positive integer, and let $\mathcal{F} \subseteq \mathrm{Sub}_k V$ be such that $V = \sum \mathcal{F}$. Suppose that there is a decomposition $V = \bigoplus_{i\in I} U_i$ of V into a direct sum of finitely generated subspaces U_i of V such that*

$$S = \bigoplus_{i\in I} (S \cap U_i) \quad \text{for all} \quad S \in \mathcal{F}. \tag{3.5}$$

Then $\mathcal{F}$ contains a weak basis of V.

Proof. With regard to Theorem 3.5, it suffices to verify that (3.5) implies that $\mathcal{U} = \{U_i \mid i \in I\} \subseteq \Delta(\mathcal{F})$. Towards this end, we get from (3.5) that for all $\mathcal{F}_0 \subseteq \mathcal{F}$ and all $j \in I$ the equality

$$U_j \cap \sum \mathcal{F}_0 = U_j \cap \sum_{S\in\mathcal{F}_0} \bigoplus_{i\in I} (S \cap U_i) = U_j \cap \bigoplus_{i\in I} \sum_{S\in\mathcal{F}_0} (U_i \cap S) = \sum_{S\in\mathcal{F}_0} (U_j \cap S)$$

holds true. This is the $\mathcal{F}$-distributivity of U_j, and so $\mathcal{U} \subseteq \Delta(\mathcal{F})$. □

4. The weak extension property

Lemma 4.1. *Let V be a vector space, let k be a positive integer, and let $\mathcal{F} \subseteq \mathrm{Sub}_k V$. Let $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$ be such that $\mathcal{G}$ is independent, $\mathcal{H}$ is weakly independent over $\sum \mathcal{G}$ and $\sum \Delta(\mathcal{F}) \subseteq \sum \mathcal{G} + \sum \mathcal{H}$. Then there is $\mathcal{G}' \subseteq \mathcal{G}$ weakly independent over $\sum \mathcal{H}$ such that $\sum \Delta(\mathcal{F}) \subseteq \sum \mathcal{G}' + \sum \mathcal{H}$.*

Proof. We put $\mathcal{G}_0 = \mathcal{G}$, $\mathcal{H}_1 = \mathcal{H}$ and continue as in the proof of Theorem 3.5. □

Definition. Let V be a vector space and let $\mathcal{F} \subseteq \mathrm{Sub}\, V$. We say that the set $\mathcal{F}$ has a *weak extension property* provided that for all $\mathcal{G} \subseteq \mathcal{F}$, every weakly independent subset of $\mathcal{G}$ extends to a weak basis of $\sum \mathcal{G}$.

We say that $\mathcal{F}$ is *good* provided that whenever $\sum \Delta(\mathcal{F}) \subseteq H = \sum \mathcal{H}$ for some $\mathcal{H} \subseteq \mathcal{F}$, the set $\mathcal{F}^H$ has a weak extension property.

Theorem 4.2. *Let V be a vector space and let k be a positive integer. Then every good $\mathcal{F} \subseteq \mathrm{Sub}_k(V)$ contains a weak basis of $\sum \mathcal{F}$.*

Proof. We proceed by induction on k. The statement is clear when $k = 1$. Let $k > 1$ and suppose that the statement holds whenever $\mathcal{F} \subseteq \mathrm{Sub}_{k-1} V$.

We start with finding $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$ such that $\mathcal{G}$ is independent, $\mathcal{H}$ is weakly independent over $\sum \mathcal{G}$ and

$$\sum \mathcal{F} = \sum \mathcal{G} + \sum \mathcal{H}. \tag{4.1}$$

We find $\mathcal{G}$ as a maximal independent subset of $\mathcal{F}$ by Zorn's lemma. It follows from the maximality of $\mathcal{G}$ that $S \cap \sum \mathcal{G} \neq 0$ for all $S \in \mathcal{F}$, whence, putting $G = \sum \mathcal{G}$, we have that $\mathcal{F}^G \subseteq \mathrm{Sub}_{k-1}(V/G)$. Thus, by the induction hypothesis, there is $\mathcal{H} \subseteq \mathcal{F}$ weakly independent over $\sum \mathcal{G}$ with $\sum \mathcal{F} = \sum \mathcal{G} + \sum \mathcal{H}$.

Next we are going to construct stepwise an decreasing class $\langle \mathcal{G}_\alpha \mid \alpha \in \mathbf{On} \rangle$ and an increasing class $\langle \mathcal{C}_\alpha \mid \alpha \in \mathbf{On} \rangle$ of subsets of $\mathcal{G}$ such that, putting

$$\mathcal{H}_\beta = \mathcal{H} \cup \bigcup_{\alpha \in \beta} C_\alpha \qquad (\beta \in \mathbf{On}),$$

the following properties are satisfied for all $\beta \in \mathbf{On}$:

(1_β) $\mathcal{G}_\beta$ is weakly independent over $\sum \mathcal{H}_\beta$ and $\sum \Delta(\mathcal{F}) \subseteq \sum \mathcal{G}_\beta + \sum \mathcal{H}_\beta$.
(2_β) $\mathcal{C}_\beta$ forms a weak basis of $\sum \mathcal{F}$ over $\sum \mathcal{G}_\beta + \sum \mathcal{H}$.

Step 0: Note that, by the definition, $\mathcal{H}_0 = \mathcal{H}$. Applying Lemma 4.1, we find $\mathcal{G}_0 \subseteq \mathcal{G}$ weakly independent over $\sum \mathcal{H}$ such that $\sum \Delta(\mathcal{F}) \subseteq \sum \mathcal{G}_0 + \sum \mathcal{H}$. This is property ($1_0$). Since $\mathcal{F}$ is good and $\sum \mathcal{F} \subseteq \sum \mathcal{G} + \sum \mathcal{H}$, there is a weak basis $\mathcal{C}_0 \subseteq \mathcal{G}$ of $\sum \mathcal{F}$ over $\sum \mathcal{G}_0 + \sum \mathcal{H}$. Thus we have ($2_0$) as well.

Step β for $\beta = \alpha + 1$: We imitate the initial step. First observe that $\mathcal{H}_\beta = \mathcal{H} \cup \mathcal{C}_\alpha$ and, applying Lemma 4.1, find $\mathcal{G}_\beta \subseteq \mathcal{G}_\alpha$ weakly independent over $\sum \mathcal{H}_\beta$ such that $\sum \Delta(\mathcal{F}) \subseteq \sum \mathcal{G}_\beta + \sum \mathcal{H}_\beta$, in particular, property (1_β) holds true. Since $\mathcal{G}_\beta \subseteq \mathcal{G}_\alpha$, it follows from property (2_α) that $\mathcal{C}_\alpha$ is weakly independent over $\sum \mathcal{G}_\alpha + \sum \mathcal{H}$. Now, since $\mathcal{F}$ is good and $\sum \mathcal{F} \subseteq \sum \mathcal{G} + \sum \mathcal{H}$, there is $\mathcal{C}_\beta$ a weak basis of $\sum \mathcal{F}$ over $\sum \mathcal{G}_\beta + \sum \mathcal{H}$ with $\mathcal{C}_\alpha \subseteq \mathcal{C}_\beta \subseteq \mathcal{G}$. Thus we have gained property (2_β).

Step β for β a limit ordinal: We put

$$\mathcal{G}_\beta = \bigcap_{\alpha \in \beta} \mathcal{G}_\alpha.$$

Since $\mathcal{G}_\alpha$ is weakly independent over $\sum \mathcal{H}_\alpha$, by (1_α), for all $\alpha \in \beta$, we infer that $\mathcal{G}_\beta$ is weakly independent over $\sum \mathcal{H}_\alpha$ for all $\alpha \in \beta$. Since, by its definition, $\mathcal{H}_\beta = \bigcup_{\alpha \in \beta} \mathcal{H}_\alpha$, we conclude that $\mathcal{G}_\beta$ is weakly independent over $\sum \mathcal{H}_\beta$. Properties (1_α), $\alpha \in \beta$, give us that $\sum \Delta(\mathcal{F}) \subseteq \sum \mathcal{G}_\alpha + \sum \mathcal{H}$, for all $\alpha \in \beta$. Applying Corollary 3.2, we infer that

$$\sum \Delta(\mathcal{F}) \subseteq \sum \mathcal{G}_\beta + \sum \mathcal{H}. \tag{4.2}$$

Thus property (1_β) holds true. It follows from (2_α), $\alpha \in \beta$, that the union $\bigcup_{\alpha \in \beta} \mathcal{C}_\alpha$ is weakly independent over $\sum \mathcal{G}_\beta + \sum \mathcal{H}$. Since $\mathcal{F}$ is good, it follows from equality (4.1) and inclusion (4.2) that there is a weak basis $\mathcal{C}_\beta$ of $\sum \mathcal{F}$ over $\sum \mathcal{G}_\beta + \sum \mathcal{H}$ satisfying $\bigcup_{\alpha \in \beta} \mathcal{C}_\alpha \subseteq \mathcal{C}_\beta \subseteq \mathcal{G}$. We conclude that property (2_β) is satisfied.

Since $\operatorname{Sub}_k V$ does not form a proper class, there is $\alpha \in \mathbf{On}$ with $\mathcal{C}_\alpha = \mathcal{C}_{\alpha+1}$. From $(1_{\alpha+1})$, we get that $\mathcal{G}_{\alpha+1}$ is weakly independent over $\sum \mathcal{C}_\alpha + \sum \mathcal{H}$ and from $(2_{\alpha+1})$ we get that $\mathcal{C}_\alpha = \mathcal{C}_{\alpha+1}$ forms a weak basis of $\sum \mathcal{F}$ over $\sum \mathcal{G}_{\alpha+1} + \sum \mathcal{H}$; in particular $\sum \mathcal{F} = \sum \mathcal{C}_\alpha + \sum \mathcal{G}_{\alpha+1} + \sum \mathcal{H}$. Since both $\mathcal{G}_{\alpha+1}$ and $\mathcal{C}_\alpha$ are included in $\mathcal{G}$, we infer that $\mathcal{H}$ is weakly independent over $\mathcal{G}_{\alpha+1} \cup \mathcal{C}_\alpha$. Summing it up, we get that the union $\mathcal{G}_{\alpha+1} \cup \mathcal{C}_\alpha \cup \mathcal{H}$ forms the desired weak basis of $\sum \mathcal{F}$ selected from $\mathcal{F}$. □

Corollary 4.3. *Let V be a vector space, let k be a positive integer, and let $\mathcal{F} \subseteq \operatorname{Sub}_k V$. Put $D = \sum \Delta(\mathcal{F})$ and suppose that $\mathcal{F}^D \subseteq \operatorname{Sub}_1(V/D)$. Then $\mathcal{F}$ contains a weak basis of $\sum \mathcal{F}$.*

A particular case of this statement extends Corollary 3.6 as follows:

Corollary 4.4. *Let V be a vector space, let k be a positive integer, and let $\mathcal{F} \subseteq \operatorname{Sub}_k V$ be such that $V = \sum \mathcal{F}$. Suppose that there is a decomposition $V = W \oplus \bigoplus_{i \in I} U_i$, where all U_i are finitely dimensional such that*

$$\dim(S \cap W) \leq 1 \quad \text{and} \quad S = (S \cap W) \oplus \bigoplus_{i \in I} (S \cap U_i) \quad \text{for all} \quad S \in \mathcal{F}.$$

Then $\mathcal{F}$ contains a weak basis of V.

In view of Corollary 4.4 we formulate the following problem, which seems to be the next step when trying to decide Problem 1.1:

Problem 4.1. *Let V be a vector space and let $\mathcal{F} \subseteq \operatorname{Sub} V$ be such that $V = \sum \mathcal{F}$. Let n be a positive integer and suppose that there is a decomposition $V = W_1 \oplus \cdots \oplus W_n$ such that for all $S \in \mathcal{F}$:*

$$\dim(S \cap W_i) \leq 1, \text{ for all } i = 1, \dots, n, \quad \text{and} \quad S = \bigoplus_{i=1}^{n} (S \cap W_i). \tag{4.3}$$

Does such $\mathcal{F}$ contain a weak basis of V?

A particular task would be the case Problem 4.1 when $n = 2$. Finally notice that assumption (4.3) can be placed with considering modules over a finite product of copies of the division ring D. A slight generalization of this, considering a product of (not necessarily same) division rings, is then the case (2) of our original problem.

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